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Recursive Heaviside step functions and beginning of the universe

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HIGHLIGHTS

- Recursive Heaviside step functions are introduced for the first time.
- The new generalized functions are analyzed by using sequence approximations.
- The hefty mass during the crunch-bounce transition period introduces a time shift.
- The time shift is suggested as a possible power source for the Big Bounce.

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ABSTRACT

This article introduces recursive Heaviside step functions, as a potential of the known universe, for the first time in the history of mathematics, science, and engineering. In modern cosmology, various bouncing models have been suggested based on the postulation that the current universe is the result of the collapse of a previous universe. However, all Big Bounce models leave unanswered the question of what powered inflation. Recursive Heaviside step functions are analyzed to represent the warpage of space-time during the crunch-bounce transition. In particular, the time shift appeared during the transition is modeled in the form of recursive Heaviside step functions and suggested as a possible answer for the immeasurable energy appeared for the Big Bounce.

1. Introduction

It is difficult and challenging to imagine the time of the Big Bang (or, Big Bounce), about 13.8 billion years ago, when the entire universe existed suddenly as a singularity (Hawking and Penrose, 1970; Silk, 2000). However, scientists and theologians have asked even more difficult question like “What existed before the Big Bang?”, in order to understand the pre-Big Bang singularity and the origin of our universe.

St. Augustine, a fourth-century Christian theologian and philosopher, asked the same question and answered with a view that *everything in the universe was created simultaneously by God, and not in seven calendar days like Genesis said literally* (Young, 1988). He claimed that both time and space were God’s creation, and there simply was nothing before the Big Bang. Such a view point on time, appeared in the 4th-century, continued to be claimed by some of best mathematical physicists in the 20th century. For example, Albert Einstein came to very similar conclusions with his general theory of relativity (GTR), in particular, the effect of mass on time. In Einstein’s theory, a planet’s massive mass warps time – making time run slower on Earth’s surface than a satellite in orbit. The heftier mass a human stands by, the slower time runs due to a stronger gravitational force, which is called the gravitational time dilation (Roos, 2003). The pre-Big Bang singularity possessed all the mass in our universe, which must have effectively brought time to a standstill.

Modern theory of the pre-Big Bang singularity has been developed with questions like: *Is our universe an offspring of another older universe?* Astronomers first observed the cosmic microwave background radiation (CMBR) in 1964 (Penzias and Wilson, 1965), which created the *inflation theory* for the Big Bang in early 1980s (Guth, 1980, 1981). This theory entails an extremely rapid expansion of the universe in the first few moments of its existence (between 10^{-35} and 10^{-32} s), explaining the temperature and density fluctuations in the CMBR (Guth, 1980). Then, some cosmologists have postulated an endless and recursive progress of inflationary bubbles, each of them becoming a universe and forming even more inflationary bubbles in an immeasurable multiverse (Byrne, 2010); although some have argued that the multiverse is a philosophical rather than a scientific hypothesis due to the lack of falsification (Steinhardt, 2014). See the review paper (Kragh, 2009) for history of cosmology and the controversy over the multiverse.

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Other models for the universe or multiverse are still being revolved in order to formulate the pre-Big Bang singularity in a different, more reliable way. Some astrophysicists speculate that our expanding universe could be the *white hole* output from a black hole in another universe (Retter and Heller, 2012). On the other hand, other scientists place the formation of the singularity inside a cycle called the *Big Bounce* in which our expanding universe will eventually collapse back in on itself in an event called the *Big Crunch* (Poplawski, 2012). Recently, a group of astrophysicists have suggested a new model of "nonsingular Big Bounce" in order to overcome problems related to the singularity (Overduin et al., 2007; Easson and Brandenberger, 2012; Vilenkin, 2013); in nonsingular Big Bounce models, the GTR can be applied over the time of crunch-bounce transition. However all Big Bounce models leave unanswered the question of what powered inflation (NASA, 2016). See Silk (2005), for cosmology and numerous topics related to our current understanding of creation and evolution of the Universe.

In this article, we will try to give a scenario that explains a power source for the transition period of the Big Crunch and the Big Bounce. Just before the Big Bounce, all the mass of the current universe must be possessed in the "small universe" in which time runs slower. Considering such a time warpage, we will suggest a mathematical model in the form of recursive Heaviside step functions which represents the state of our universe during the time of crunch-bounce transition. The generalized step functions are analyzed to estimate the energy appeared for the Big Bounce, the jump in the state of our universe.

The article is organized as follows. In the next section, we will introduce recursive Heaviside step functions and consider their differential properties, as preliminaries. Section 3 considers a potential, in terms of recursive Heaviside step functions, to represent the state of our universe seen as a part of multiverse. In Section 4, we summarize our work and present conclusions.

2. Recursive Heaviside step functions

Consider recursive step functions of the form

$$\begin{aligned} U_0(t,\tau) &= 1, \\ U_n(t,\tau) &= H(t-\tau\, U_{n-1}(t,\tau)), \quad n \geq 1, \end{aligned} \tag{1}$$

where $\tau \neq 0$ and H denotes the *Heaviside step function* defined as

$$H(t) = \begin{cases} 0, & t < 0, \\ 1, & t \geq 0. \end{cases} \tag{2}$$

We call U_n the *recursive Heaviside step function of degree n, $n \geq 1$.*

High-degree recursive step functions U_n, $n \geq 2$, have never been introduced in the history of mathematics, science, and engineering. In this article, we will study their properties, using their sequence representations, and applications to modern cosmology. Such recursive step functions are not functions but *generalized functions* (distributions), having special properties.

For example, for $U_1(t,\tau) = H(t-\tau)$, the (standard) Heaviside step function. Its derivative is *formally* defined to be the Dirac delta function $\delta(t-\tau)$:

$$\frac{dH(t-\tau)}{dt} = \delta(t-\tau) \quad \text{and} \quad \int_{-\infty}^{t} \delta(t-\tau)\, dt = H(t-\tau), \tag{3}$$

which can be found from most of introductory textbooks of differential equations. See Duff and Naylor (1966), for example. However, high-degree recursive Heaviside step functions ($n \geq 2$) may not be treated in the same way as in (3). That is,

$$U_n(t,\tau) \neq \int_{-\infty}^{t} \frac{dU_n(t,\tau)}{dt}\, dt, \quad n \geq 2. \tag{4}$$

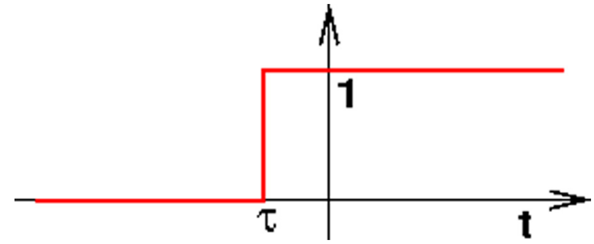

Fig. 1. The plot of the recursive Heaviside step functions $U_n(t,\tau)$, $n \geq 1$, for $\tau < 0$.

These high-degree recursive Heaviside step functions have never been studied in the history of mathematics, science, and engineering. In this article, the right-side of (4) will be estimated by applying sequence approximations; see Section 3 for details.

We will close the section with the following remark. When $\tau < 0$, it is not difficult to check that all of U_n assign exactly the same value for each t regardless of the degree $n \geq 1$. See Fig. 1, where

$$U_m(t,\tau) = U_n(t,\tau), \quad \tau < 0, \tag{5}$$

for all choices of degrees m, $n \geq 1$. However, their *formal* derivatives are different from each other. Details will be presented in the following.

3. Potential, force, and the energy

This section introduces a potential which represents the state of our universe seen as a part of multiverse. The potential is defined only in time, because it is not much meaningful to point out the location of the universe. Then, we will study the derivative of the potential, which becomes force. Integrating the force over time, we will see a huge (possibly, infinite) amount of energy involved during the change of state of the universe.

3.1. Potential of the known universe

We will begin with the postulation that the universe was made by the Big Bounce right after a Big Crunch (Poplawski, 2012; Vilenkin, 2013). The sudden change of state of the universe can be expressed by using the Heaviside step function

$$U_1(t,\tau) = H(t-\tau), \tag{6}$$

where t is the time of the universe and τ denotes the beginning of the universe. (One may consider $\tau = 0$ momentarily.)

In the final stages of the Big Crunch, the parent universe must have become compacted into a very small volume and the warpage of spacetime came to be so chaotic that time in particular shattered into droplets not to be able to run continuously. Then, right after the critical moment, the time of the current universe appears and speeds up to run regularly. Such a time discontinuity can be explained as a time shift, which may produce or absorb energy, as we will see below. This is an analogue of the energy requirement for a sudden space shift in classical mechanics.

The critical moment cannot be explained by the theory of relativity due to the singularity; nonsingular bouncing models have been developed to expand the theory of relativity over the critical moment (Overduin et al., 2007; Easson and Brandenberger, 2012; Vilenkin, 2013). Similarly, we may expand the time of universe t up to the moment just before the Big Bounce. When the time axis of the current university is assumed to be uniform, the event of time discontinuity makes the Big Bounce look to begin a little bit earlier than time zero ($t = 0$). This phenomenon can be equivalently explained as follows: *Appearing at $t = 0$, the time of the current universe instantaneously shifted back to $t = \tau < 0$ and began to run forward*. Then, the potential U_1 in (6) must be reformulated in order to incorporate with such a postulation properly; a logical way is to replace τ in (6) with $\tau H(t-\tau)$. That is,

$$U_2(t,\tau) := H(t-\tau H(t-\tau)), \quad \tau < 0. \tag{7}$$

However every moment of time discontinuity must invoke a time shift, which implies that the time shift must happen many times *recursively*, by a small amount each, till the time axis looks uniform. Thus, the last τ in (7) must be replaced by $\tau H(t-\tau)$ and this logic can be applied again and again, recursively. Therefore, in general, the potential of the universe can be formulated as follows. For $n \geq 1$,

$$U_n(t,\tau) := H(t-\tau U_{n-1}(t,\tau)), \quad \tau < 0. \tag{8}$$

Here we have set $U_0(t,\tau) = 1$.

3.2. The force

When the potential U_n is differentiated with respected to t, the result becomes force. The *formal* t-derivative of (7), for example, can be formulated by using the chain rule of differentiation as follows.

$$\frac{\partial U_2}{\partial t} = \delta(t-\tau H(t-\tau)) \cdot [1-\tau\delta(t-\tau)], \tag{9}$$

and therefore

$$\left.\frac{\partial U_2}{\partial t}\right|_{t\geq\tau} = \delta(t-\tau)\cdot[1-\tau\delta(t-\tau)]. \tag{10}$$

Similarly, the t-derivative of U_n reads

$$\begin{aligned}\frac{\partial U_n}{\partial t} &= \delta(t-\tau U_{n-1})\cdot\left(1-\tau\frac{\partial U_{n-1}}{\partial t}\right)\\ &= \delta(t-\tau U_{n-1})\cdot\left[1-\tau\cdot\left\{\delta(t-\tau U_{n-2})\right.\right.\\ &\quad\left.\left.\times\left(1-\tau\frac{\partial U_{n-2}}{\partial t}\right)\right\}\right],\end{aligned} \tag{11}$$

and therefore

$$\begin{aligned}\left.\frac{\partial U_n}{\partial t}\right|_{t\geq\tau} &= \delta(t-\tau)\cdot\left[1-\tau\cdot\left\{\delta(t-\tau)\right.\right.\\ &\quad\left.\left.\times\left(1-\tau\left.\frac{\partial U_{n-2}}{\partial t}\right|_{t\geq\tau}\right)\right\}\right]\\ &= \delta(t-\tau)\cdot\sum_{j=0}^{n-1}(-\tau\delta(t-\tau))^j.\end{aligned} \tag{12}$$

While it is possible to carry out the formal differentiation using the chain rule as in the above, it is interesting to figure out the derivatives by replacing the Heaviside step function with a Heaviside step sequence H_k.

$$H_k(t-\tau) := \frac{1+\tanh k(t-\tau)}{2}, \tag{13}$$

where, by definition, $\tanh z = \sinh z/\cosh z = (e^z - e^{-z})/(e^z + e^{-z}) \in (-1, 1)$. It is known that

$$H(t-\tau) = \lim_{k\to\infty} H_k(t-\tau). \tag{14}$$

Similarly, the Dirac delta function can be approximated by a delta function sequence δ_k, which is the derivative of H_k, given as

$$\delta_k(t-\tau) = \frac{d}{dt}\frac{1+\tanh k(t-\tau)}{2} = \frac{k}{2}\mathrm{sech}^2 k(t-\tau). \tag{15}$$

Then, we have

$$\delta(t-\tau) = \lim_{k\to\infty}\delta_k(t-\tau). \tag{16}$$

Now, we can approximate U_n in (8) by replacing H with H_k in (13). Define $U_{n,k}$ as

$$U_{n,k}(t,\tau) = H_k(t-\tau U_{n-1,k}(t,\tau)). \tag{17}$$

Then, it is easy to see that

$$U_n(t,\tau) = \lim_{k\to\infty} U_{n,k}(t,\tau). \tag{18}$$

The t-derivative of $U_{n,k}$ reads similarly as in (11):

$$\begin{aligned}\frac{\partial U_{n,k}}{\partial t} &= \delta_k(t-\tau U_{n-1,k})\cdot\left(1-\tau\frac{\partial U_{n-1,k}}{\partial t}\right)\\ &= \delta_k(t-\tau U_{n-1,k})\cdot\left[1-\tau\cdot\left\{\delta_k(t-\tau U_{n-2,k})\right.\right.\\ &\quad\left.\left.\times\left(1-\tau\frac{\partial U_{n-2,k}}{\partial t}\right)\right\}\right].\end{aligned} \tag{19}$$

When $t \geq \tau$, using the formulas in (12) and (15), the force $\partial U_{n,k}/\partial t$ can be expressed as

$$\begin{aligned}\left.\frac{\partial U_{n,k}}{\partial t}\right|_{t\geq\tau} &= \delta_k(t-\tau)\cdot\sum_{j=0}^{n-1}(-\tau\delta_k(t-\tau))^j\\ &= \frac{k}{2}\mathrm{sech}^2 k(t-\tau)\cdot\sum_{j=0}^{n-1}\left(-\tau\frac{k}{2}\mathrm{sech}^2 k(t-\tau)\right)^j.\end{aligned} \tag{20}$$

3.3. Estimation of the energy

In classical mechanics, the energy (work) is defined as the integration of the force over the distance. This concept is quite clear because the force is considered for the purpose of displacement. However, in our study of cosmology, the potential is defined only in time and the force (the derivative of the potential) is due to the time shift. Thus the energy can be defined as the integration of the force over the time. That is, the integration of the force over the time represents the total energy appeared for the Big Bounce, the change of state of the current universe.

As aforementioned, modern theory of the pre-Big Bang singularity postulated that the universe expanded extremely rapidly in the first few moments of its existence (between 10^{-35} and 10^{-32} s). Whatever it is, the duration of rapid expansion can be expressed as $[\tau_0, 0]$ for some $\tau_0 < 0$, concerning the time shift. For convenience, we select $\tau = \tau_0 = -10^{-32}$. Note that $\partial U_{n,k}/\partial t$ in (20) is symmetric with respected to $t = \tau$. Thus the total energy released during the Big Bounce due to the time shift can be estimated by integrating $\partial U_{n,k}/\partial t$ over the time period $[\tau, 0]$, for large n and k, and multiplying the result by 2.

Define the energy $E(n, k)$ as

$$\begin{aligned}E(n,k) &:= 2\int_{\tau_0}^{0}\frac{\partial U_{n,k}}{\partial t}\,dt\\ &= 2\int_{\tau_0}^{0}\frac{k}{2}\mathrm{sech}^2 k(t-\tau)\\ &\quad\times\sum_{j=0}^{n-1}\left(-\tau\frac{k}{2}\mathrm{sech}^2 k(t-\tau)\right)^j dt.\end{aligned} \tag{21}$$

We introduce a change of variable, $t = -\tau_0 s$, for a convenient evaluation of the integral which depends on k and n. Then the energy in (21) becomes

$$\begin{aligned}E(n,\sigma) &= \int_{-1}^{0}\sigma\,\mathrm{sech}^2(\sigma(1+s))\\ &\quad\times\sum_{j=0}^{n-1}\left(\frac{\sigma}{2}\mathrm{sech}^2(\sigma(1+s))\right)^j ds,\end{aligned} \tag{22}$$

where $\sigma = -\tau_0 k$.

There is no compact formula for the integral in (22). Let $E_{\mathrm{comp}}(n,\sigma)$ be an computed value of $E(n, \sigma)$, using Maple. Then we have observed that for each fixed σ,

$$\frac{E_{\mathrm{comp}}(n+1,\sigma)}{E_{\mathrm{comp}}(n,\sigma)} \approx \frac{E_{\mathrm{comp}}(n,\sigma)}{E_{\mathrm{comp}}(n-1,\sigma)}, \quad n \geq 1, \tag{23}$$

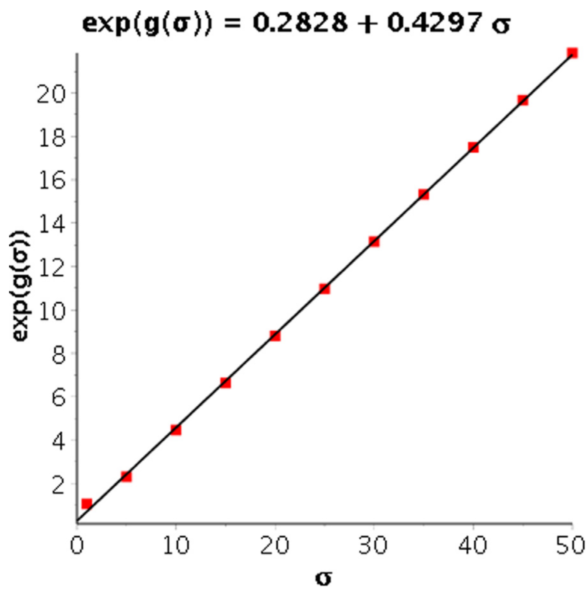

Fig. 2. Points $(\sigma,\ e^{g(\sigma)})$, for selected σ's, and their least-squares fitting line.

which implies that if the total energy is formulated as

$$E(n,\sigma) \approx C\, e^{g(\sigma)\cdot(n-1)}, \tag{24}$$

for a constant $C > 0$, then the function g depends only on σ. It is easy to see that $C = 1$, because the total energy $\lim_{\sigma\to\infty} E(1,\sigma) = 1$ (when $n = 1$). Furthermore, it follows from (24) that

$$\frac{E(n+1,\sigma)}{E(n,\sigma)} \approx \frac{e^{g(\sigma)\cdot n}}{e^{g(\sigma)\cdot(n-1)}} = e^{g(\sigma)}, \quad n \ge 1. \tag{25}$$

Fig. 2 depicts points $(\sigma,\ e^{g(\sigma)})$, for selected σ's, and their least-squares fitting line reads

$$e^{g(\sigma)} = 0.2828 + 0.4297\,\sigma. \tag{26}$$

Thus it follows from (24), (26), and $C = 1$ that

$$E(n,\sigma) \approx (0.2828 + 0.4297\,\sigma)^{n-1}, \tag{27}$$

where $\sigma = -\tau_0 k \ge 1$.

Remarks. For $n \ge 2$, the total energy becomes larger as σ (and k) increases. When the force profile of the Big Bounce is experimentally known, the constant k in $\partial U_{n,\,k}/\partial t$ can be decided to estimate the total energy. When $\sigma < 1.669$, the total energy produced may approach zero as n increases. We do not know what determines the degree n. However, the set of recursive Heaviside step functions is a possible model for the warpage of spacetime during the time of crunch-bounce transition.

4. Conclusions

This article has introduced recursive Heaviside step functions, for the first time in mathematics and science, as a potential of the known universe and as a model for the warpage of spacetime during the time of crunch-bounce transition. The new generalized functions are analyzed by using sequence approximations. In particular, the time shift appeared during the transition period is suggested as a possible answer for the huge amount of energy appeared at the beginning of the Big Bounce.

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