Multivariable Calculus

(following "J. R. Stewart, *Calculus, 8th Ed.*")

Lectures on YouTube:
https://www.youtube.com/channel/UCmRbK4vlGDht-joOQ5g0J2Q

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Prologue

This lecture note is closely following the part of multivariable calculus in *Stewart's book* [7]. In organizing this lecture note, I am indebted by *Cedar Crest College Calculus IV Lecture Notes*, Dr. James Hammer [1].

Two projects are included for students to experience **computer algebra**. Computer algebra (also called **symbolic computation**) is a scientific area that refers to the study and development of algorithms and software for manipulating mathematical expressions and other mathematical objects; it emphasizes **exact computation** with expressions containing variables that have no given value and are manipulated as symbols. In practice, you can use computer algebra to effectively handle complex math equations and problems that would be simply too complicated/time-consuming to do by hand. The projects are organized using **Maple**.

Through the lecture note, I tried to make figures using Maple. Also added are some of programming scripts written in Maple. The end of each section includes exercise problems. For problems indicated by the *Computer Algebra System* (CAS) sign CAS, you are recommended to use a CAS to solve the problem.

Currently the lecture note is not fully grown up; other useful techniques and interesting examples would be soon incorporated. Any questions, suggestions, comments will be deeply appreciated.

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June 17, 2022

Contents

CHAPTER **12**

Vectors and the Geometry of Space

In this chapter, we study vectors and equations in the 3-dimensional (3D) space. In particular, you will learn

- vectors
- dot product
- cross product
- equations of lines and planes, and
- cylinders and quadric surfaces

Contents of Chapter 12

This chapter corresponds to Chapter 12 in STEWART, ***Calculus*** (8th Ed.), 2015.

12.1. Vector Operations

There exists a lot to cover in the class of multivariable calculus; however, it is important to have a good foundation before we trudge forward. In that vein, let's review vectors and their geometry in space ($\mathbb{R}^3$) briefly.

12.1.1. 3D coordinate systems

Recall: Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points in $\mathbb{R}^2$. Then the **distance** from P to Q is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{12.1}$$

Definition 12.1. *Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. Then the **distance** from P to Q is*

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{12.2}$$

Problem 12.2. Find the distance between $P(-3, 2, 7)$ and $Q(-1, 0, 6)$.

Solution.

Ans: 3

Recall: A **circle** in $\mathbb{R}^2$ is defined to be all of the points in the plane ($\mathbb{R}^2$) that are equidistant from a central point $C(a, b)$.

$$(x-a)^2 + (y-b)^2 = r^2. \qquad (12.3)$$

A natural generalization of this to 3-D space would be to say that a sphere is defined to be all of the points in $\mathbb{R}^3$ that are equidistant from a central point C. This is exactly what the following definition does!

Definition 12.3. *Let $C(h, k, l)$ be a point in $\mathbb{R}^3$. Then the **sphere** centered at C with radius r is defined by the equation*

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2. \qquad (12.4)$$

That is to say that this defines all points $(x, y, z) \in \mathbb{R}^3$ that are at the same distance r from the center $C(h, k, l)$.

Problem 12.4. Show that $x^2 + y^2 + z^2 - 4x + 2y - 6z + 10 = 0$ is the equation of a sphere, and find its center and radius.

Solution.

Ans: $C(2, -1, 3)$ and $r = 2$

12.1.2. Vectors and vector operations

Definition 12.5. *A **vector** is a mathematical object that stores both length (which we will often call magnitude) and direction.*

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the *vector* with initial point P and terminal point Q (denoted $\overrightarrow{PQ}$) is defined by

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \overrightarrow{OQ} - \overrightarrow{OP},$$

where O is the origin, $O = (0, 0, 0)$. The vector $\overrightarrow{OP}$ is called the **position vector** of the point P. For convenience, we use bold-faced lower-case letters to denote vectors. For example, $\mathbf{v} =< v_1, v_2, v_3 >$ is a (position) vector in $\mathbb{R}^3$ associated with the point (v_1, v_2, v_3).

Definition 12.6. *Two vectors are said to be **equal** if and only if they have the same length and direction, regardless of their position in $\mathbb{R}^3$. That is to say that a vector can be moved (with no change) anywhere in space as long as the magnitude and direction are preserved.*

Definition 12.7. *Let $\mathbf{v} =< v_1, v_2, v_3 >$. Then the **magnitude** (a.k.a. **length** or **norm**) of $\mathbf{v}$ (denoted $|\mathbf{v}|$ or sometimes $||\mathbf{v}||$) is defined by*

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{12.5}$$

Definition 12.8. *(Vector addition) Let $\mathbf{u} =< u_1, u_2, u_3 >$ and $\mathbf{v} =< v_1, v_2, v_3 >$. Then*

$$\mathbf{u} + \mathbf{v} =< u_1 + v_1, u_2 + v_2, u_3 + v_3 > .$$

Definition 12.9. *(Scalar multiplication) Let* $\mathbf{v} =< v_1, v_2, v_3 >$ *and* $k \in \mathbb{R}$. *Then*

$$k\,\mathbf{v} =< kv_1, kv_2, kv_3 > .$$

Problem 12.10. If $\mathbf{a} =< 0, 3, 4 >$ and $\mathbf{b} =< 1, 5, 2 >$, find $|\mathbf{a}|$, $2\mathbf{a} - 3\mathbf{b}$, and $|2\mathbf{a} - 3\mathbf{b}|$.

Solution.

Ans: $|\mathbf{a}| = 5$; $2\mathbf{a} - 3\mathbf{b} =< -3, -9, 2 >$; $|2\mathbf{a} - 3\mathbf{b}| = \sqrt{94}$

Definition 12.11. *A* ***unit vector*** *is a vector whose magnitude is 1. Note that given a vector* $\mathbf{v}$, *we can form a unit vector (of the same direction) by dividing by its magnitude. That is, let* $\mathbf{v} =< v_1, v_2, v_3 >$. *Then*

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{12.6}$$

is a unit vector in the direction of $\mathbf{v}$.

Definition 12.12. *Any vector can be denoted as the linear combination of the* ***standard unit vectors***

$$\mathbf{i} =< 1, 0, 0 >, \quad \mathbf{j} =< 0, 1, 0 >, \quad \mathbf{k} =< 0, 0, 1 > .$$

So given a vector $\mathbf{v} =< v_1, v_2, v_3 >$, one can express it with respect to the standard unit vectors as

$$\mathbf{v} =< v_1, v_2, v_3 >= v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}. \tag{12.7}$$

This text, however, will more often than not use the angle brace notation.

Definition 12.13. *Let* $\mathbf{u} =< u_1, u_2, u_3 >$ *and* $\mathbf{v} =< v_1, v_2, v_3 >$*. Then the* ***dot product*** *is*

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + u_3 v_3, \tag{12.8}$$

which is sometimes referred as the ***Euclidean inner product****. Note that* $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$*.*

Theorem 12.14. *Let* θ *be the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *(so* $0 \le \theta \le \pi$*). Then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\,\cos(\theta). \tag{12.9}$$

Corollary 12.15. *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if* $\mathbf{u} \cdot \mathbf{v} = 0$*.*

Problem 12.16. Find the angle between the vectors $\mathbf{a} =< 2, 2, 1 >$ and $\mathbf{b} =< 3, 0, 3 >$.

Solution.

Ans: $\pi/4\,(= 45^\circ)$

Definition 12.17. *Let* $\mathbf{u} =< u_1, u_2, u_3 >$ *and* $\mathbf{v} =< v_1, v_2, v_3 >$*. Then the* ***cross product*** *is the determinant of the following matrix:*

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ &= \det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \mathbf{k} \\ &= < u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 > . \end{aligned} \tag{12.10}$$

Problem 12.18. Find the cross product $\mathrm{a} \times \mathrm{b}$, when $\mathrm{a} =< 1, 3, 4 >$ and $\mathrm{b} =< 3, -1, -2 >$.

Solution.

Ans: $< -2, 14, -10 >$

Theorem 12.19. *The vector* $\mathrm{a} \times \mathrm{b}$ *is orthogonal to both* a *and* b*.*

Theorem 12.20. *Let* θ *be the angle between* a *and* b *(so* $0 \le \theta \le \pi$*). Then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,\sin(\theta). \tag{12.11}$$

Claim 12.21. *The length of the cross product* $a \times b$ *is equal to the area of the parallelogram determined by* a *and* b.

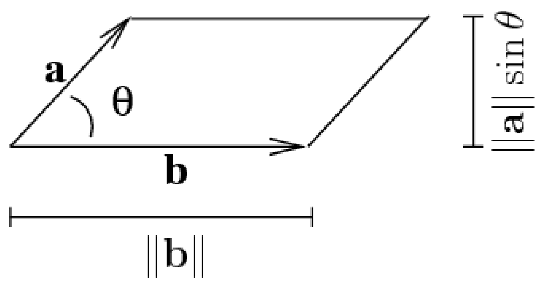


Figure 12.1

Problem 12.22. Prove that two nonzero vectors a and b are parallel if and only if $a \times b = 0$.

Solution.

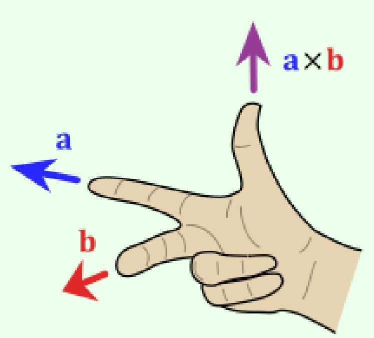


Figure 12.2: Finding the direction of the cross product by the right-hand rule.

The cross product $a \times b$ is defined as a vector that is perpendicular (orthogonal) to both a and b, with a direction given by the **right-hand rule** and a magnitude equal to the area of the parallelogram that the vectors span.

If the fingers of your right hand curl in the direction of a rotation (through an angle less than $180°$) from a to b, then the thumb points in the direction of $a \times b$. See Figure 12.2.

Exercises 12.1

1. Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

   (a) $\mathbf{a} =< 1, 2, -1 >$, $\mathbf{b} =< 2, 0, -3 >$
   (b) $\mathbf{a} =< 1, t, 1/t >$, $\mathbf{b} =< t^2, t, 1 >$

2. Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

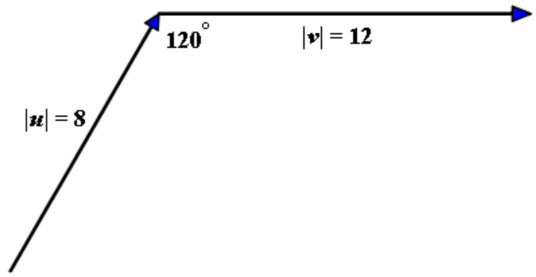


Figure 12.3

3. (i) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (ii) find the area of the triangle PQR.

   (a) $P(1, 0, 1),\ \ Q(2, 1, 3),\ \ R(-3, 2, 5)$
   Ans: $< 0, -12, 6 >, 3\sqrt{5}$
   (b) $P(1, -1, 0),\ \ Q(-3, 1, 2),\ \ R(0, 3, -1)$
   Ans: $< -10, -6, -14 >, \sqrt{83}$

4. Find the angle between $\mathbf{a}$ and $\mathbf{b}$, when $\mathbf{a} \cdot \mathbf{b} = -\sqrt{3}$ and $\mathbf{a} \times \mathbf{b} =< 2, 2, 1 >$.
   Ans: $120°$

Note: Exercise problems are added for your homework; answers would be provided for some of them. However, you have to verify them, by showing solutions in detail.

12.2. Equations in the 3D Space

Objective: To build equations of lines, line segments, and planes.

Parametrization of a Line. Let $P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, and $\mathbf{v} = \langle a, b, c\rangle$ be a vector in $\mathbb{R}^3$. Then the line through P_0 parallel to $\mathbf{v}$ is

$$\mathbf{r} = P_0 + t\,\mathbf{v}, \quad t \in \mathbb{R}. \tag{12.12}$$

This can also be written as

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct; \quad t \in \mathbb{R}. \tag{12.13}$$

or as the symmetric equation

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{12.14}$$

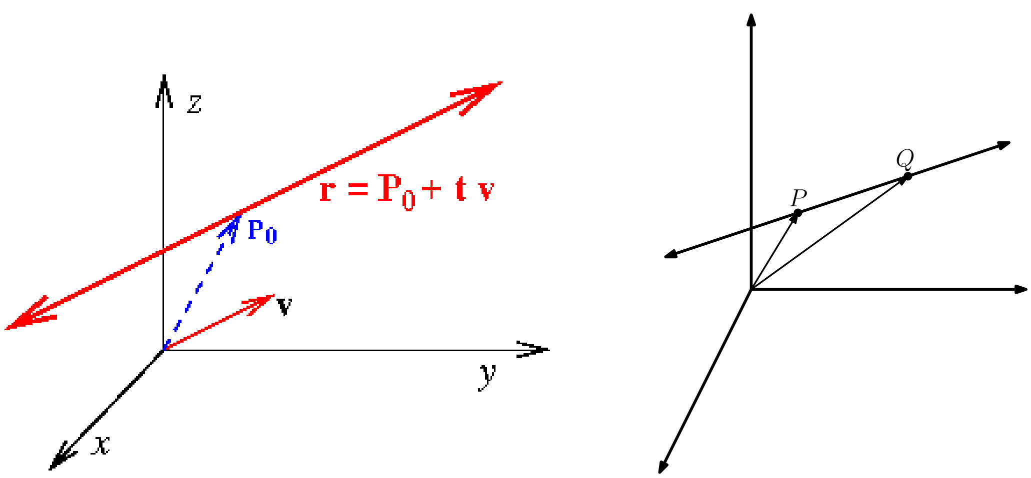


Figure 12.4: Parametrization: (left) line and (right) line segment.

Parametrization of a Line Segment. Let P and Q be respectively the initial and terminal points of a line segment. Then the line segment $\overline{PQ}$ can be parametrized as

$$\mathbf{r}(t) = (1 - t)\,\overrightarrow{OP} + t\,\overrightarrow{OQ}, \quad 0 \le t \le 1. \tag{12.15}$$

Problem 12.23. Find a vector equation and parametric equation for the line that passes through the point $(5, 1, 3)$ and is parallel to $\langle 1, 4, -2\rangle$.

Solution.

Ans: $x = 5 + t$, $y = 1 + 4t$, $z = 3 - 2t$

Problem 12.24. Find the parametric equation of the line segment from $(2, 4, -3)$ to $(3, -1, 1)$.

Solution.

Ans: $\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t\rangle$, $0 \leq t \leq 1$.

Planes. Let $\mathbf{x}_0 = (x_0, y_0, z_0)$ be a point in the plane and $\mathbf{n} = \langle a, b, c\rangle$ be a vector normal to the plane. Then the equation of the plane is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0. \qquad (12.16)$$

Problem 12.25. Find an equation of the plane that passes through the points $P(1, 2, 3)$, $Q(3, 2, 4)$, and $R(1, 5, 2)$.

Solution.

Ans: $-3(x - 1) + 2(y - 2) + 6(z - 3) = 0$.

Exercises 12.2

1. Find an equation of the line which passes through $(1, 0, 3)$ and *perpendicular* to the plane $x - 3y + 2z = 4$.

2. Find the line of the intersection of planes $x + 2y + 3z = 6$ and $x - y + z = 1$. (***Hint***: The intersection is a line; consider how the direction of the line is related to the normal vectors of the planes.)

   Ans: $\mathbf{r} = P_0 + t\mathbf{v} = < 1, 1, 1 > + t < 5, 2, -3 >$

3. Find the vector equation for the line segment from $P(1, 2, -4)$ to $Q(5, 6, 0)$.

4. Find an equation of the plane.

   (a) The plane through the point $(0, 1, 2)$ and *parallel* to the plane $x - y + 2z = 4$.
   (b) The plane through the points $P(1, -2, 2)$, $Q(3, -4, 0)$, and $R(-3, -2, -1)$.

   Ans: $3(x - 1) + 7(y + 2) - 4(z - 2) = 0$.

5. Use intercepts to help sketch the plane $2x + y + 5z = 10$.

12.3. Cylinders and Quadric Surfaces

Objective: To visualize surfaces, given their equations.

Definition 12.26. *A **cylinder** is a surface that consists of all lines that are parallel to a given line and pass through a given plane curve.*

Problem 12.27. Sketch $z = x^2$ in $\mathbb{R}^3$.

Problem 12.28. Sketch $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

Problem 12.29. Sketch $y^2 + z^2 = 1$ in $\mathbb{R}^3$.

Definition 12.30. *A **quadric surface** is the graph of a second-degree equation in three variables $x, y,$ and z. By translation and rotation, we can write the standard form of a quadric surface as*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \textit{or} \quad Ax^2 + By^2 + Iz = 0. \tag{12.17}$$

Definition 12.31. *The **trace** of a surface in $\mathbb{R}^3$ is the graph in $\mathbb{R}^2$ obtained by allowing one of the variables to be a specific real number. For example, $x = a$.*

Problem 12.32. Use the traces to sketch $x^2 + \frac{y^2}{9} + \frac{z^2}{4} = 1$.

Problem 12.33. Use the traces to sketch $z = 4x^2 + y^2$.

Exercises 12.3

1. Sketch the surface.

   (a) $x^2 + y^2 = 1$
   (b) $x^2 + y^2 - 2y = 0$
   (c) $z = \sin x$

2. Use traces to sketch and identify the surface.

   (a) $z = y^2 - x^2$
   (b) $4y^2 + 9z^2 = x^2 + 36$

3. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 2 - x^2 - y^2$.

4. Sketch the surface obtained by rotating the line $\mathbf{r}(t) = \langle 0, 1, 3 \rangle t$ about the z-axis; find an equation of it. (***Hint***: The line can be expressed as $\{z = 3y,\ \ x = 0\}$.)

   Ans: $|z| = 3\sqrt{x^2 + y^2}$ or $z^2 = 9(x^2 + y^2)$

CHAPTER **14**

Partial Derivatives

In mathematics, a **partial derivative** of a function of several variables is its derivative with respect to one of those variables, with the others held constant. In this chapter, you will learn about the partial derivatives and their applications.

Subjects	Applications
Limits and continuity Partial derivatives	
	Tangent planes & linear approximations
Chain rule Directional derivatives and the Gradient Vector	
	Maximum and minimum values Method of Lagrange multipliers

Contents of Chapter 14

This chapter corresponds to Chapter 14 in STEWART, ***Calculus*** (8th Ed.), 2015.

14.1. Functions of Several Variables

14.1.1. Domain and range

Definition 14.1. *A* ***function of two variables****, f, is a rule that assigns each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is called the* ***domain*** *of f and its* ***range*** *is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in D\}$.*

Definition 14.2. *Let f be a function of two variables, and $z = f(x, y)$. Then x and y are called* ***independent variables*** *and z is called a* ***dependent variable****.*

Problem 14.3. Let $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$. Evaluate $f(3, 2)$ and give its domain.

Ans: $f(3, 2) = \sqrt{6}/2$; $D = \{(x, y) : x + y + 1 \geq 0,\ \ x \neq 1\}$

Problem 14.4. Find the domain of $f(x, y) = x \ln\left(y^2 - x\right)$.

Problem 14.5. Find the domain and the range of $f(x, y) = \sqrt{9 - x^2 - y^2}$.

14.1.2. Graphs

Definition 14.6. *If f is a function of two variables with domain D, then the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for all $(x, y) \in D$.*

Problem 14.7. Sketch the graph of $f(x, y) = 6 - 3x - 2y$.

Solution. The graph of f has the equation $z = 6 - 3x - 2y$, or $3x + 2y + z = 6$, which is a plane. Now, we can find intercepts to graph the plane.

Problem 14.8. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution. The graph of g has the equation $z = \sqrt{9 - x^2 - y^2}$, or $x^2 + y^2 + z^2 = 9$, $z \geq 0$, which is a upper hemi-sphere.

14.1.3. Level curves

Definition 14.9. *The **level curves** of a function of two variables, f, are the curves with equations $f(x, y) = k$, for $k \in K \subset$ Range(f).*

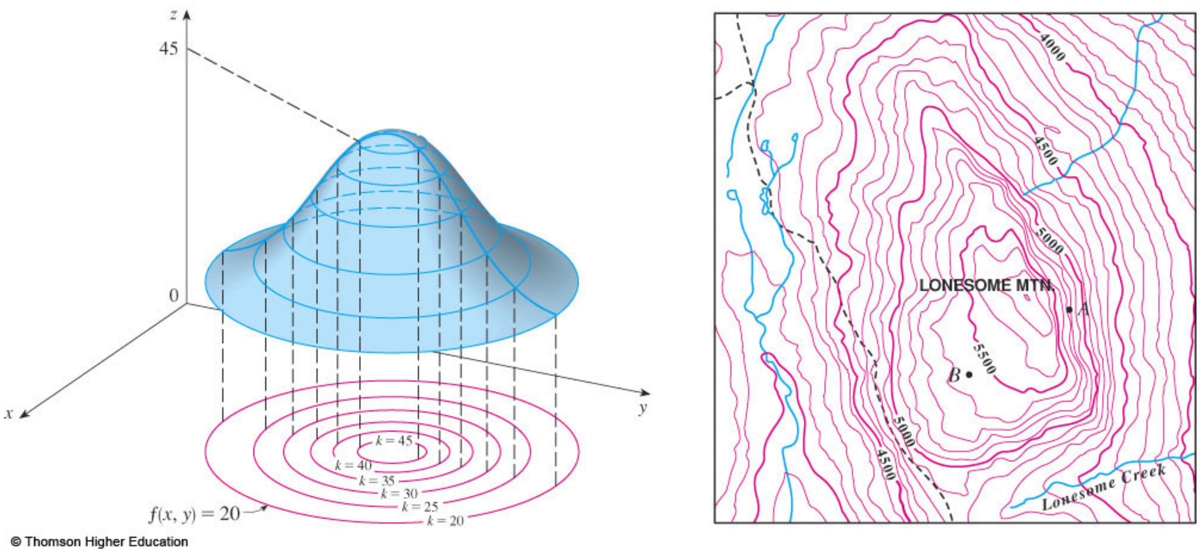


Figure 14.1: Level curves: (left) the graph of a function vs. level curves and (right) a topographic map of a mountainous region. Level curves are often considered for an effective visualization.

Problem 14.10. Sketch the level curves of $f(x, y) = 6 - 3x - 2y$ for $k \in \{-6, 0, 6, 12\}$.

Problem 14.11. Sketch the level curves of $g(x, y) = \sqrt{9 - x^2 - y^2}$ for $k \in \{0, 1, 2, 3\}$

Problem 14.12. Sketch the level curves of $h(x, y) = 4x^2 + y^2 + 1$.

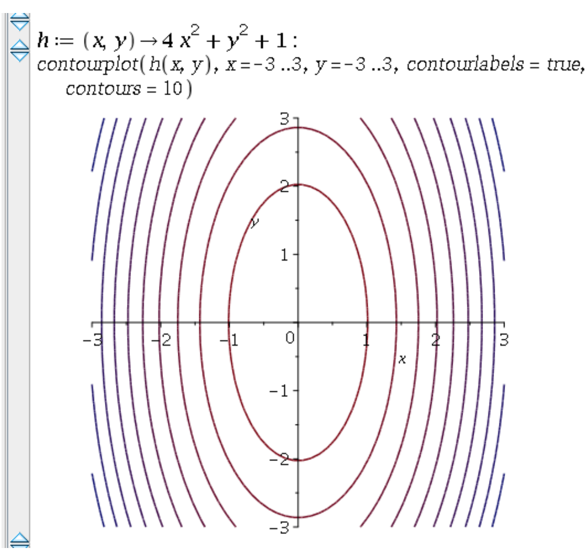


Figure 14.2

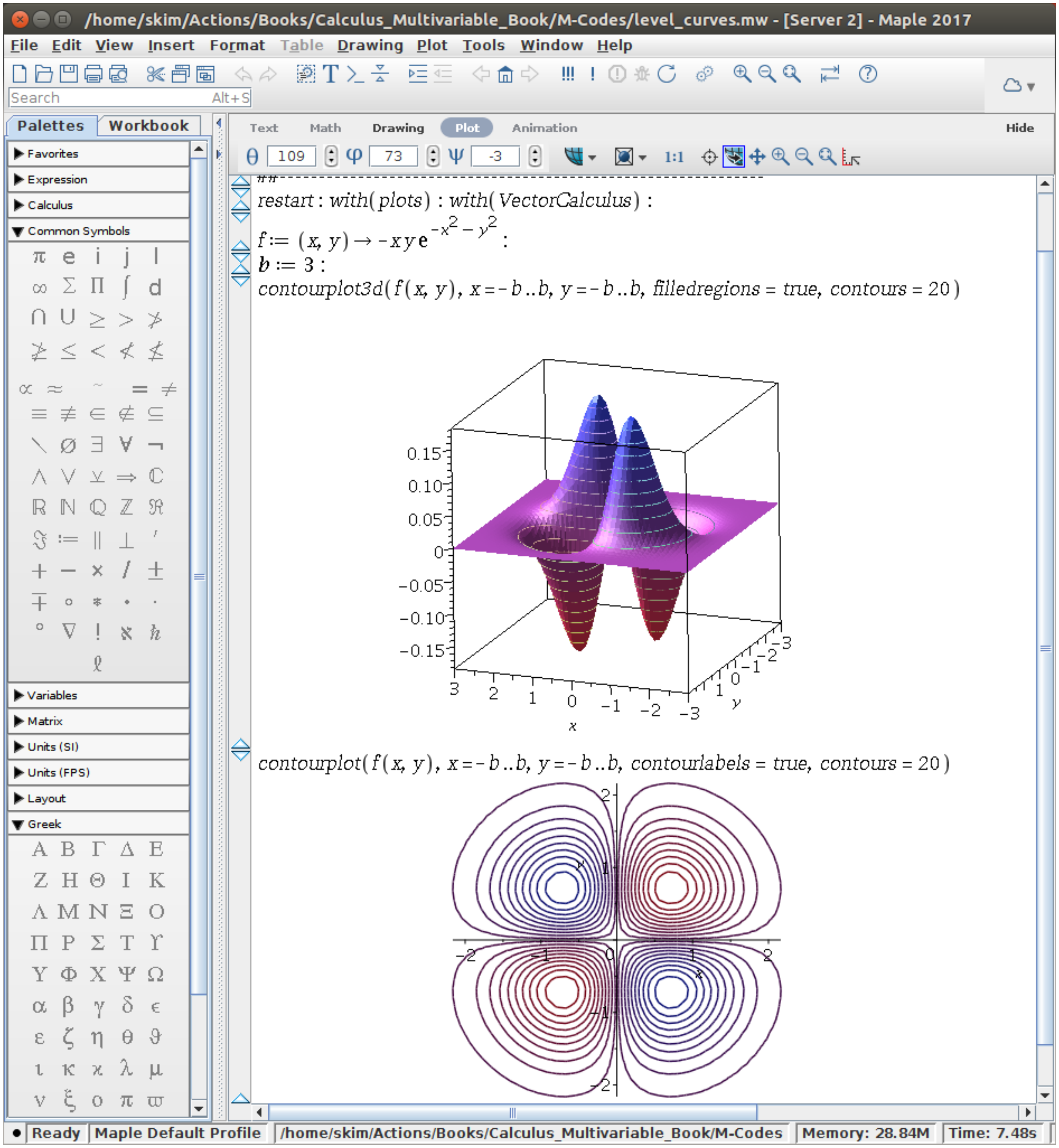


Figure 14.3: Computer-generated level curves.

Function visualization is now easy with e.g., Mathematica, Maple, and Matlab, as shown in Figure 14.3.[1]

[1]For plotting with Maple, you may exploit `plot`, `plot3d`, `contourplot3d`, and `contourplot`, which are available from the `plots` package. Maple can include packages with the `with` command, as in Figure 14.3.

14.1.4. Functions of three or more variables

Definition 14.13. *A **function of three variables**, f, is a rule that assigns each ordered pair of real numbers (x, y, z) in a set $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.*

Problem 14.14. Find the domain of f if $f(x, y, z) = \ln(z - y) + xy \sin z$.

Problem 14.15. Find the **level surfaces (:=**$f(x, y, z) = k$**)** of $f(x, y, z) = x^2 + y^2 + z^2$.

Solution.

Note: A level surface is the surface where the function values are all the same as k. Thus **the outer normal** is the **fastest increasing direction** of f.

Exercises 14.1

1. Find and sketch the domain of the function

   (a) $f(x,y) = \ln(9 - 9x^2 - y^2)$

   (b) $g(x,y) = \dfrac{\sqrt{x - y^2}}{1 - y^2}$

2. Let $f(x,y) = \sqrt{4 - x^2 - 4y^2}$.

   (a) Find the *domain* of f.

   (b) Find the *range* of f.

   (c) Sketch the *graph* of the function.

3. Match the function with its contour plot (labeled I–VI). Give reasons for your choices.

   (a) $f(x,y) = x^2 - y^2$

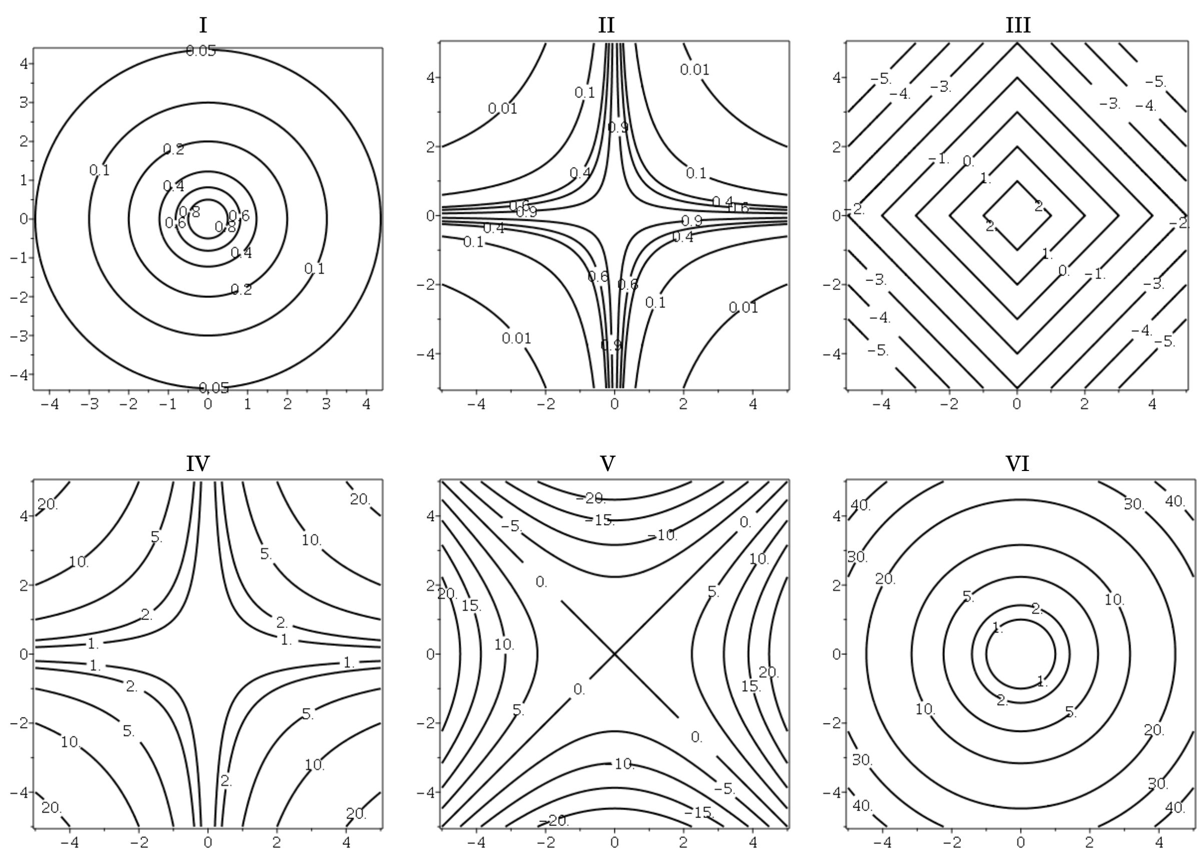
   (b) $f(x,y) = x^2 + y^2$

   (c) $f(x,y) = 3 - |x| - |y|$

   (d) $f(x,y) = |xy|$

   (e) $f(x,y) = \frac{1}{1+x^2+y^2}$

   (f) $f(x,y) = \frac{1}{1+x^2y^2}$



4. Describe the level surfaces of the function $f(x,y,z) = x^2 + y^2 - z^2$.

14.2. Limits and Continuity

Limits

Recall: For $y = f(x)$, then we say that the **limit** of $f(x)$, as $x \to a$, is L, if

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x),$$

or, equivalently, if $\forall \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\text{if } 0 < |x-a| < \delta \text{ then } |f(x) - L| < \varepsilon,$$

which is called the **ε-δ argument**. In this case, we write

$$\lim_{x\to a} f(x) = L. \tag{14.1}$$

Definition 14.16. *Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the **limit** of $f(x,y)$, as (x,y) approaches (a,b), is L:*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \tag{14.2}$$

if $\forall\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\text{if } (x,y) \in D \text{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \text{ then } |f(x,y) - L| < \varepsilon.$$

Arc length of $f(B_\delta(a,b)) \to 0$, as $\delta \to 0$.

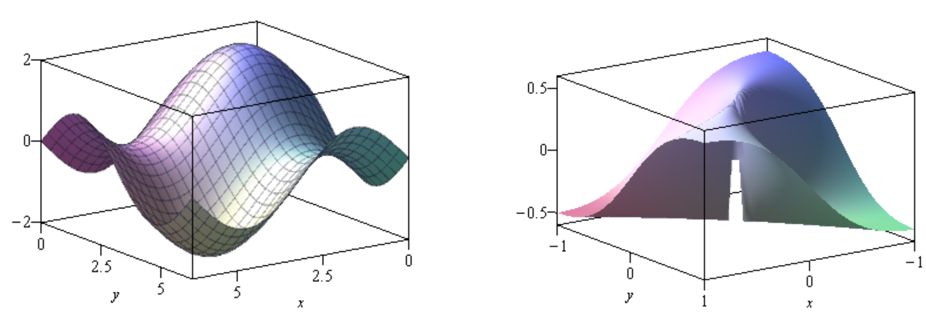


Figure 14.4: Plots of $z = \sin x + \sin y$ (left) and $z = \dfrac{xy}{x^2+y^2}$ (right).

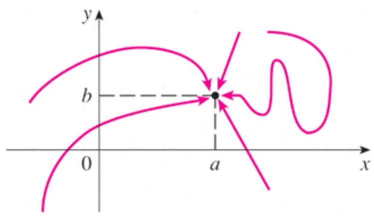


Figure 14.5

Claim 14.17. If $f(x, y) \to L_1$ as $(x, y) \to (a, b)$ along a path C_1 and $f(x, y) \to L_2$ as $(x, y) \to (a, b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ does not exist.

Problem 14.18. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2 - y^2}{x^2 + y^2}$ does not exist.

Solution. Consider two paths: e.g., $C_1 : \{y = 0\}$ and $C_2 : \{x = 0\}$.

Ans: no

Problem 14.19. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2 + y^2}$ exist?

Solution. Consider a path $C : \{x = y\}$ with another.

Ans: no

Problem 14.20. Does $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+y^2}$ exist?

Solution.

Ans: yes: $L = 1$

Problem 14.21. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^4}$ exist?

Solution. Consider a path $C : \{x = y^2\}$ with another.

Ans: no. See Figure 14.6 on p. 31 below.

Problem 14.22. Use the **squeeze theorem** to show $\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0$.

Solution.

Continuity

Recall: A function (of a single variable) f is **continuous** at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

The above means that

1. the limit on the left side exists,
2. $f(a)$ is defined, and
3. they are the same.

Definition 14.23. *A function of two variables f is called **continuous at point** $(a, b) \in \mathbb{R}^2$ if*

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b). \quad (14.3)$$

*If f is continuous at every point (x, y) in a region $D \subset \mathbb{R}^2$, then we say that f is **continuous on** D.*

Problem 14.24. Is $f(x, y) = \frac{2xy}{x^2+y^2}$ continuous at $(0, 0)$? What about at $(1, 1)$? Why?

Solution. See Problems 14.19 and 14.20.

Ans: no; yes

Problem 14.25. Is the following function continuous at $(0,0)$? What about at elsewhere?

$$g(x,y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \textbf{if } (x,y) \neq (0,0) \\ 0 & \textbf{if } (x,y) = (0,0) \end{cases} \tag{14.4}$$

Solution. See Problem 14.22.

Ans: It is continuous everywhere.

Problem 14.26. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\, \ln(x^2+y^2)$.

Solution. Consider $\lim_{x\to 0} x \ln x$ and introduce a new variable $s = x^2 + y^2$.

Ans: $L = 0$

Exercises 14.2

1. Find the limit, if it exists, or show that the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} x\cos(x-y)$

   Ans: 0

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

   Ans: 0

2. Use polar coordinates to find the limit.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{4+x^2+y^2}-2}$

   Ans: (a) 1; (b) 4

3. CAS Use a computer graph of the function to explain why the limit does not exist.[2]

   $$\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+4y^2}{3x^2+y^2}$$

4. Determine and ***verify*** whether the following functions are continuous at $(0,0)$ or not.

   (a) $f(x,y) = \begin{cases} \dfrac{x^4\sin y}{x^4+y^4}, & \text{if } (x,y)\neq(0,0), \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

   Ans: continuous

   (b) $g(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2}, & \text{if } (x,y)\neq(0,0), \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

   Ans: discontinuous

[2]You have to perform a computer implementation for problems indicated by the *Computer Algebra System* sign CAS. Of course, you must print hard copies of your computer work to be attached.

Computer algebra

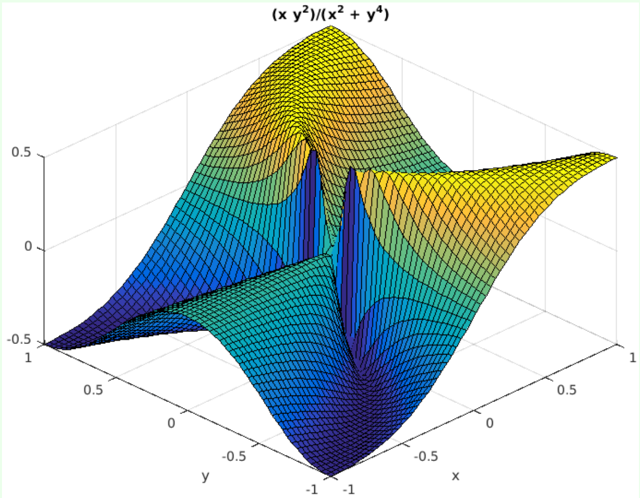


Figure 14.6: Matlab plot: using `ezsurf` for Problem 14.21, p. 27.

In computational mathematics, **computer algebra** (also called **symbolic computation**) is a scientific area that refers to the study and development of algorithms and software for manipulating mathematical expressions and other mathematical objects; it emphasizes **exact computation** with expressions containing variables that have no given value and are manipulated as symbols.

There have been about 40 computer algebra systems available; search "List of computer algebra systems" in Wikipedia. Popular ones in computational mathematics are Maple, Mathematica, and Matlab.

Matlab script

```
1 syms x y
2
3 f = x*y^2/(x^2+y^4);
4 ezsurf(f,[-1,1,-1,1])
5 view(-45,45)
6 print('-r100','-dpng','matlab_ezsurf.png');
```

The above Matlab script results in Figure 14.6. Line 1 declares symbolic variables `x y`; line 3 defines the function f; line 4 plots a figure over the rectangular domain $[-1,1]\times[-1,1]$; line 5 changes the view angle to $(-45°, 45°)$ in the horizontal and vertical directions, respectively; and the final line saves the figure to `matlab_ezsurf.png` with the resolution level of 100.

14.3. Partial Derivatives

14.3.1. First-order partial derivatives

Recall: A function $y = f(x)$ is **differentiable** at a if

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

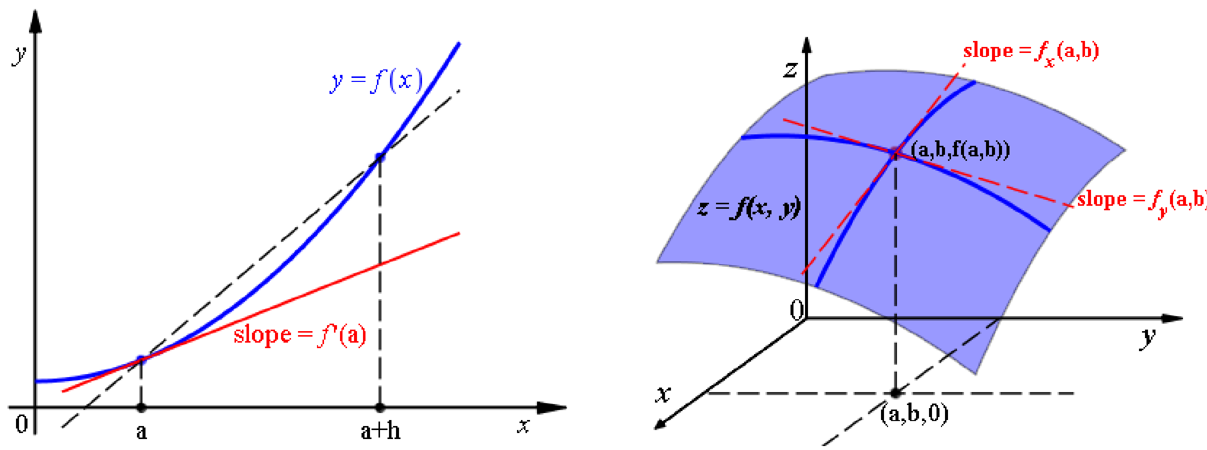


Figure 14.7: Ordinary derivative $f'(a)$ and partial derivatives $f_x(a,b)$ and $f_y(a,b)$.

Let f be a function of two variables (x, y). Suppose we let only x vary while keeping y fixed, say $\boldsymbol{y = b}$. Then $\boldsymbol{g(x) := f(x, b)}$ is a function of a single variable. If g is differentiable at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denoted by $f_x(a, b)$.

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} =: f_x(a, b). \end{aligned} \tag{14.5}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained keeping x fixed, say $\boldsymbol{x = a}$, and finding the ordinary derivative at b of $\boldsymbol{G(y) := f(a, y)}$:

$$\begin{aligned} G'(b) &= \lim_{h\to 0} \frac{G(b+h) - G(b)}{h} \\ &= \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h} =: f_y(a, b). \end{aligned} \tag{14.6}$$

Problem 14.27. Find $f_x(0, 0)$, when $f(x, y) = \sqrt[3]{x^3 + y^3}$.
Solution. Using the definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h}$$

Ans: 1

Definition 14.28. *If f is a function of two variables, its* ***partial derivatives*** *are the functions $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ defined by:*

$$\begin{aligned} f_x(x, y) &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \textit{and} \\ f_y(x, y) &= \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}. \end{aligned} \tag{14.7}$$

Observation 14.29. The partial derivative with respect to x represents the slope of the tangent lines to the curve that are parallel to the xz-plane (i.e. in the direction of $\langle 1, 0, \cdot\rangle$). Similarly, the partial derivative with respect to y represents the slope of the tangent lines to the curve that are parallel to the yz-plane (i.e. in the direction of $\langle 0, 1, \cdot\rangle$).

Rule for finding Partial Derivatives of $z = f(x, y)$

- To find f_x, regard y as a constant and differentiate f w.r.t. x.
- To find f_y, regard x as a constant and differentiate f w.r.t. y.

Problem 14.30. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.

Solution.

Ans: $f_x(2, 1) = 16$; $f_y(2, 1) = 8$

Problem 14.31. Let $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$. Find the first partial derivatives of $f(x, y)$.

Solution.

Problem 14.32. Find the first partial derivatives of $f(x, y) = x^y$.

Solution. Use $\frac{d}{dx} a^x = a^x \ln a$.

Recall: **(Implicit differentiation)**. When $y = y(x)$ and $x^2 + y^3 = 3$, you have $2x + 3y^2 y' = 0$ so that $y' = -2x/(3y^2)$.

Problem 14.33. Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

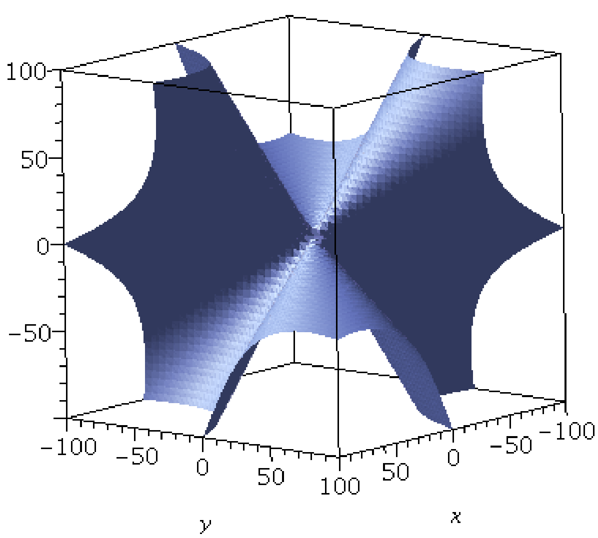


Figure 14.8: `implicitplot3d` in Maple: a plot of surface defined in Problem 14.33.

Problem 14.34. (Revisit of Problem 14.27). Find $f_x(x, y)$, when $f(x, y) = \sqrt[3]{x^3 + y^3}$. Can you evaluate $f_x(0, 0)$ easily?

Solution.

Functions of more than two variables

Problem 14.35. Let $f(x, y, z) = e^{xy} \ln z$. Find f_x, f_y, and f_z.

Solution.

14.3.2. Higher-order partial derivatives

Second partial derivatives of $\mathrm{z} = \mathrm{f}(\mathrm{x}, \mathrm{y})$

$$\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{11} \\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{12} \\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{21} \\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{22}
\end{aligned}$$

Problem 14.36. Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

Solution.

Theorem 14.37. ***(Clairaut's theorem)*** *Suppose f is defined on a disk $D \subset \mathbb{R}^2$ that contains the point (a, b).* ***If both*** f_{xy} ***and*** f_{yx} ***are continuous on*** *D, then*

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{14.8}$$

Claim 14.38. It can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.

Problem 14.39. Verify Clairaut's theorem for $f(x, y) = xye^y$.

Problem 14.40. Calculate $f_{yzxx}(x, y, z)$, given $f(x, y, z) = \sin(3x + yz)$.
Solution.

Exercises 14.3

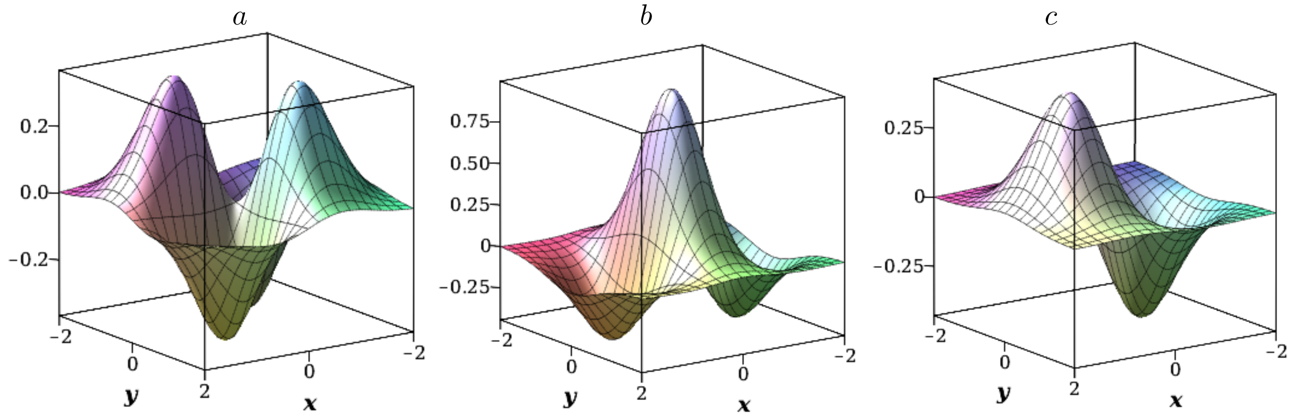
1. The temperature T (in °F) at a location in the Northern Hemi-sphere depends on the longitude x, latitude y, and time t; so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

   (a) What do the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$ mean?

   (b) **Mississippi State University** (MSU)[3] has longitude $88.8°$ W and latitude $33.5°$ N. Suppose that at noon on January first, the wind is blowing warm air to northeast, so the air to the west and south is warmer than that in the north and east. Would you expect $f_x(88.8, 33.5, 12)$, $f_y(88.8, 33.5, 12)$, and $f_t(88.8, 33.5, 12)$ to be positive or negative? Explain.

2. The following surfaces, labeled a, b, and c, are graphs of a function f and its partial derivatives f_x and f_y. Identify each surface and give reasons for your choices.

a b c



3. Find the partial derivatives of the function.

   (a) $z = y\cos(xy)$

   (b) $f(u, v) = (uv - v^3)^2$

   (c) $w = \ln(x + 2y + 3z)$

   (d) $u = \sin(x_1^2 + x_2^2 + \cdots + x_n^2)$

   Ans: (d) $\partial u/\partial x_i = 2x_i \cdot \cos(x_1^2 + x_2^2 + \cdots + x_n^2)$

4. Let $f(x, y, z) = xy^2z^3 + \arccos(x\sqrt{y}) + \sqrt{1 + xz}$. Find f_{xyz}, by using a different order of differentiation for each term.

   Ans: $6yz^2$

5. Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2u_{xx}$.

   (a) $u = \sin(kx)\sin(akt)$

   (b) $u = (x + at)^3 + (x - at)^6$

   (c) $u = \sin(x + at) + \ln(x - at)$

   (d) $u = f(x + at) + g(x - at)$

   where f and g are twice differentiable functions.

[3] MSU, the **land-grant research university**, has an elevation of 118 meters, or 387 feet.

14.4. Tangent Planes & Linear Approximations

Recall: As one zooms into a curve $y = f(x)$, the more the curve resembles a line. More specifically, the curve looks more and more like the **tangent line**. It is the same for surface: the surface looks more and more like the **tangent plane**. Some functions are difficult to evaluate at a point; so, the equation of the tangent plane (which is much simpler) is used to approximate the value of that curve at a given point.

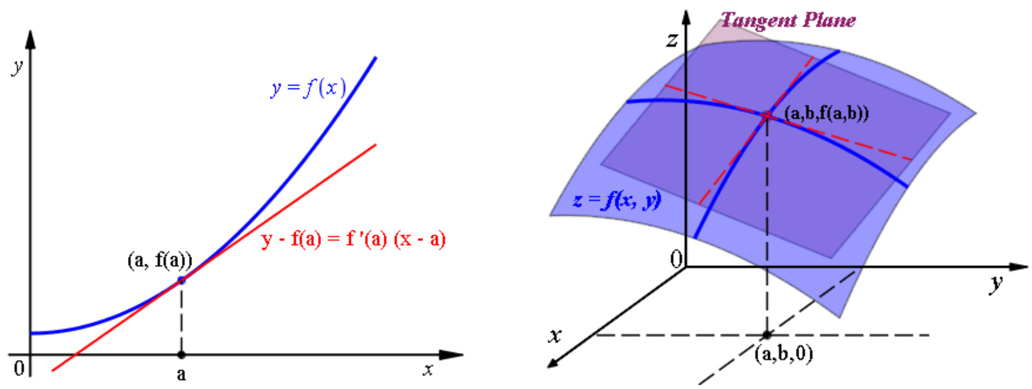


Figure 14.9: A tangent line and a tangent plane.

Tangent plane for $z = f(x, y)$ at (x_0, y_0, z_0): Any tangent plane passing through $P(x_0, y_0, z_0)$, $z_0 = f(x_0, y_0)$, has an equation of the form

$$A(x - x_0) + B(y - y_0) + C(z - z_0) = 0, \qquad \mathbf{n} =< A, B, C > .$$

By dividing the equation by $C\,(\neq 0)$ and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0). \tag{14.9}$$

Then, the intersection of the plane with $y = y_0$ must be the x-**directional tangent line** at (x_0, y_0, z_0), having the slope of $f_x(x_0, y_0)$:

$$\boxed{z - z_0 = a(x - x_0), \ \text{where} \quad y = y_0.}$$

Therefore $\boxed{a = f_x(x_0, y_0)}$. Similarly, we can conclude $\boxed{b = f_y(x_0, y_0)}$.

Summary 14.41. *Suppose that $f(x, y)$ has continuous partial derivatives. An equation of the* ***tangent plane*** *(equivalently, the* ***linear approximation****) to the surface $z = f(x, y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0), \tag{14.10}$$

where $z_0 = f(x_0, y_0)$.

Problem 14.42. Find an equation for the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution.

Ans: $z = 4x + 2y - 3$

Linear approximation (linearization) of f at (a, b):

$$f(x, y) \approx L(x, y) := f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b). \tag{14.11}$$

Problem 14.43. Give the linear approximation of $f(x, y) = xe^{xy}$ at $(1, 0)$. Then use this to approximate $f(1.1, -0.1)$.

Solution.

Ans: $L(x, y) = x + y$; $L(1.1, -0.1) = 1$, while $f(1.1, -0.1) = 0.9854\cdots$.

Differentiability for functions of multiple variables:

Recall: A function $y = f(x)$ is **differentiable** at a if

$$\lim_{\Delta x \to 0} \frac{f(a+\Delta x) - f(a)}{\Delta x} \quad \text{exists.} \quad (=: f'(a))$$

Thus, if f is differentiable at a, then $\frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a) + \varepsilon$ and

$$\Delta y \equiv f(a+\Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x, \qquad (14.12)$$

where $\varepsilon \to 0$ as $\Delta x \to 0$. $\left(\because \ \frac{f(a+\Delta x)-f(a)}{\Delta x} = f'(a) + \varepsilon\right)$

Now, for $z = f(x, y)$, suppose that (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. Then the corresponding **change** of z is

$$\Delta z = f(a+\Delta x, b+\Delta y) - f(a, b).$$

Definition 14.44. *A function $z = f(x, y)$ is* ***differentiable*** *at (a, b) if Δz can be expressed in the form*

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \qquad (14.13)$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

It is sometimes hard to use Definition 14.44 directly to check the differentiability of a function.

Theorem 14.45. *If f_x and f_y* ***exist*** *near (a, b) and are* ***continuous*** *at (a, b), then $z = f(x, y)$ is differentiable at (a, b).*

Note: The above theorem implies that if **partial derivatives of f are continuous**, then **the slope of f exists for all directions**.

Problem 14.46. Let $f(x, y) = y + \sin(x/y)$. Explain why the function is differentiable at $(0, 3)$.

Differentials

Recall: For $y = f(x)$, let dx be the differential of x (an independent variable). The **differential** of y is then defined as

$$dy = f'(x)\,dx. \tag{14.14}$$

Note: $\boldsymbol{\Delta y}$ represents the change in height of the curve $y = f(x)$, while $\boldsymbol{dy}$ represents **the change in height of the tangent line**; when x changes by $\Delta x = dx$.

Definition 14.47. *For $z = f(x, y)$, we define **differentials** dx and dy to be independent variables. Then the **differential** dz is defined by*

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy, \tag{14.15}$$

*which is also called the **total differential**.*

Problem 14.48. Let $z = f(x, y) = x^2 + 3xy - y^2$.

(a) Find the differential dz.

(b) If (x, y) changes from $(2, 3)$ to $(2.1, 2.9)$, compare the values of Δz and dz.

Solution.

Ans: (a) $dz = (2x + 3y)dx + (3x - 2y)dy$; (b) $dz = 1.3$, $\Delta z = 1.27$

Problem 14.49. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter, if the metal in the top and bottom is 0.1 cm thick and the metal in the side is 0.05 cm thick.

Solution. $V(r,z) = \pi r^2 z$. Therefore

$$dV = V_r\,dr + V_z\,dz = 2\pi rz\,dr + \pi r^2\,dz,$$

where $dr = 0.05$ and $dz = 2 \cdot 0.1 = 0.2$.

Ans: $dV = 2.8\pi = 8.796459431\cdots$ ($\Delta V = 9.0022337\cdots$)

Exercises 14.4

1. Find an equation of the tangent plane to the given surface at the specified point.

   (a) $z = \sin(2x+3y)$, $(-3,2,0)$
   (b) $z = x^2 + 2y^2 - 3y$, $(1,-1,6)$

2. Explain why the function is **differentiable** at the given point. Then, find the linearization $L(x,y)$ of the function at that point.

   (a) $f(x,y) = 5 + x\ln(xy-1)$, $(1,2)$
   (b) $f(x,y) = xy + \sin(y/x)$, $(2,0)$

3. Given that f is a differentiable function with $f(5,2) = 4$, $f_x(5,2) = 1$, and $f_y(5,2) = -1$, use a linear approximation to estimate $f(4.9,2.2)$.

   Ans: 3.7

4. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 16 cm if the tin is 0.05 cm thick.

   Ans: 8π

14.5. The Chain Rule

14.5.1. Chain rule

Recall: **Chain Rule for functions of a single variable**: If $y = f(x)$ and $x = g(t)$, where f and g are differentiable, then y is a differentiable function of t and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \tag{14.16}$$

Theorem 14.50. ***The Chain Rule (Case 1)***. *Suppose that $z = f(x, y)$ is a differentiable function, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{14.17}$$

Observation: Let $z = f(x, y) = x\,y$ and x and y be functions of t:

$$z = f(x, y) = x\,y = x(t)\,y(t).$$

Then

$$\begin{aligned} \frac{dz}{dt} &= x'(t)\,y(t) + x(t)\,y'(t), \quad \text{(product rule)} \\ \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= y\,x'(t) + x\,y'(t). \end{aligned}$$

Problem 14.51. If $z = x^2y + xy^3$, where $x = \cos t$ and $y = \sin t$, find dz/dt at $t = 0$.

Solution.

Ans: 1

Now, we will solve the above problem using the following script in Maple.

Maple script and answers

```
1 z := x*y^3+x^2*y:
2 x := cos(t):  y := sin(t):
3 zt := diff(z, t)
4     zt := -2 cos(t) sin(t) + cos(t) - sin(t) + 3 cos(t) sin(t)
5 simplify(%)
6     -4 cos(t) + 3 cos(t) + 5 cos(t) - 2 cos(t) - 1
7 eval(zt, t = 0)
8     1
```

Lines 4, 6, and 8 are answers from Maple.

Theorem 14.52. ***The Chain Rule (Case 2).*** *Suppose that $z = f(x, y)$ is a differentiable function, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}. \tag{14.18}$$

Problem 14.53. If $z = e^x \sin(y)$, where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution.

Ans:
$$\begin{aligned} z_s &= t^2 e^{st^2} \sin(s^2 t) + 2\, e^{st^2} st \cos(s^2 t) \\ z_t &= 2\, st e^{st^2} \sin(s^2 t) + e^{st^2} s^2 \cos(s^2 t) \end{aligned}$$

Functions of three and more variables:

Theorem 14.54. ***The Chain Rule (General Version)***. *Suppose that u is a differentiable function of n variables, $x_1, x_2, \ldots, x_n$, each of which has m variables, $t_1, t_2, \ldots, t_m$. Then for each $i \in \{1, 2, \ldots, m\}$,*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$

Problem 14.55. Write the chain rule for $w = f(x, y, z, t)$, where $x = x(u, v), y = y(u, v), z = z(u, v)$, and $t = t(u, v)$. That is, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

Problem 14.56. If $g(s, t) = f\left(s^2 - t^2, t^2 - s^2\right)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Solution. Let $x = s^2 - t^2$ and $y = t^2 - s^2$.
Then $g_s = f_x\, x_s + f_y\, y_s$ and $g_t = f_x\, x_t + f_y\, y_t$.

Problem 14.57. If $z = x^3 + x^2 y$, where $x = s + 2t - u$ and $y = stu$, find the values of z_s, z_t, and z_u, when $s = 2$, $t = 0$, $u = 1$.

Solution.

Ans: $z_s = 3$, $z_t = 8$, and $z_u = -3$

14.5.2. Implicit differentiation

Consider $F(x, y) = 0$, where y is a function of x, i.e., $y = f(x)$. Then,

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0.$$

Thus,we have

$$y' = -\frac{F_x}{F_y}. \tag{14.19}$$

Problem 14.58. Find y' if $x^3 + y^3 = 6xy$.

Solution. Let $F = x^3 + y^3 - 6xy$. Then, use (14.19).

Ans: $y' = -(x^2 - 2y)/(y^2 - 2x)$

Note: You can solve the above problem using the technique you learned earlier in Calculus I. That is, applying ***x*-derivative** to $x^3 + y^3 = 6xy$ reads

$$3x^2 + 3y^2\, y' = 6y + 6xy'.$$

Thus

$$3y^2\, y' - 6xy' = -3x^2 + 6y \;\;\Rightarrow\;\; y' = -\frac{3x^2 - 6y}{3y^2 - 6x}.$$

Claim 14.59. Let $z = f(x, y)$ and $F(x, y, z) = 0$. Then $F_x \frac{\partial x}{\partial x} + F_y \frac{\partial y}{\partial x} + F_z \frac{\partial z}{\partial x} = 0$ and $F_x \frac{\partial x}{\partial y} + F_y \frac{\partial y}{\partial y} + F_z \frac{\partial z}{\partial y} = 0$. Thus

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z}. \tag{14.20}$$

Problem 14.60. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, if

$$x^3 + y^3 + z^3 + 6xyz = 1. \tag{14.21}$$

Solution.

Ans: $z_x = -\frac{3x^2+6yz}{3z^2+6xy} = -\frac{x^2+2yz}{z^2+2xy}$. See Figure 14.8, p. 35, for a figure of (14.21).

Exercises 14.5

1. Use the Chain Rule to find dz/dt or dw/dt.

   (a) $z = \cos x \sin y; \quad x = t^3, \;\; y = 1/t$
   (b) $w = (x + y^2 + z^3)^2; \quad x = 1 + 2t, \;\; y = -2t, \;\; z = t^2$

2. Suppose f is a differentiable function of x and y, and $g(u, v) = f(u + \cos v,\; u^2 + 1 + \sin v)$. Use the table of values to find $\boldsymbol{g_u(0,0)}$ and $\boldsymbol{g_v(0,0)}$.

	f	g	f_x	f_y
$(0,0)$	1	2	-1	10
$(0,1)$	3	5	10	5
$(1,1)$	2	7	20	2

Ans: $g_u(0,0) = 20$ & $g_v(0,0) = 2$

3. Use the Chain Rule to find the indicated partial derivatives.

   (a) $z = x^2 + y^4; \quad x = s + 2t - 3u, \;\; y = stu; \quad \dfrac{\partial z}{\partial s}, \; \dfrac{\partial z}{\partial t}, \; \dfrac{\partial z}{\partial u}$
   when $s = 3$, $t = 1$, and $u = 1$

   Ans: $z_s(3,1,1) = 112$ & $z_u(3,1,1) = 312$

   (b) $w = xy + yz + zx; \quad x = r\cos\theta, \;\; y = r\sin\theta; \quad \dfrac{\partial w}{\partial r}, \; \dfrac{\partial w}{\partial \theta}, \; \dfrac{\partial w}{\partial z}$
   when $r = 2$, $\theta = \pi/2$, and $z = 1$

   Ans: $w_z = 2$

4. Use the formulas in (14.20) to find $\partial z/\partial x$ and $\partial z/\partial y$, where z is function of (x, y).

   (a) $x^2 + 2y^2 + 3z^2 - 4 = 0$
   (b) $e^z = xy + z$

   Ans: (b) $z_x = y/(e^z - 1)$

14.6. Directional Derivatives and the Gradient Vector

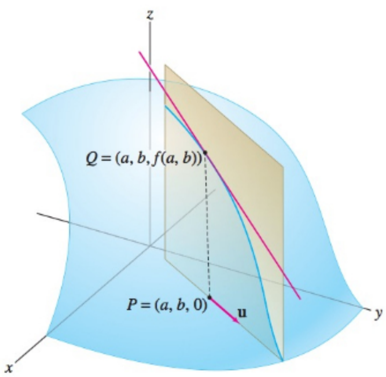


Figure 14.10

Recall: If $z = f(x, y)$, then the partial derivatives f_x and f_y represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

Note: It would be nice to be able to find the slope of the tangent line to a surface S in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b\rangle$.

Definition 14.61. *The* ***directional derivative*** *of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b\rangle$ is*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}, \tag{14.22}$$

if the limit exists.

Note that

$$\begin{aligned} f(x_0 + ha, y_0 + hb) - f(x_0, y_0) &= f(x_0 + ha, y_0 + hb) - f(x_0, y_0 + hb) \\ &+ f(x_0, y_0 + hb) - f(x_0, y_0) \end{aligned}$$

Thus

$$\begin{aligned} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} &= a\,\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0 + hb)}{ha} \\ &+ b\,\frac{f(x_0, y_0 + hb) - f(x_0, y_0)}{hb}, \end{aligned}$$

which converges to "$a\, f_x(x_0, y_0) + b\, f_y(x_0, y_0)$" as $h \to 0$.

Theorem 14.62. *If f is a differentiable function of x and y, then f has a* ***directional derivative*** *in the direction of any unit vector $\mathbf{u} = \langle a, b \rangle$ and*

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= f_x(x,y)\, a + f_y(x,y)\, b \\ &= \langle f_x(x,y), f_y(x,y) \rangle \cdot \langle a, b \rangle \\ &= \langle f_x(x,y), f_y(x,y) \rangle \cdot \mathbf{u}. \end{aligned} \qquad \textbf{(14.23)}$$

Problem 14.63. Find the directional derivative $D_{\mathbf{u}} f(x,y)$, if $f(x,y) = x^3 + 2xy + y^4$ and $\mathbf{u}$ is the unit vector given by the angle $\theta = \frac{\pi}{4}$. What is $D_{\mathbf{u}} f(2,3)$?

Solution. $\mathbf{u} = \langle \cos(\pi/4), \sin(\pi/4) \rangle = \left\langle 1/\sqrt{2}, 1/\sqrt{2} \right\rangle$.

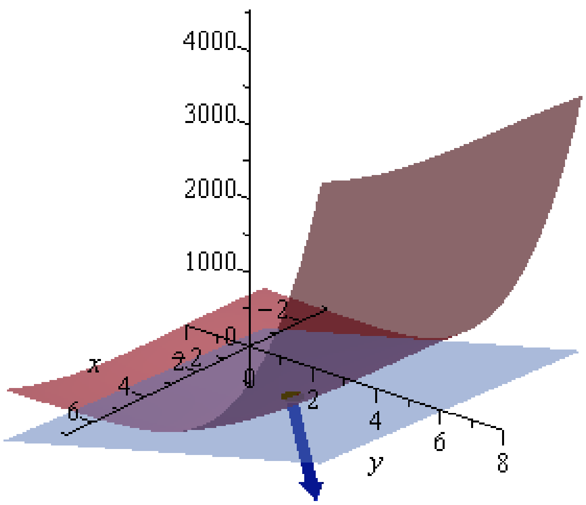


Figure 14.11

Ans: $65\sqrt{2}$

Note: 1. The only reason we are restricting the directional derivative to the unit vector is because we care about the rate of change in f per unit distance. Otherwise, the magnitude is irrelevant.
2. If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$. Thus

$$D_{\mathbf{u}} f(x, y) = f_x(x, y)\cos\theta + f_y(x, y)\sin\theta. \tag{14.24}$$

Self-study **14.64.** Find the directional derivative of $f(x, y) = x + \sin(xy)$ at the point $(1, 0)$ in the direction given by the angle $\theta = \pi/3$.

Solution.

Ans: $(1 + \sqrt{3})/2$

Problem **14.65.** If $f(x, y, z) = x^2 - 2y^2 + z^4$, find the directional derivative of f at $(1, 3, 1)$ in the direction of $\mathbf{v} = \langle 2, -2, -1 \rangle$.

Solution.

Ans: 8

Gradient Vector

Definition 14.66. *Let f be a differentiable function of two variables x and y. Then the **gradient** of f is the vector function*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}. \tag{14.25}$$

Problem 14.67. If $f(x,y) = \sin(x) + e^{xy}$, find $\nabla f(x,y)$ and $\nabla f(0,1)$.

Solution.

Ans: $\langle 2, 0 \rangle$

Note: With this notation of the gradient vector, we can rewrite

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y) \cdot \mathbf{u} = f_x(x,y)a + f_y(x,y)b, \quad \text{where } \mathbf{u} = \langle a, b \rangle\,. \tag{14.26}$$

Problem 14.68. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ and in the direction of the vector $\vec{v} = \langle 3, 4 \rangle$.

Solution.

Ans: 4

Maximizing the Directional Derivative

Note that

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta,$$

where θ is the angle between ∇f and $\mathbf{u}$; the maximum occurs when $\theta = 0$.

Theorem 14.69. *Let f be a differentiable function of two or three variables. Then*

$$\max_{\mathbf{u}} D_{\mathbf{u}}f(\mathbf{x}) = |\nabla f(\mathbf{x})| \tag{14.27}$$

and it occurs when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$.

Problem 14.70. Let $f(x,y) = xe^y$.

(a) Find the rate of change of f at $P(1,0)$ in the direction from P to $Q(-1,2)$.

(b) In what direction does f have the maximum rate of change? What is the maximum rate of change?

Solution.

Ans: (a) 0; (b) $\sqrt{2}$

Remark 14.71. Let $\mathbf{u} = \dfrac{\nabla f(\mathbf{x})}{|\nabla f(\mathbf{x})|}$, the unit vector in the gradient direction. Then

$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u} = \nabla f(\mathbf{x}) \cdot \frac{\nabla f(\mathbf{x})}{|\nabla f(\mathbf{x})|} = |\nabla f(\mathbf{x})|. \tag{14.28}$$

This implies that the directional derivative is maximized in the gradient direction.

Claim 14.72. *The gradient direction is the direction where the function changes fastest, more precisely, increases fastest!*

The Gradient Vector of Level Surfaces

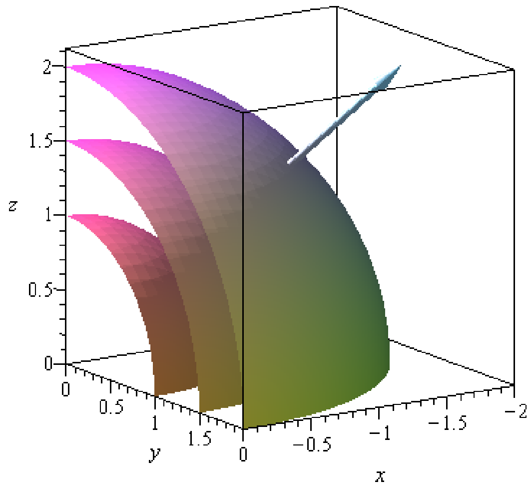


Figure 14.12: Level surfaces $x^2+y^2+z^2 = k^2$, where $k = 1, 1.5, 2$, and the gradient vector at $P(-1, 1, \sqrt{2})$, when $k = 2$.

Suppose S is a surface with equation

$$\boxed{F(x, y, z) = k}$$

and $P(x_0, y_0, z_0) \in S$. Let C be any curve that lies on the surface S, passes through P, and is described by a continuous vector function

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle\,. \qquad (14.29)$$

Then, any point $\langle x(t), y(t), z(t)\rangle$ must satisfy

$$F(x(t), y(t), z(t)) = k. \qquad (14.30)$$

Apply the Chain Rule to have

$$\frac{d}{dt}F = F_x\frac{dx}{dt} + F_y\frac{dy}{dt} + F_z\frac{dz}{dt} = \nabla F \cdot \mathbf{r}'(t) = 0.$$

In particular, letting $t = t_0$ be such that $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$,

$$\nabla F(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) = 0, \qquad (14.31)$$

where $\mathbf{r}'(t_0)$ is the tangent vector at $P(x_0, y_0, z_0)$.

Summary 14.73. ***(Gradient Vector)***. *Given a level surface $F(x, y, z) = k$, the gradient vector $\nabla F(x, y, z)$ is **normal** to the surface and pointing the **fastest increasing direction**.*

Tangent Plane to a Level Surface

Suppose S is a surface given as $F(x, y, z) = k$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ is on S. Then the **tangent plane** to S at $\mathbf{x}_0$ is

$$\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = F_x(\mathbf{x}_0)(x-x_0)+F_y(\mathbf{x}_0)(y-y_0)+F_z(\mathbf{x}_0)(z-z_0) = 0. \quad \textbf{(14.32)}$$

The **normal line** to S at $\mathbf{x}_0$ is

$$\frac{x-x_0}{F_x(\mathbf{x}_0)} = \frac{y-y_0}{F_y(\mathbf{x}_0)} = \frac{z-z_0}{F_z(\mathbf{x}_0)}. \quad \textbf{(14.33)}$$

Problem 14.74. Find the equations of the tangent plane and the normal line at $P(-1, 1, 2)$ to the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 3.$$

Solution.

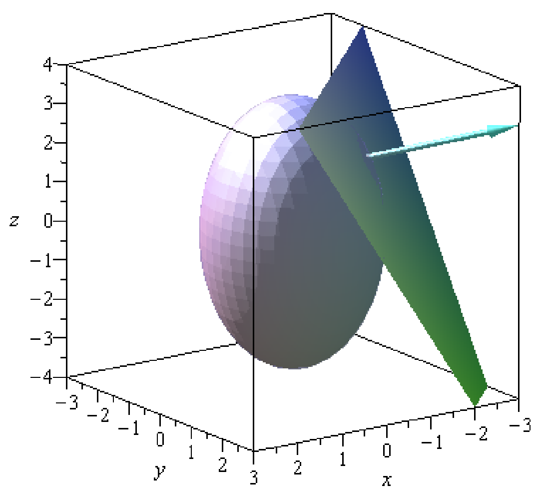


Figure 14.13

Ans: $-2x - 6 + 2y + z = 0$; $\frac{x+1}{-2} = \frac{y-1}{2} = \frac{z-2}{1}$

Exercises 14.6

1. Find the directional derivative of f at the point P in the direction indicated by either the angle θ or a vector v.

   (a) $f(x,y) = x\sin(xy), \quad P(0,1), \quad \theta = \pi/4$
   (b) $f(x,y,z) = y^2 e^{xyz}, \quad P(0,1,-1), \quad \mathbf{v} =< -1,2,2 >$

   Ans: (b) $5/3$

2. Find the maximum rate of change of f at the given point and the direction in which it occurs.

   (a) $f(x,y) = \sin(xy), \quad (0,1)$
   (b) $f(x,y,z) = \dfrac{z}{x+y}, \quad (1,1,4)$

   Ans: (b) $|\nabla f(1,1,4)| = 3/2, \quad \nabla f(1,1,4) =< -1,-1,1/2 >$

> **Note**: We know that a differentiable function f increases most rapidly in the direction of ∇f. Thus, it is natural to claim that the function ***decreases most rapidly*** in the direction opposite to the gradient vector, that is, $-\nabla f$.

3. Find the direction in which the function $f(x,y,z) = x^2 + y^2 + z^2$ decreases fastest at the point $(1,1,1)$.

4. Find directions (unit vectors) in which the directional derivative of $f(x,y) = x^2 + xy^2$ at the point $(1,2)$ has value 0.

   Ans: $\mathbf{u} = \pm\frac{<2,-3>}{\sqrt{13}}$

5. Find the equations of (i) the tangent plane and (ii) the normal line to the given surface at the specified point.

   (a) $(x-1)^2 + (y-2)^2 + (z-3)^2 = 3, \quad (2,1,4)$
   (b) $xy + yz + zx - 5 = 0, \quad (1,1,2)$

   Ans: (b) $3(x-1) + 3(y-1) + 2(z-2) = 0$ & $\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-2}{2}$

14.7. Maximum and Minimum Values

Recall: To find the **absolute maximum and minimum values** of a continuous function f on a closed interval $[a, b]$:

1. Find values of f at the critical points of f in (a, b).
2. Find values of f at the end points of the interval.
3. The largest is the absolute maximum value;
   the smallest is the absolute minimum value.

Recall: **(Second Derivative Test for** $\mathrm{y} = \mathrm{f}(\mathrm{x})$**)** Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a **local minimum** at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a **local maximum** at c.

14.7.1. Local extrema

Definition 14.75. *Let f be a function of two variables x and y.*

- *It has a **local minimum** at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b).*
- *It has a **local maximum** at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).*

Theorem 14.76. ***(First Derivative Test)**. If f has a local extreme at (a, b) and the first order partial derivatives exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$, that is, $\nabla f(a, b) = 0$.*

Problem 14.77. Find the **critical points** of

$$f(x, y) = 2x^3 - 3x^2 + y^2 + 4y + 1.$$

Ans: $(0, -2)$, $(1, -2)$

Theorem 14.78. ***(Second Derivative Test)***. *Suppose that the second order partial derivatives of f are continuous near (a,b) and suppose that $\nabla f(a,b) = 0$. Let*

$$D = D(a,b) = f_{xx}(a,b) f_{yy}(a,b) - [f_{xy}(a,b)]^2 .$$

- *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*
- *If $D < 0$, then $f(a,b)$ is a saddle point.*

Note:

1. If $D = 0$, then no conclusion can be drawn from this test.
2. $D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - (f_{xy})^2$
3. Let $D > 0$. Then, $f_{xx}(a,b) \gtrless 0$ is equivalent to $f_{yy}(a,b) \gtrless 0$.

Problem 14.79. Find all local extrema of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution.

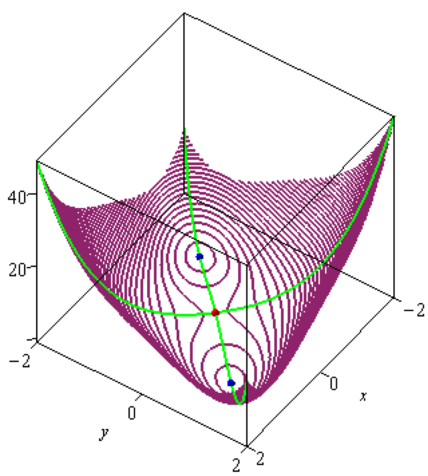


Figure 14.14

Ans: local min: $(\pm 1, \pm 1)$; saddle point: $(0,0)$

Problem 14.80. Find the shortest distance from the point $(1, 0, 2)$ to the plane $2x + 2y - z + 2 = 0$.

Solution. (a) You may use the formula $d = |ax_0 + by_0 + cz_0 + d|/\sqrt{a^2 + b^2 + c^2}$.
(b) Let d be the distance. Then

$$f(x, y) = d^2 = (x - 1)^2 + (y - 0)^2 + (z - 2)^2 = (x - 1)^2 + y^2 + (2x + 2y)^2$$

Ans: $2/3$

Problem 14.81. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the plane $3x + 2y + z = 6$.

Hint: Maximize $V = xyz$, subject to $3x + 2y + z = 6$. Thus $V = xy(6 - 3x - 2y)$. Try to find the maximum by setting $\nabla V = 0$.

Solution.

Ans: $(x, y, z) = (2/3, 1, 2)$; $V = xyz = 4/3$

14.7.2. Absolute extrema

Theorem 14.82. ***(Existence)***. *If f is continuous on a closed and bounded set $D \subset \mathbb{R}^2$, then f attains an **absolute minimum value** $f(x_0, y_0)$ and an **absolute maximum value** $f(x_1, y_1)$ at some points $(x_i, y_i) \in D$, $i = 0, 1$.*

Strategy 14.83. To find absolute extrema,

1. Find critical points and values of f at those critical points.
2. Find the extreme values that occur on the boundary.
3. Compare all of those values for the largest and smallest values.

Problem 14.84. Find the absolute extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $R = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.

Solution.

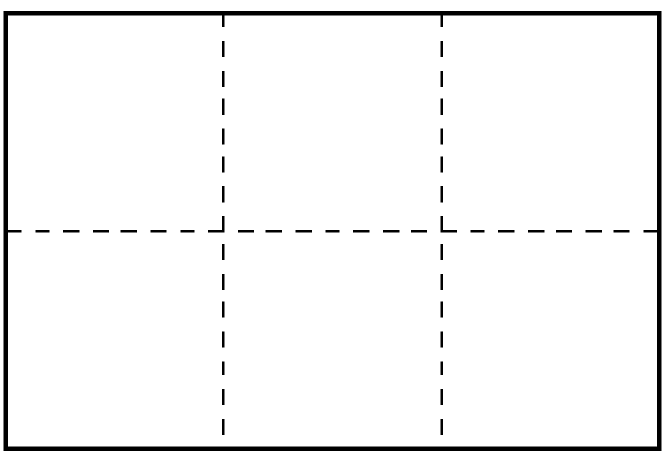


Figure 14.15: $R = [0, 3] \times [0, 2]$

Ans: abs.min=0; abs.max=9

Exercises 14.7

1. (i) Find the local maxima and minima and saddle points of the function. (ii) CAS Use Maple's **plot3d** and **contourplot** functions to verify them.

   (a) $f(x,y) = x^3 - 3xy^2$
   (b) $f(x,y) = (2x^2 + y^2)e^{-x^2-y^2}$

   (**Note**: You may use Mathematica, if you want.)

2. Find the *absolute* maximum and minimum values of f on D.

   (a) $f(x,y) = x^2 + y^2 - 2x$; D is the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.
   Ans: max: $f(0,\pm 2) = 4$; min: $f(1,0) = -1$
   (b) $f(x,y) = 4x^2 + y^4$; $D = \{(x,y) | x^2 + y^2 \le 1\}$.
   Ans: max: $f(\pm 1,0) = 4$; min: $f(0,0) = 0$

3. Find three positive numbers whose sum is 60 and whose product is maximum.
   Hint: The problem can read: $\max_{(x,y,z)} xyz$, subject to $x + y + z = 60$. Thus for example it can be reformulated as: $\max_{(x,y)} xy(60 - x - y)$, with each component being positive. From this, you may conclude $x = y$.

4. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $2x + 5y + z = 30$. ***Clue***: Try to use the hint given for Problem 14.81.

14.8. Lagrange Multipliers

In Problem 14.81, on p. 62, we maximized a volume function $V = xyz$ subject to the **constraint** $3x + 2y + z = 6$, which was the plane having a vertex of the rectangular box.

In this section, we consider Lagrange's method to solve the problem of the form

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{subj.to} \quad g(\mathbf{x}) = c. \tag{14.34}$$

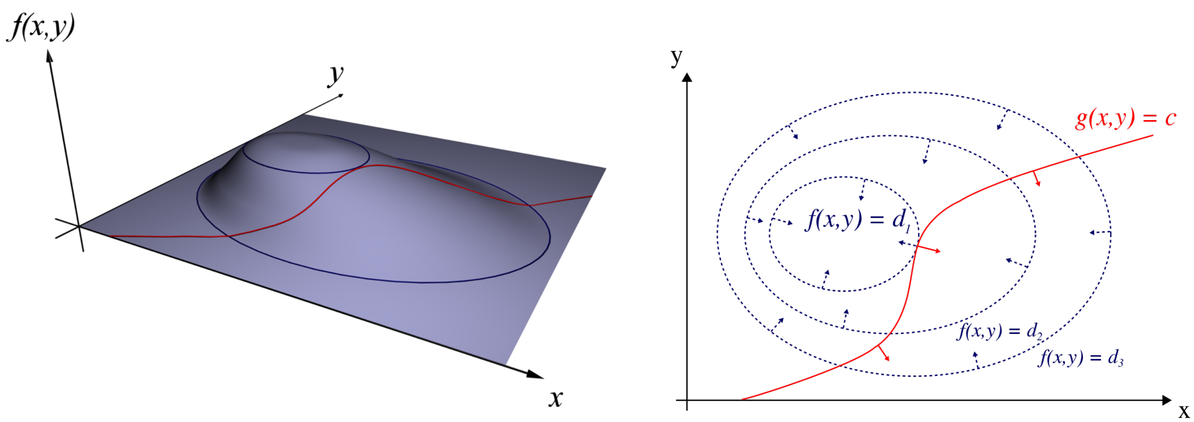


Figure 14.16: The method of Lagrange multipliers in $\mathbb{R}^2$: **∇f // ∇g, at maximum**.

Strategy 14.85. (Method of Lagrange multipliers). For the maximum and minimum values of $f(x, y, z)$ **subject to** $g(x, y, z) = c$,

(a) Find all values of (x, y, z) and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z) \ \text{ and } \ g(x, y, z) = c. \tag{14.35}$$

(b) Evaluate f at all these points, to find the maximum and minimum.

Problem 14.86. (Revisit of Problem 14.81, p. 62). Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the plane $3x + 2y + z = 6$, using the method of Lagrange multipliers.

Solution.

Ans: $4/3$

Problem 14.87. A topless rectangular box is made from 12m^2 of cardboard. Find the dimensions of the box that maximizes the volume of the box.

Solution. Maximize $V = xyz$ subj.to $2xz + 2yz + xy = 12$.

Ans: 4 $(x = y = 2z = 2)$

Problem 14.88. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution. $\nabla f = \lambda \nabla g \Longrightarrow \begin{bmatrix} 2x \\ 4y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Therefore, $\begin{cases} 2x = 2x\,\lambda & ① \\ 4y = 2y\,\lambda & ② \\ x^2 + y^2 = 1 & ③ \end{cases}$

From ①, $x = 0$ or $\lambda = 1$.

Ans: min: $f(\pm 1, 0) = 1$; max: $f(0, \pm 1) = 2$

Problem 14.89. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \leq 1$.

Solution. ***Hint***: You may use Lagrange multipliers when $x^2 + y^2 = 1$.

Ans: min: $f(0,0) = 0$; $f(0, \pm 1) = 2$

Two Constraints

Consider the problem of the form

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \textbf{subj.to} \quad g(\mathbf{x}) = c \ \textbf{ and } \ h(\mathbf{x}) = d. \tag{14.36}$$

Then, at extrema we must have

$$\nabla f \ \in \ \text{Plane}(\nabla g, \nabla h) := \{c_1 \nabla g + c_2 \nabla h\}. \tag{14.37}$$

Thus (14.36) can be solved by finding all values of (x, y, z) and (λ, μ) such that

$$\begin{array}{rcl} \nabla f(x,y,z) & = & \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) & = & c \\ h(x,y,z) & = & d \end{array} \tag{14.38}$$

Problem 14.90. Find the maximum value of the function $f(x, y, z) = z$ on the curve of the intersection of the cone $2x^2 + 2y^2 = z^2$ and the plane $x + y + z = 4$.

Solution. Letting $g = 2x^2 + 2y^2 - z^2 = 0$ ④ and $h = x + y + z = 4$ ⑤, we have

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 4x \\ 4y \\ -2z \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \left\{ \begin{array}{rcrl} 0 & = & 4\lambda x + \mu & ① \\ 0 & = & 4\lambda y + \mu & ② \\ 1 & = & -2\lambda z + \mu & ③ \end{array} \right.$$

From ① and ②, we conclude $x = y$; using ④, we have $z = \pm 2x$.

Ans: 2

Exercises 14.8

1. Use Lagrange multipliers to find extreme values of the function subject to the given constraint.

   (a) $f(x, y) = xy; \quad x^2 + 4y^2 = 2$
   (b) $f(x, y) = x + y + 2z; \quad x^2 + y^2 + z^2 = 6$

   Ans: max: $f(1, 1, 2) = 6$; min: $f(-1, -1, -2) = -6$

2. Find extreme values of f subject to both constraint.

$$f(x, y, z) = x^2 + y^2 + z^2; \quad x - y = 3, \;\; x^2 - z^2 = 1.$$

   Ans: $f(1, -2, 0) = 5$

   Note: The value just found for Problem 2 is the minimum. Why? See the figure below.

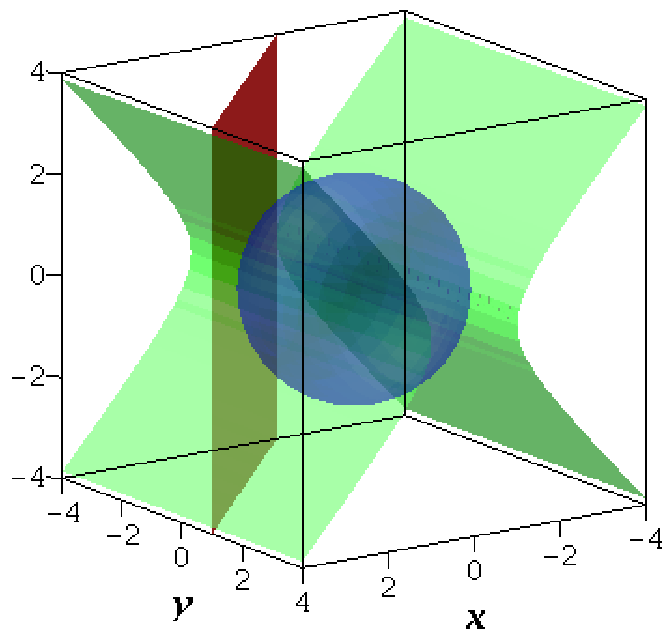


Figure 14.17: implicitplot3d. red: $x - y = 3$; green: $x^2 - z^2 = 1$; blue: $f(x, y, z) = 5$.

3. Use Lagrange multipliers to solve Problem 3 in Exercise 14.7. (See p. 64.)
4. Use Lagrange multipliers to solve Problem 4 in Exercise 14.7.

R.14. Review Problems for Ch. 14

1. Let $f(x, y) = \sqrt{4 - x^2 - 4y^2}$.

   (a) Find the **domain** of f.

   (b) Sketch the **graph** of the function.

Ans: (a) $\{(x, y) \mid x^2 + 4y^2 \le 4\}$

2. **Determine and verify** whether the following functions are continuous at $(0, 0)$ or not.

   (a) $f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$ Circle: continuous discontinuous

   (b) $g(x, y) = \begin{cases} \dfrac{\sin(x^2 + y^2)}{x^2 + y^2}, & \text{if } (x, y) \ne (0, 0), \\ 0, & \text{if } (x, y) = (0, 0). \end{cases}$ Circle: continuous discontinuous

Ans: (a) continuous; (b) discontinuous

3. Let $f(x, y) = \sin(2x - 3y)$.

   (a) Find $\boldsymbol{f_x(3, 2)}$ and $\boldsymbol{f_y(3, 2)}$.

   (b) Find $\boldsymbol{f_{xyx}(3, 2)}$ and $\boldsymbol{f_{yxx}(3, 2)}$.

Ans: (a) 2, −3; (b) 12, 12

4. Let $f(x, y) = 1 + x \ln(xy - 5)$.

   (a) Explain why f is **differentiable** at $(2, 3)$.

   (b) Find the **linearization** $L(x, y)$ of f at $(2, 3)$.

Ans: (a): $f_x = \ln(xy - 5) + \frac{xy}{xy-5}$ and $f_y = \frac{x^2}{xy-5}$ are continuous near $(2, 3)$.
(b) $L(x, y) = 1 + 6(x - 2) + 4(y - 3)$.

5. Suppose f is a differentiable function of x and y, and $g(u, v) = f(u + e^v + \cos v,\ u^2 + \sin v)$. Use the table of values to find $\boldsymbol{g_u(0, 0)}$ and $\boldsymbol{g_v(0, 0)}$.

	f	g	f_x	f_y
$(0, 0)$	2000	2	100	9
$(2, 0)$	3	4	9	12

Ans: $g_u(0,0) = 9$, $g_v(0,0) = 21$

6. Let $f(x,y) = x + \sin(xy)$ and $P(1,0)$.
   (a) Find the **directional derivative** of f at P in the direction given by the angle $\theta = \pi/3$.
   (b) Determine $\max_{\mathbf{u}} \boldsymbol{D_{\mathbf{u}} f(P)}$, where $\mathbf{u}$ is a unit vector.

Ans: (a) $< 1, 1 > \cdot < \cos(\pi/3), \sin(\pi/3) >= (1 + \sqrt{3})/2$. (b) $\sqrt{2}$

7. Consider the ellipsoid $\dfrac{x^2}{4} + y^2 + \dfrac{z^2}{9} = 3$. Find the **tangent plane** to it at point $(2, 1, -3)$.

Ans: $(x-2) + 2(y-1) - \frac{2}{3}(z+3) = 0$

8. Find the **absolute maximum and minimum values** of f over D:
$$f(x,y) = 3x^2 + y^2, \quad D = \{(x,y) \mid x^2 + y^2 \le 1\}.$$

Ans: min:0, max:3

9. Use the **method of Lagrange multipliers** to find the maximum and minimum values of f subject to the given constraint:
$$f(x,y) = x^2 - y^2; \quad x^2 + y^2 = 1$$

Hint: $\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 2x = \lambda \cdot 2x & \text{①} \\ -2y = \lambda \cdot 2y & \text{②} \\ x^2 + y^2 = 1 & \text{③} \end{cases}$ Then, it follows from ① and ② that $\lambda = \pm 1$. When $\lambda = -1$, for example, $x = 0$ from ①, with which ③ makes $y = \pm 1$.

Ans: min:-1, max:1

10. Use the **method of Lagrange multipliers** to find three positive numbers whose sum is 15 and the sum of whose squares is as small as possible.

Clue: $\min x^2 + y^2 + z^2$, subject to $x + y + z = 15$.

Ans: $x = y = z = 5$

Project 1. Linear and Quadratic Approximations

This project is designed for students to experience computer algebra, while solving some calculus problems with computer coding. Although it includes examples written in Maple only, students can finish the project using Maple, Mathematica, or MathCad.

Getting familiar with Computer Algebra CAS

For a smooth function of one variable, f, its **Taylor series** about a is given as

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots . \quad \mathbf{(14.39)}$$

As with any convergent series, $f(x)$ is the limit of the sequence of partial sums. That is,

$$f(x) = \lim_{n\to\infty} T_n(x), \quad \mathbf{(14.40)}$$

where $T_n(x)$ is called the **Taylor polynomial of degree n**:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Example 14.91. Let

$$f(x) = \arctan(x) - \frac{1}{3}. \quad \mathbf{(14.41)}$$

Then, when it is expanded about $a = 1/2$, $T_n(x)$ can be obtained using Maple:

```
a  := 1/2:
Tn := x-> convert(taylor(f(x),x=a,n+1),polynom):
```

See Figure 14.18[4] (p. 73). For the function in (14.41),

$$\begin{aligned} T_1(x) &= \arctan\left(\frac{1}{2}\right) - \frac{1}{3} + \frac{4}{5}\left(x - \frac{1}{2}\right), \\ T_2(x) &= \arctan\left(\frac{1}{2}\right) - \frac{1}{3} + \frac{4}{5}\left(x - \frac{1}{2}\right) - \frac{8}{25}\left(x - \frac{1}{2}\right)^2. \end{aligned} \quad \mathbf{(14.42)}$$

[4]In Maple, `taylor(f(x),x=a,n+1)` returns a polynomial of $(n+1)$ terms plus the remainder, $T_n(x) + \mathcal{O}((x-a)^{n+1})$; while the command `convert(g,polynom)` converts `g` into a polynomial form, which is $T_n(x)$. In Mathematica, `Series[f[x],x,a,n]` produces the same result as for `taylor(f(x),x=a,n+1)` in Maple. Now, the Mathematica-command `Normal` can be used to convert the result into normal expressions of polynomials.

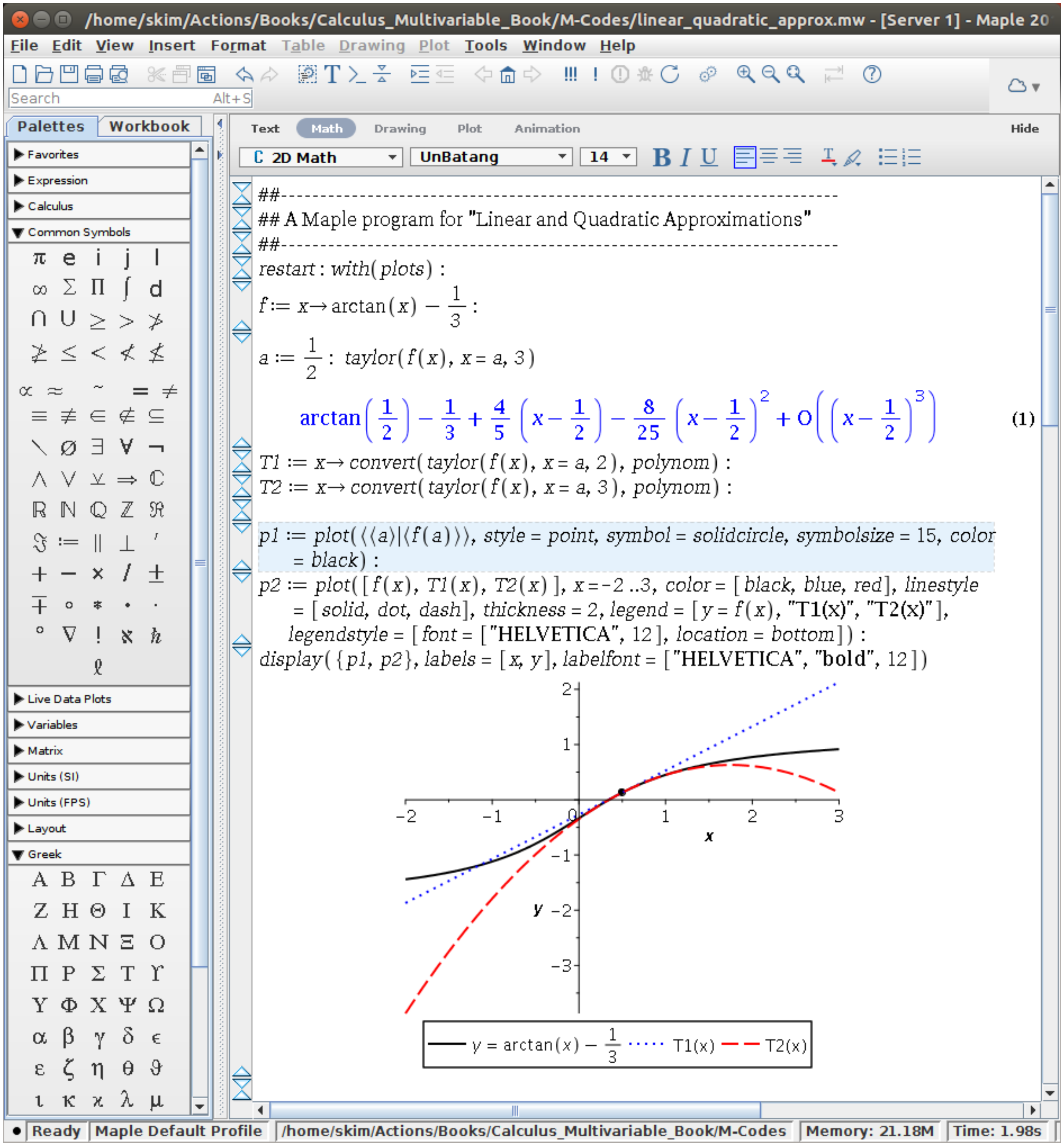


Figure 14.18: Screen-shot of Maple window, which plots linear and quadratic approximations of $f(x) = \arctan(x) - \frac{1}{3}$ about $a = 1/2$.

Maple: 3D Plots

First load the `plots` package, along with other frequently used packages, using the entry:

```
with(plots): with(plottools):
with(VectorCalculus): with(Student[MultivariateCalculus]):
```

1. Plot $z = f(x, y)$ in **Cartesian coordinates**, using

$$plot3d(\ f(x,y), x = a..b, y = c..d, options)$$

   Consider the options
   (a) *style* = *patchcontour* Puts contour curves on the surface.
   (b) *axes* = *boxed* Puts the axes on the edges of a box enclosing the surface.
   (c) *scaling* = *constrained* Makes the scale on the three axes the same.
   (d) *orientation* =$[40, 70]$ Orients the viewpoint so it is closer to what you see in your text.

2. Plot $F(x, y, z) = 0$ in **Cartesian coordinates**, using

$$implicitplot3d(\ F(x,y,z) = 0, x = a..b, y = c..d, z = s..t, options)$$

   Consider the options listed above along with the following.
   (a) *grid* = $[m, n, k]$ Where m, n, k are positive integers, try $[30, 30, 30]$ for example. This plots 30 points in each direction for a smoother surface.
   (b) *axes* = *framed* Puts axes along the edges of a frame around the plot.
   (c) *orientation* = $[-50, 60]$ Another nice viewing angle.

3. Plot $r = f(\theta, z)$ in **cylindrical coordinates**, using

$$plot3d(\ f(\theta,z), \theta = a..b, z = s..t, coords = cylindrical, options)$$

   To plot $z = g(r, \theta)$, use

$$plot3d(\ [r, \theta, g(r,\theta)], r = a..b, \theta = \alpha..\beta, coords = cylindrical, options)$$

   Options are more or less the same as the above.

4. Plot $\rho = f(\theta, \phi)$ in **spherical coordinates**, using

$$plot3d(\ f(\theta,\phi), \theta = \alpha..\beta, \phi = \gamma..\delta, coords = spherical, options)$$

5. Implicit plots can also be made in **cylindrical or spherical coordinates**. For example, to plot the equation $r^2 + 2z^2 = r\cos\theta$ in cylindrical coordinates, use

$$implicitplot3d(\ r^2 + 2\,z^2 = r\cos(\theta), r = a..b, \theta = \alpha..\beta, z = s..t, coords = cylindrical, options)$$

6. **(Contour plots in $2D$)**. For $z = f(x, y)$, use

$$contourplot(f(x,y), x = a..b, y = c..d, options)$$

P.1.1. Newton's method

As one can see from Figure 14.18, the first-degree Taylor series $T_1(x)$ is the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. One of popular applications exploiting the tangent line is **Newton's method** for the problem of root-finding.

$$\boxed{\text{Given a differentiable function } f(x), \text{ find } r \text{ such that } f(r) = 0,} \tag{14.43}$$

where r is an x-intercept of the curve $y = f(x)$.

The idea behind Newton's method:

- The tangent line is close to the curve and so its x-intercept must be close to the x-intercept of the curve.
- Let x_0 be the initial approximation close to r. Then, the tangent line at $(x_0, f(x_0))$ reads

$$L(x) = f'(x_0)(x - x_0) + f(x_0). \tag{14.44}$$

  Let x_1 be the x-intercept of $y = L(x)$. Then,

$$0 = f'(x_0)(x_1 - x_0) + f(x_0).$$

  Solving the above equation for x_1 becomes

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{14.45}$$

  which hopefully is a better approximation for the root r.
- Repeat the above till the convergence.

Algorithm 14.92. ***(Newton's method for solving*** $\mathrm{f(x) = 0}$***).*** *For x_0 chosen close to a root r, compute $\{x_n\}$ repeatedly satisfying*

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n \ge 1.} \tag{14.46}$$

Problem 14.93. Consider the function $f(x) = \arctan(x) - \frac{1}{3}$ in (14.41).

1. Implement a code for Newton's method to approximate a root of $f(x) = 0$.
   (You can use Maple, Mathematica, or MathCad.)

2. Run a few iterations, starting from $x_0 = 0.5$, and measure how the error decreases as the iteration count grows.
   (Note that the exact root $r = \tan(1/3) \approx 0.34625354951057549103$.)

P.1.2. Estimation of critical points

The second part of the project involves a min-max analysis of a function in (x, y) that is based on each student's ID number, so that each student has his/her own function to work with. If a student's ID number is 123-45-6789, then he/she will study the behavior of the function

$$f(x,y) = 1*\sin(x-2)+3*\cos(y-4)+5*x^2-6*xy+7*y^2-8*x+9*y, \quad (14.47)$$

where the alternating signs are used to create a little more "action". We will call such a function the **ID function**.

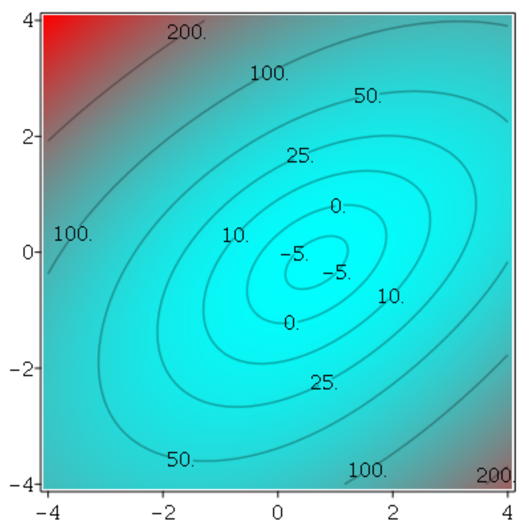


Figure 14.19: Contour plot of $f(x, y)$ in (14.47).

Problem 14.94. Create your ID function. Then,

1. Include a variety of surface plots with different views and contour plots with different windows to provide a good picture of the behavior of your ID function.[5]

[5] In Maple, you can use the commands `plot3d` and `countourplot`. In Mathematica, `Plot3D` and `CountourPlot` are available.

2. Label the figures and refer to them in your write-up, as you discuss the kinds of critical points you observe. (If you have no or one critical point, change the signs and/or shuffle the digits in your ID function to get more action.)
3. Zoom in sufficiently so that you can estimate the coordinates of each of the critical points.

P.1.3. Quadratic approximations

We have discussed the **linear approximation** (or, **linearization**) of a function f of two variables at a point (a, b):

$$L(x,y) \;=\; f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b), \tag{14.48}$$

which is also called the **first-degree Taylor polynomial** of f at (a, b). If f has continuous second-order partial derivatives at (a, b), then the **second-degree Taylor polynomial** of f at (a, b) is

$$\begin{aligned} Q(x,y) \;=\;& f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ & + \frac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2} f_{yy}(a,b)(y-b)^2, \end{aligned} \tag{14.49}$$

and the approximation $f(x, y) \approx Q(x, y)$ is called the **quadratic approximation** of f at (a, b).

Problem 14.95. Answer the following.

1. Verify that the quadratic approximation Q has the same first- and second-order partial derivatives as f at (a, b). (This is the only portion of the project that you can finish without using computer implementation.) ***Hint***: The partial derivatives evaluated at (a, b), appeared in Q, are all constant.
2. Use computer algebra to find the first- and second-degree Taylor polynomials L and Q for your ID function f at a critical point $C(x_0, y_0)$ that you estimated from Problem 14.94.
3. Compare the values of f, L, and Q at $(x_0 + 0.1, y_0 - 0.1)$.

4. Graph f, L, and Q; comment on how well L and Q approximate f.

Report. Submit hard copies of your experiences.

- Solve Problems 14.93, 14.94, and 14.95, using computer programming.
- Make hard copies of your work, and **collect them in order**.
- Attach a "summary" or "conclusion" page at the beginning of report.

You may work in a small group; however, you must report individually.

CHAPTER **15**

Multiple Integrals

The **multiple integral** is a definite integral of a function of more than one real variable, for example, $f(x,y)$ or $f(x,y,z)$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ are called **double integrals**, and integrals of a function of three variables over a region of $\mathbb{R}^3$ are called **triple integrals**.

In this chapter, you will learn double integrals and triple integrals in rectangular coordinates, polar coordinates, cylindrical coordinates, and spherical coordinates. Also, you will learn how to perform integration by changing variables between or inside coordinates.

Contents of Chapter 15

This chapter corresponds to Chapter 15 in STEWART, ***Calculus*** (8th Ed.), 2015.

15.1. Double Integrals over Rectangles

Recall: (**Review on Definite Integrals**). We know the following:

- We defined the integral in terms of **Riemann Sum**.
- That is, we found the area underneath the curve $y = f(x)$ by dividing the area into rectangles. We then added up their areas to get the area under the curve.
- We then found the exact area of this by evaluating

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

- We could also get it using the **Fundamental Theorem of Calculus** (Part 2):

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{15.1}$$

where F is a function such that $F' = f$ (antiderivative).

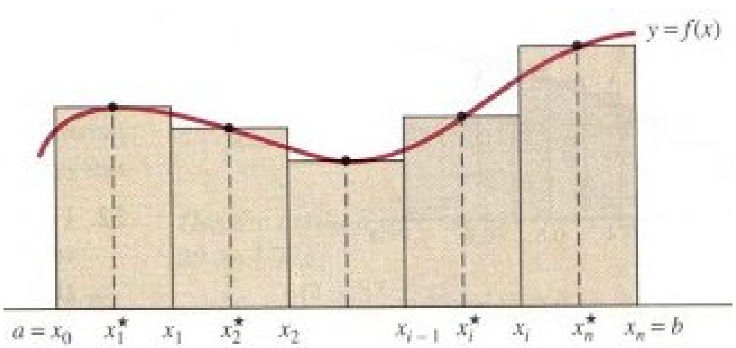


Figure 15.1: Riemann Sum.

15.1.1. Volumes as double integrals

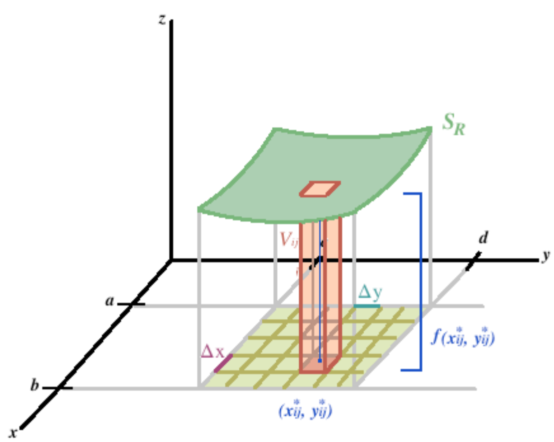


Figure 15.2

Let $R = [a, b] \times [c, d] \subset \mathbb{R}^2$ be a rectangle. Define

$$\Delta x = (b - a)/m, \quad \Delta y = (d - c)/n,$$

for some $m, n > 0$. Let

$$\begin{aligned} x_i &= a + i\Delta x, \quad i = 0, 1, \cdots, m, \\ y_j &= c + j\Delta y, \quad j = 0, 1, \cdots, n, \end{aligned}$$

and

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

Let $S_R = \{(x, y, z) \mid 0 \le z \le f(x, y), (x, y) \in R\}$ define the solid that lies above R. Let $\Delta A = \Delta x \, \Delta y$ denote the area of each R_{ij}. Then we can express this volume of S_R as

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \, \Delta A, \tag{15.2}$$

where (x^*_{ij}, y^*_{ij}) is a **sample point** in each division R_{ij}.

Definition 15.1. *The* ***double integral*** *of f over the rectangle R is*

$$\iint_R f(x, y) \, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x^*_{ij}, y^*_{ij}) \, \Delta A. \tag{15.3}$$

We can simplify this if we choose each sample point to be the point in the upper right corner of each sub-rectangle, $(x^*_{ij}, y^*_{ij}) = (x_i, y_j)$:

$$\iint_R f(x, y) \, dA = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j) \, \Delta A. \tag{15.4}$$

Problem 15.2. Estimate the volume of the solid that lies above the square $R = [0, 2] \times [0, 2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the **upper right corner** of each square R_{ij}. Approximate the Volume.

Solution.

Ans: 34

Problem 15.3. (Midpoint rule). Estimate the volume of the solid that lies above the square $R = [0, 2] \times [1, 2]$ and below the function $f(x, y) = 5x^2 - 4y$. Divide R into four equal rectangles and choose the sample point to be the **midpoint** of each rectangle R_{ij}. Approximate the volume.

Solution.

Ans: 1

15.1.2. Iterated integrals

Okay; so, taking these Riemann Sums is a bit of a pain.

Recall: Earlier in Calculus, we equated these Riemann sums to the definition of an integral. We will attempt to do the same here; however, we will use two partial integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

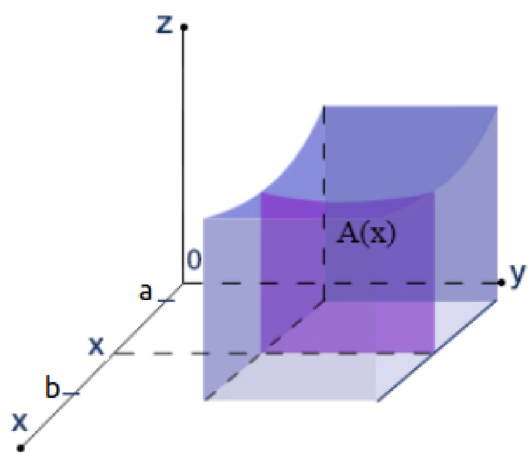


Figure 15.3: $A(x)$.

Definition 15.4. *We define*

$$A(x) = \int_c^d f(x, y)\, dy \qquad (15.5)$$

as the ***partial integral with respect to*** y*. We evaluate this integral by treating* x *as a constant, and integrate* $f(x, y)$ *with respect to* y*.*

Definition 15.5. *We define*

$$B(y) = \int_a^b f(x, y)\, dx \qquad (15.6)$$

as the ***partial integral with respect to*** x*. We evaluate this integral by treating* y *as a constant, and integrate* $f(x, y)$ *with respect to* x*.*

Note: The *Fundamental Theorem of Calculus, Part 2*, Equation (15.1) on p. 80, can be used to evaluate the partial integrals.

Definition 15.6. *The* ***iterated integral*** *is defined as follows:*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_a^b A(x)\,dx. \tag{15.7}$$

In other words, we work this integral ***from the inside out****.*

Problem 15.7. Evaluate the integrals

$$\text{(a)} \int_0^3 \int_1^2 x^2 y\,dy\,dx \qquad \text{and} \qquad \text{(b)} \int_1^2 \int_0^3 x^2 y\,dx\,dy.$$

Solution. $R = [0,3] \times [1,2]$.

Ans: $27/2$

Theorem 15.8. ***(Fubini's Theorem).*** *If f is continuous on the rectangle $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$, then*

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \tag{15.8}$$

Problem 15.9. (Revisit of Problem 15.3). Evaluate the double integral $\iint_R (5x^2 - 4y)\, dA,$, where $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Solution.

Ans: $4/3$

Problem 15.10. Evaluate $\iint_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Solution. Let's try the iterated integrals with x-first and y-first.

Ans: 0

Separable functions $f(x,y) = g(x)\,h(y)$:
Let $R = [a,b] \times [c,d]$. Then

$$\begin{aligned}\iint\limits_R f(x,y)\,dA &= \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \Big(\int_a^b g(x)\, \underline{h(y)}\,dx \Big)\,dy \\ &= \int_c^d h(y) \underline{\Big(\int_a^b g(x)\,dx \Big)}\,dy \\ &= \Big(\int_a^b g(x)\,dx \Big) \int_c^d h(y)\,dy,\end{aligned}$$

where the underlined (in maroon) are treated as constants.

$$\iint\limits_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy, \quad R = [a,b] \times [c,d]. \tag{15.9}$$

Problem 15.11. Evaluate $\iint_R e^{x+3y}\,dA$, where $R = [0,3] \times [0,1]$.
Solution.

Ans: $(e^3 - 1)^2/3$

Average Value

Recall: The **average value** of a function f of one variable defined on an interval $[a, b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

Definition 15.12. *In a similar fashion, we define the* **average value** *of f of two variables defined on R to be*

$$f_{\text{ave}} = \frac{1}{A(R)}\iint_R f(x, y)\,dA, \tag{15.10}$$

where $A(R)$ is the area of R.

Problem 15.13. Find the average value of $f(x, y) = x^2 + \sin(2y)$ over $R = [0, 3] \times [0, \pi]$.

Solution. Use symmetry, for a simpler calculation!

Ans: 3

Exercises 15.1

1. Estimate the volume of the solid that lies below the surface $z = x^2 + y$ and above the rectangle
$$R = \{(x, y) \mid 0 \le x \le 4,\ \ 0 \le y \le 6\}.$$
Use a Riemann sum with $m = 2$, $n = 3$, and the *Midpoint Rule*.

Ans: $48 \cdot 4 = 192$

2. Let V be the volume of the solid that lies under the surface $z = 30 - 4x - y^2$ and above the rectangle $R = \{(x, y) \mid 2 \le x \le 6,\ \ 0 \le y \le 2\}$. Use the lines $x = 4$ and $y = 1$ to divide R into four subrectangles. Let L and U be the Riemann sums computed respectively using lower left corners and upper right corners. Without using the actual numbers V, L, and U, arrange them in increasing order and describe your reasoning.

3. Evaluate the double integral by first identifying it as the volume of a solid.

   (a) $\displaystyle\iint_R (x + 1)\, dA, \quad R = \{(x, y) \mid 0 \le x \le 2,\ \ 0 \le y \le 2\}$

   (b) $\displaystyle\iint_R (4 - 2y)\, dA, \quad R = [0, 1] \times [0, 1]$

4. Calculate the iterated integral.

   (a) $\displaystyle\int_1^3 \int_0^2 (6xy^2 - 12x^2)\, dx\, dy$

   (b) $\displaystyle\int_0^2 \int_1^3 (6xy^2 - 12x^2)\, dy\, dx$

   (c) $\displaystyle\int_0^1 \int_0^2 2\pi xy \sin(\pi xy^2)\, dy\, dx$

   (d) $\displaystyle\int_0^2 \int_0^1 2\pi xy \sin(\pi xy^2)\, dx\, dy$

Ans: (a) 40, (c) 1

5. Calculate the double integral.

   (a) $\displaystyle\iint_R y \sec^2(x)\, dA, \quad R = [0, \pi/4] \times [0, 4]$

   (b) $\displaystyle\iint_R xe^{-xy}\, dA, \quad R = [0, 2] \times [0, 1]$

Ans: (a) 8; (b) $1 - e^{-2}$

6. Find the volume of the solid in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

Ans: 36

15.2. Double Integrals over General Regions

Okay. So, now we know how to find the volume of the solid under a surface, when the projection of the solid down to the xy-plane is a rectangular region.

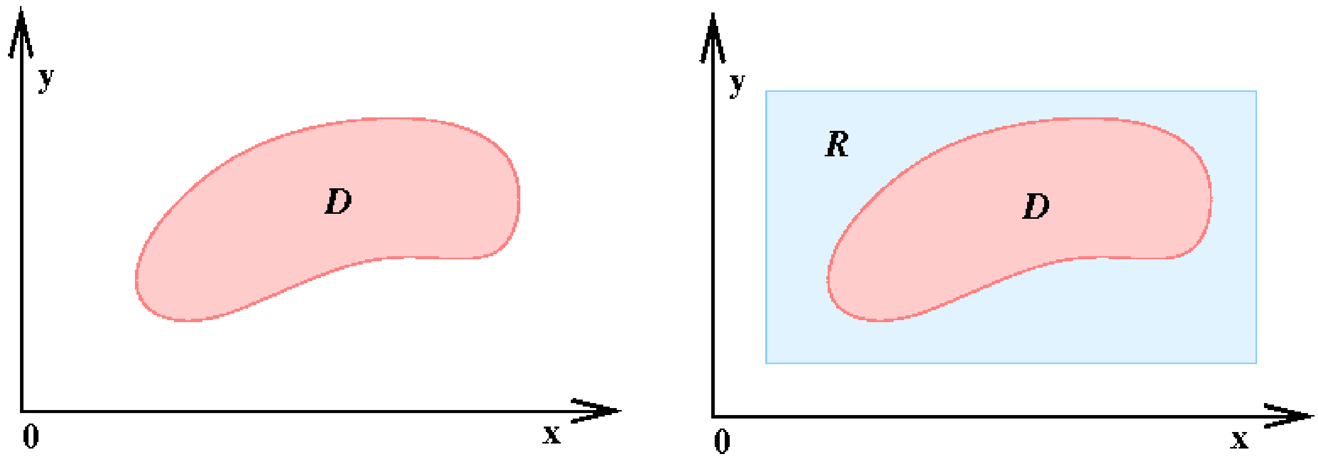


Figure 15.4: A general region D and its surrounding rectangle R.

Let $D \subset \mathbb{R}^2$ be a bounded region of general shape as in Figure 15.4. For a bounded function f defined over D, define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \notin D. \end{cases}$$

Then,

$$\iint_D f(x,y)\, dA = \iint_R F(x,y)\, dA, \tag{15.11}$$

which implies the following.

- The integral $\iint_D f(x,y)\, dA$ exists.
- The **iterated integral** can be applied to get the double integral over general regions.

Quesiton. What if the region D is not rectangular but defined as ***the boundary between two functions***?

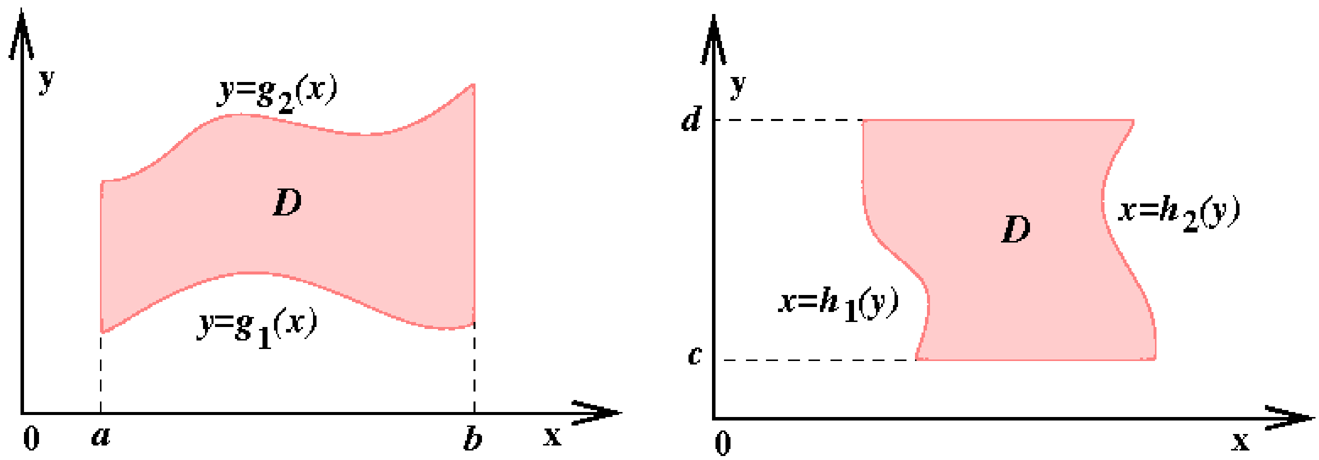


Figure 15.5: General regions D: Type 1 (left) and Type 2 (right).

Let the region D be given as

$$\begin{aligned} D_1 &= \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\} \\ D_2 &= \{(x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y), c \le y \le d\}, \end{aligned}$$

Then

$$\begin{aligned} \iint\limits_{D_1} f(x,y)\,dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \\ \iint\limits_{D_2} f(x,y)\,dA &= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \end{aligned} \tag{15.12}$$

Strategy 15.14. Double integral over general regions D:

1. ***Visualize*** to recognize the region.
2. Decide the ***order of integration***.
3. If the calculation becomes complicated, ***try the other order***.

Problem 15.15. Evaluate $\iint\limits_D (x + 2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution. First, visualize the region.

Ans: $\frac{32}{15}$

Problem 15.16. Find the volume of the solid that lies under the plane $z = 1 + 2y$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. Try for both orders.

Ans: $28/5$

Note: Here, the main concern is how to access the domain D; the iterated integration must access points in D, ***once–and–only–once***.

Problem 15.17. Evaluate $\iint_D 2xy\,dA$, where D is the region bounded by the line $y = x - 2$ and the parabola $x = y^2$.

Solution.

Ans: $45/4$

Problem 15.18. Evaluate $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\, dy\, dx$.

Solution. Visualize the region and try to change the order of integration.

Ans: $(1-\cos 1)/2$

Proposition 15.19. ***(Properties of double integrals)***. *Let f and g be continuous functions defined in D and $c \in \mathbb{R}$. Then*

① $\displaystyle\iint_D \big[f(x,y)+g(x,y)\big]\, dA = \iint_D f(x,y)\, dA + \iint_D g(x,y)\, dA$

② $\displaystyle\iint_D c\, f(x,y)\, dA = c\iint_D f(x,y)\, dA$

③ $\displaystyle\iint_D f(x,y)\, dA \ge \iint_D g(x,y)\, dA,$ ***if*** $f(x,y) \ge g(x,y),\ \forall\, (x,y) \in D$

④ $\displaystyle\iint_D f(x,y)\, dA = \iint_{D_1} f(x,y)\, dA + \iint_{D_2} f(x,y)\, dA,$ ***when*** $D = D_1 \dot{\bigcup} D_2$

⑤ $\displaystyle\iint_D 1\, dA = A(D)$

⑥ $\displaystyle m\cdot A(D) \le \iint_D f(x,y)\, dA \le M\cdot A(D),$ ***if*** $m \le f(x,y) \le M,\ \forall\, (x,y) \in D$

Problem 15.20. Show that ⑤ $\iint_D 1\, dA = A(D)$, where $A(D)$ denotes the area of the region D.

Hint: Consider a solid cylinder whose base is D and whose height is 1.

Problem 15.21. Use Property ⑥ in Proposition 15.19 to estimate the integral $I = \iint_D e^{\sin x \cos y} dA$, where D is the disk with center the origin and radius 2.

Solution.

Ans: $4\pi/e \le I \le 4\pi e$

Problem 15.22. Evaluate the integral $\displaystyle\int_0^1 \int_x^1 2\, e^{x/y}\, dy\, dx$ by reverting the order of integration.

Solution.

Ans: $e - 1$

Exercises 15.2

1. Evaluate the double integral, by setting up an iterated integral in the easier order.

   (a) $\displaystyle\iint_D 2e^{-x^2}\,dA, \quad D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x\}$

   (b) $\displaystyle\iint_D x\,dA$, D is bounded by $y = x + 2$ and $y = x^2$

   (c) $\displaystyle\iint_D y \sin \pi x\,dA$, D is bounded by $x = 0$, $x = y^2$, and $y = 2$

   Ans: (a) $1 - e^{-4}$; (c) $2/\pi$

2. Evaluate the volume of the solid that lies under the surface $z = x(y + 2)$ and above the triangle with vertices $P(1, 1)$, $Q(3, 1)$, and $R(1, 3)$.

   Ans: 12

3. Sketch the region of the integral and change the order of integration.

   (a) $\displaystyle\int_0^1 \int_0^{y^2} f(x,y)\,dx\,dy$ (b) $\displaystyle\int_1^e \int_0^{\ln x} f(x,y)\,dy\,dx$

   Ans: (b) $\int_0^1 \int_{e^y}^e f(x,y)\,dx\,dy$

4. Evaluate the integral by reversing the order of integration:

   (a) $\displaystyle\int_0^1 \int_{x^2}^1 \sqrt{y}\cos(y^2)\,dy\,dx$ (b) $\displaystyle\int_0^4 \int_0^{\sqrt{4-y}} e^{12x - x^3}\,dx\,dy$

   Ans: (a) $\frac{\sin 1}{2}$

5. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^y f(x,y)\,dx\,dy + \int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy.$$

   (a) Sketch the region D.

   (b) Express the double integral as a ***single*** iterated integral with **reversed** order of integration.

15.3. Double Integrals in Polar Coordinates

We have spent most of our lives in the Cartesian/Rectangular coordinate system, which was invented by none other than *René Descartes*, who was because he thought. Sometimes, however, functions (and consequently integrals) become simpler when expressed in different coordinate systems. There are many different coordinate systems. Here, we will focus on one that was invented by *Sir Isaac Newton* – the **polar coordinate system**.

Definition 15.23. ***(Polar point)***. *Points in polar coordinate system are defined by two parameters (r, θ), where r is the distance the point is from the origin and θ is the angle between the polar axis (positive x-axis) and the line that connects the point to the origin.*

Since a picture is worth a thousand words, here is a picture describing what was just described:

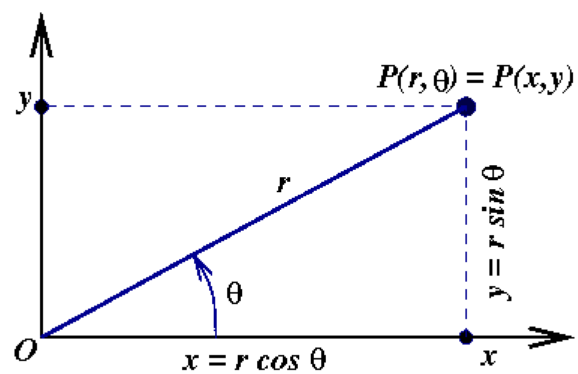


Figure 15.6: Point in Rectangular/Cartesian and Polar coordinates.

Naturally, there is a conversion from the Polar Coordinates to the Rectangular Coordinate system and vice versa. That conversion looks like:

$(x,y)_R \leftarrow (r,\theta)_P$	$(r,\theta)_P \leftarrow (x,y)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$

(15.13)

Frequently used trigonometric formulas

$$\begin{aligned} &\sin^2 x + \cos^2 x = 1 && 1 + \tan^2 x = \sec^2 x \\ &\sin 2x = 2\sin x\cos x && \cos 2x = \cos^2 x - \sin^2 x \\ &\sin^2 x = \frac{1-\cos 2x}{2} && \cos^2 x = \frac{1+\cos 2x}{2} \end{aligned} \tag{15.14}$$

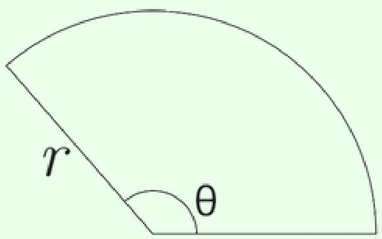


Figure 15.7

Sectors: arc length and area

$$\begin{aligned} &\text{Arc length:} && \ell = r\theta \\ &\text{Area:} && A = \frac{1}{2}r\ell = \frac{1}{2}r^2\theta \end{aligned} \tag{15.15}$$

More study on trigonometry and sectors

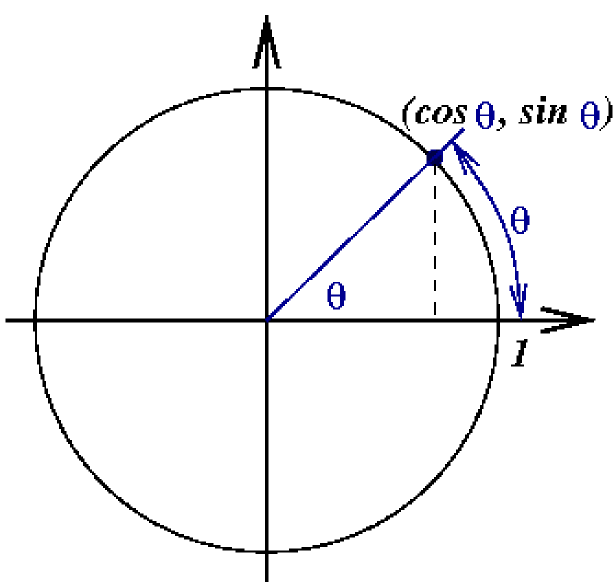


Figure 15.8

Double integrals with polar coordinates

Polar rectangle:

$$R = \{(r,\theta) \mid a \le r \le b,\ \ \alpha \le \theta \le \beta\}.$$

Let $\Delta r = (b-a)/m$ and $\Delta\theta = (\beta-\alpha)/n$, for some m, n, and

$$\begin{aligned} r_i &= a + i\Delta r, \quad i = 0, 1, \cdots, m, \\ \theta_j &= \alpha + j\Delta\theta, \quad j = 0, 1, \cdots, n. \end{aligned}$$

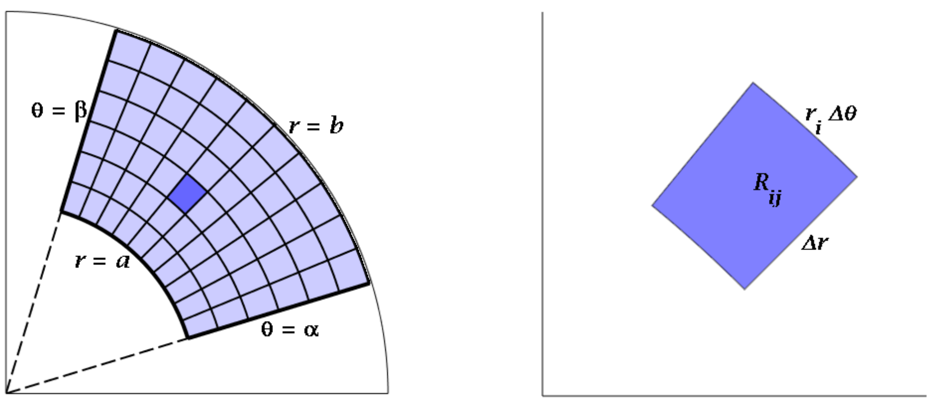


Figure 15.9: Dividing the polar rectangle $R = ([a,b]\times[\alpha,\beta])_P$: (left) polar subrectangles and (right) zoom-in of $R_{ij} = ([r_{i-1}, r_i]\times[\theta_{j-1},\theta_j])_P$.

The area of R_{ij} is

$$\Delta A_{ij} = \frac{1}{2}r_i^2\Delta\theta - \frac{1}{2}r_{i-1}^2\Delta\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^*\Delta r\Delta\theta, \qquad (15.16)$$

where $r_i^* = (r_i + r_{i-1})/2$.

Theorem 15.24. ***(Polar version of iterated integral)***. *If f is continuous on the polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta. \qquad (15.17)$$

Note: ① Do not forget the "$\boldsymbol{r}$" before the $\boldsymbol{dr\,d\theta}$!
② It follows from Figure 15.9 that $\boldsymbol{\Delta A_{ij} \approx \Delta r \cdot r_j\Delta\theta = r_j\,\Delta r\,\Delta\theta}$.

Problem 15.25. Evaluate $\displaystyle\iint_R (3x + 4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution.

Ans: $15\pi/2$

Problem 15.26. Find the volume of the solid bounded by the plane $z = 0$ and the paraboloid $z = 1 - x^2 - y^2$.

Solution. Volume $V = \iint_D (1 - x^2 - y^2)\,dA$, where $D = \{(x, y) \mid x^2 + y^2 \leq 1\}$.

Ans: $\pi/2$

Theorem 15.27. ***(Polar version of (15.12), p. 90).*** *If f is continuous on a polar region of the form $D = \{(r,\theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\}$, then*

$$\iint\limits_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta. \tag{15.18}$$

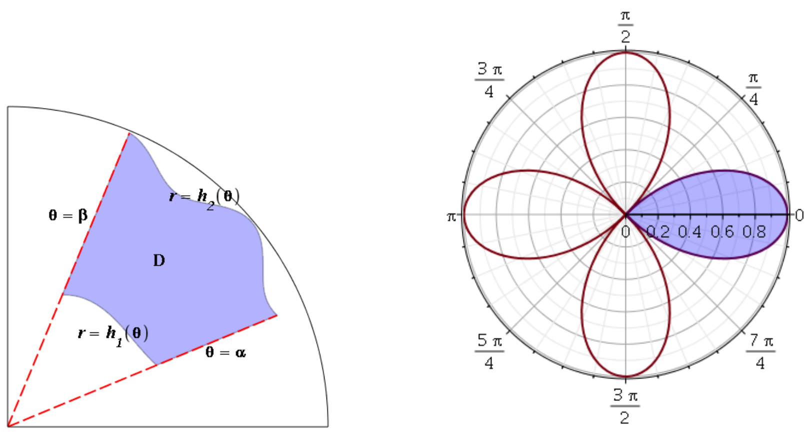


Figure 15.10

Problem 15.28. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

Solution. $A(D) = \iint_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta$.

Ans: $\pi/8$

Problem 15.29. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

Hint.

- First, find what the "polar region" looks like.
- That is to say, translate $x^2 + y^2 = 2x$ into polar coordinates and see what that region looks like. (Also, you may refer to $(x-1)^2 + y^2 = 1$.)
- Then, look at $z = x^2 + y^2$ as a polar function and use it as your integrand.
- Evaluate.
- Don't forget the r in "$r\,dr\,d\theta$"!

Solution.

Ans: $\frac{3}{2}\pi$

Problem 15.30. (A variant of Problem 15.29). Evaluate the double integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\, dy\, dx$, by recognizing the region and converting it to polar coordinates.

Solution. ***Hint***: $D = \{\theta = 0..\pi/2,\;\; r = 0..2\cos\theta\}$

Ans: $16/9$

Volume of n-Ball: The **unit interval** $[-1,1]$ can be rewritten as

$$B_1 \stackrel{\text{def}}{=\!=} \{x \mid x^2 \le 1\} \subset \mathbb{R}. \tag{15.19}$$

Similarly, the **unit circle** and the **unit sphere** (of radius 1) read

$$B_2 \stackrel{\text{def}}{=\!=} \{(x,y) \mid x^2+y^2 \le 1\} \subset \mathbb{R}^2 \ \text{ and } \ B_3 \stackrel{\text{def}}{=\!=} \{(x,y,z) \mid x^2+y^2+z^2 \le 1\} \subset \mathbb{R}^3. \tag{15.20}$$

In general, an n-**dimensional ball** (or n-**Ball**) **of radius** r is defined as

$$B_{n,r} = \{(x_1, x_2, \cdots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2\} \subset \mathbb{R}^n. \tag{15.21}$$

It is possible to define **volume of n-Ball of radius** r, $V_{n,r}$; in $\mathbb{R}$ it is **length**, in $\mathbb{R}^2$ it is **area**, in $\mathbb{R}^3$ it is **ordinary volume**, and in $\mathbb{R}^n$, $n \ge 4$, it is called a **hypervolume**. For example,

$$V_{1,r} = V(B_{1,r}) = 2r, \ \ V_{2,r} = V(B_{2,r}) = \pi r^2, \ \ V_{3,r} = V(B_{3,r}) = \frac{4}{3}\pi r^3. \tag{15.22}$$

Note that $V_{n,r} = V_{n,1} \cdot r^n$, $n \ge 1$.

Challenge 15.31. *Let $B_n = B_{n,1}$ and $V_n = V(B_{n,1})$. Use* ***polar coordinates*** *to find V_n, the volume of the unit n-Ball B_n, $n \ge 4$.*

Solution. See Figure 15.11. Then,

$$V_n = \int_0^{2\pi} \int_0^1 \left(V_{n-2}\big(\sqrt{1-r^2}\big)^{n-2} \right) r\,dr\,d\theta$$

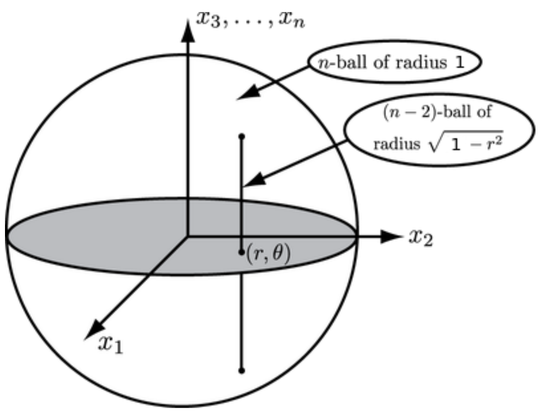


Figure 15.11: The unit n-Ball, $B_{n,1}$.

Ans: $V_n = \frac{2\pi}{n} V_{n-2}$. (You will solve this problem differently in Project 2, p. 143.)

Exercises 15.3

1. Use polar coordinates to evaluate the double integral, or the volume of the solid.

   (a) $\iint_D e^{x^2+y^2}\,dA$, where D is the region bounded by the semi-circle $x = \sqrt{1-y^2}$ and the y-axis.

   (b) The solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 8$.

   Ans: (a) $(e-1)\pi/2$; (b) $\frac{32(\sqrt{2}-1)\pi}{3}$

2. A swimming pool is circular with 60–ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the east end to 8 ft at the west end. Find the volume of water in the pool, using a double integral in polar coordinates. ***Hint***: $V = \iint_D (5 + \frac{x}{10})\,dA$, where D is the circle of radius 30 and centered at the origin.

3. Use polar coordinates to evaluate

$$\iint_{D_a} e^{-x^2-y^2}\,dA, \tag{15.23}$$

   where D_a is the disk of radius a centered at the origin.

   Ans: $\pi(1-e^{-a^2})$

4. We may define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I := \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \lim_{a\to\infty}\iint_{D_a} e^{-x^2-y^2}\,dA. \tag{15.24}$$

   (a) Use the result from the previous problem (Problem 3, Exercises 15.3) to conclude

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{15.25}$$

   (b) Let $\sigma > 0$. Use the change of variable $x = \sigma t$ to evaluate

$$\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\,dx. \tag{15.26}$$

15.4. Applications of Double Integrals

Objectives. Find the **mass** and **center of mass** of a planar lamina and **moments of inertia**, using double integrals. Then, apply them for **probability** and **mean** values.

Density and Mass

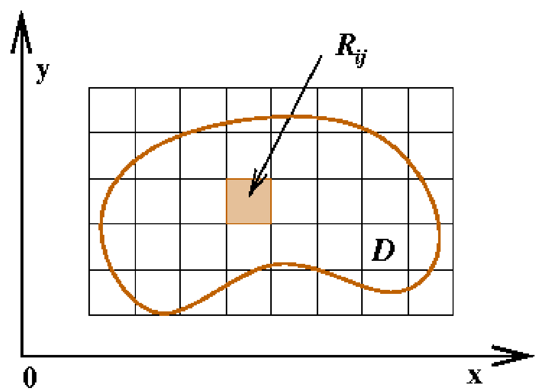


Figure 15.12

Let a lamina occupy a region D in xy-plane. Then its **density** is defined as

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}, \tag{15.27}$$

where Δm and ΔA the mass and the area of a small rectangle that contains (x,y). Thus, the mass of the lamina over D approximates

$$m = \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*) \Delta A.$$

By increasing the number of subrectangles, we obtain the **total mass** of the lamina

$$m = \lim_{m,n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*) \Delta A = \iint_D \rho(x,y)\, dA. \tag{15.28}$$

Problem 15.32. Find the mass of the triangular lamina with vertices $(0,0)$, $(2,2)$, and $(0,4)$, given that the density at (x,y) is $\rho(x,y) = 2x + y$.

Solution.

Ans: $40/3$

Definition 15.33. *The **moment** of a particle about an axis is the product of its mass and its **directed distance** from the axis. Say,* $M_x = m \cdot y$, $M_y = m \cdot x$.

Theorem 15.34. *The **moments (first moments)** of the entire lamina about x- and y-axes:*

$$M_x = \iint\limits_D y\,\rho(x,y)\,dA, \quad M_y = \iint\limits_D x\,\rho(x,y)\,dA. \tag{15.29}$$

*When we define the **center of mass** $(\overline{x}, \overline{y})$ so that $m\,\overline{x} = M_y$ and $m\,\overline{y} = M_x$, then*

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m}\iint_D x\,\rho(x,y)\,dA, \quad \overline{y} = \frac{M_x}{m} = \frac{1}{m}\iint_D y\,\rho(x,y)\,dA. \tag{15.30}$$

Problem 15.35. (Revisit of Problem 15.32). Find the center of mass for the triangular lamina with vertices $(0,0)$, $(2,2)$, and $(0,4)$, given that the density at (x,y) is $\rho(x,y) = 2x + y$.

Solution. We know $m = 40/3$.

Ans: $(\overline{x}, \overline{y}) = (4/5,\ 11/5)$

Probability

Recall: The **probability density function** f of a continuous random variable X is a function such that

$$f(x) \geq 0, \ \ \forall\, x \in \mathbb{R}, \ \ \text{and} \ \ \int_{-\infty}^{\infty} f(x)\, dx = 1.$$

The **probability** that X lies between a and b is

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

Definition 15.36. *The* ***joint density function*** *of a pair of random variables X and Y is a function f such that*

$$f(x,y) \geq 0, \ \ \forall\, (x,y) \in \mathbb{R}^2, \ \ \textit{and} \ \ \iint\limits_{R^2} f(x,y)\, dA = 1.$$

The ***probability*** *that (X,Y) lies in a region D is*

$$P((X,Y) \in D) = \iint\limits_{D} f(x,y)\, dA.$$

Problem 15.37. If the joint density function for X and Y is given by

$$f(x,y) = \begin{cases} k(3x - x^2)\,(2y - y^2), & \text{if } (x,y) \in [0,3] \times [0,2], \\ 0, & \text{otherwise}. \end{cases}$$

find the constant k. Then, find $P(X \leq 2, Y \geq 1)$.

Solution.

Ans: $k = 1/6$; $P(X \leq 2, Y \geq 1) = 10/27$

Expected Values of X and Y

Recall: If f is a probability density function of a random variable X, then its **mean** is

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Definition 15.38. *Let $f(x,y)$ be a joint density function of random variables X and Y. We define the X**-mean** and Y**-mean**, also called the **expected values**, of X and Y, to be*

$$\mu_1 = \iint_{\mathbb{R}^2} x\, f(x,y)\, dA, \quad \mu_2 = \iint_{\mathbb{R}^2} y\, f(x,y)\, dA.$$

Problem 15.39. Let $f(x,y) = \begin{cases} \frac{4-2x^2-2y^2}{3\pi}, & \text{if } x^2+y^2 \le 1, \\ 0, & \text{otherwise.} \end{cases}$

(a) Verify f is a joint density function.

(b) Find $P(X \le 0, Y \ge 0)$.

(c) Find the expected values of X and Y.

Solution.

Ans: (b) $1/4$; (c) $\mu_1 = \mu_2 = 0$

Exercises 15.4

1. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

   (a) D is the triangle with vertices $(0, 0)$, $(4, 0)$, and $(2, 2)$; $\rho(x, y) = y$

   (b) D is the part of the disk $x^2 + y^2 \leq 4$ in the first quadrant; ρ is proportional to its distance from the origin ***Hint***: Set $\rho(x, y) = k\sqrt{x^2 + y^2}$ and use polar coordinates for the integrals.

   Ans: (a) $m = 8/3$, $(\overline{x}, \overline{y}) = (2, 1)$; (b) $m = 4k\pi/3$, $(\overline{x}, \overline{y}) = (3/\pi, 3/\pi)$

2. CAS Use a computer algebra system (Maple, Mathematica, etc.) to find the mass and center of mass of the lamina that occupies the region D and has the given density function.

   (a) $D = \{(x, y) \mid 0 \leq x \leq ye^{-y},\ \ 0 \leq y \leq 1\}$; $\quad \rho(x, y) = (1 + x^2)\cos y$

   Ans: $m \approx 0.2167$, $\ (\overline{x}, \overline{y}) \approx (0.1507, 0.5697)$

   (b) D is the region closed by the right loop of the four-leaved rose $r = \cos 2\theta$ (as shown in Figure 15.10 on page 101); $\rho(x, y) = \sqrt{x^2 + y^2}$

3. Suppose X and Y are random variable with joint density function

$$f(x, y) = \begin{cases} k(x + 1)y, & \text{if } 0 \leq x \leq 2,\ \ 0 \leq y \leq 1, \\ 0, & \text{otherwise}. \end{cases}$$

   (a) Find the value of the constant k.

   (b) Find $P(x \leq 1, y \leq 1)$

   (c) Find $P(x - y \geq 1)$

   (d) Find X-mean and Y-mean.

   Ans: (a) $k = 1/2$; (c) $11/48$

15.5. Surface Area

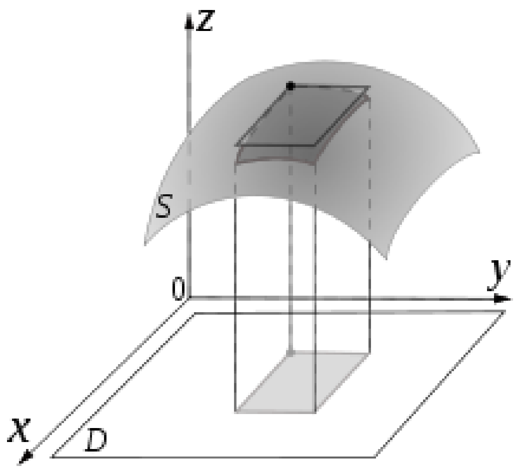


Figure 15.13

We may define the **surface area** of S to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}, \qquad (15.31)$$

where

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|.$$

Here,

$$\begin{aligned} \mathbf{a} &= \langle \Delta x, 0, f_x(\mathbf{x}_{ij})\Delta x \rangle, \\ \mathbf{b} &= \langle 0, \Delta y, f_y(\mathbf{x}_{ij})\Delta y \rangle. \end{aligned}$$

Since

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x\Delta x \\ 0 & \Delta y & f_y\Delta y \end{bmatrix} = \langle -f_x, -f_y, 1\rangle \, \Delta x\Delta y, \qquad (15.32)$$

we have ($\Delta A = \Delta x\Delta y$)

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{f_x^2 + f_y^2 + 1}\,\Delta A. \qquad (15.33)$$

Definition 15.40. *The **surface area** of S with $z = f(x, y)$, $(x, y) \in D$, where ∇f is continuous, is*

$$A(S) = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA. \qquad (15.34)$$

Recall: For $y = f(x)$, $x \in [a, b]$, the **arc length** is obtained as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (15.35)$$

Note: The surface area will be considered again when we learn Parametric Surfaces and Their Areas; see §16.6.3, p. 207.

Problem 15.41. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.
Solution. (See Problem 16.88 on p. 210.)

Ans: $\frac{\pi}{6}(37\sqrt{37} - 1)$

Problem 15.42. Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.
Solution.

Ans: $\frac{2\pi}{3}(2\sqrt{2} - 1)$

Exercises 15.5

1. Find the area of the surface.

   (a) The part of the plane $2x + y + 5z = 10$ that lies in the first octant

   (b) The part of the sphere $x^2 + y^2 + z^2 = 2$ that lies above the plane $z = 1$

   Ans: (b) $2\sqrt{2}\pi\,(\sqrt{2} - 1)$

2. Find the area of the finite part of the paraboloid $z = x^2 + y^2$ cut of by the plane $z = 9$.

3. CAS Use your calculator (or, a computer algebra system) to estimate the area of the surface correct to four decimal places.

   The part of the surface $z = \sin(x^2 + y^2)$ that lines in the cylinder $x^2 + y^2 = 4$.

   Hint: If you use Maple for numeric integration for $\int_a^b f(x)\,dx$, the command looks:
   `int(f(x),x=a..b,numeric)`

   Ans: 27.7291

15.6. Triple Integrals

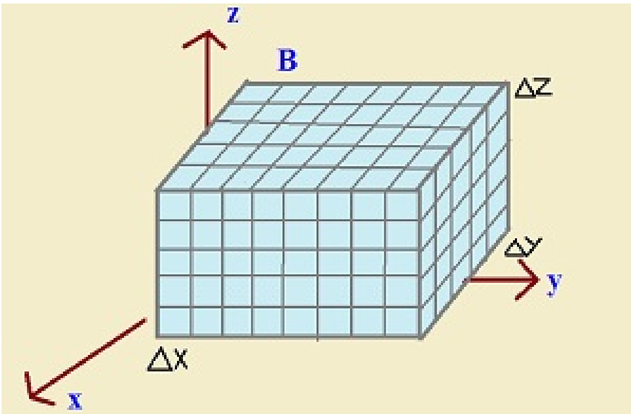


Figure 15.14

Let's begin with a function of three variables defined on a rectangular box:

$$w = f(x, y, z), \quad (x, y, z) \in B,$$

where

$$B = [a, b] \times [c, d] \times [r, s].$$

In defining a triple integral, the first step is to divide B into sub-boxes.

For some positive integers $\ell, m, n > 0$,

$$\Delta x = \frac{b-a}{\ell}, \quad \Delta y = \frac{d-c}{m}, \quad \Delta z = \frac{s-r}{n}.$$

Let B_{ijk} be the (ijk)-th sub-box:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{\ell-1}, z_\ell];$$

each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.

Definition 15.43. *The **triple integral** of f over the box B is*

$$\iiint_B f(x, y, z)\, dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\mathbf{x}^*_{ijk}) \Delta V.$$

where $\mathbf{x}^*_{ijk} = (x^*_i, y^*_j, z^*_k) \in B_{ijk}$.

Theorem 15.44. ***(Fubini's Theorem for Triple Integrals).*** *If f is continuous on $B = [a, b] \times [c, d] \times [r, s]$, then*

$$\iiint_B f(x, y, z)\, dV = \int_a^b \int_c^d \int_r^s f(x, y, z)\, dz\, dy\, dx; \tag{15.36}$$

the integration order can be changed for five other choices.

Problem 15.45. Evaluate the triple integral $\iiint_B xyz^2\,dV$, where $B = [0,1]\times[-1,2]\times[0,3]$.

Solution.

Ans: $27/4$

Triple integral over a general bounded region E:

Strategy 15.46. To evaluate a given triple integral over E:

1. Recognize (visualize in your brain) the domain E.
2. Separate the domain, e.g., $E = D\times[u_1(x,y), u_2(x,y)]$, $D\subset\mathbb{R}^2$. Then, $\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$.
3. The principle is: you must find a scheme to **cover the whole domain** E (without missing, without duplicating).

Let's go on a journey!!

Problem 15.47. Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution. $E = D \times [0, 1 - x - y]$, where D is the unit right triangle in the xy-plane.

Ans: $1/24$

Problem 15.48. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solution. $E = D \times [x^2 + y^2, 4]$, where D is the disk of center the origin and radius 2.

Ans: $128\pi/15$

Applications of Triple Integrals

Claim 15.49. Let $f(x, y, z) = 1$ for all points in E. Then triple integral of f over E represents the **volume** of E:

$$V(E) = \iiint_E 1\, dV. \tag{15.37}$$

Problem 15.50. Use the triple integral to find the volume of the tetrahedron T bounded by the four planes $x = 0$, $\boldsymbol{y = x}$, $z = 0$, and $x + y + z = 2$.

Solution.

Ans: $2/3$

Changing the order of integration

Problem 15.51. Write a couple of other iterated integrals that are equivalent to

$$\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dx\, dz\, dy$$

Hint: Change the order for adjacent two variables in the integral, keeping the other the same. For example, start with $x \leftrightarrow z$ or $z \leftrightarrow y$.

Solution.

Exercises 15.6

1. Evaluate the iterated integral.

   (a) $\displaystyle\int_0^2 \int_0^1 \int_0^{\ln x} xe^{-y}\,dy\,dx\,dz$ (b) $\displaystyle\int_0^{\pi} \int_0^2 \int_0^{\sqrt{4-z^2}} z\cos x\,dy\,dz\,dx$

   Ans: (a) -1; (b) 0

2. Evaluate the triple integral.

   (a) $\displaystyle\iiint_E e^{z/x}\,dV, \quad E = \{(x,y,z) \mid 0 \le x \le 1,\ x \le y \le 1,\ 0 \le z \le x\}$

   (b) $\displaystyle\iiint_E y\,dV,$ E is determined by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$

   Ans: (a) $(e-1)/6$; (b) $64\pi/3$

3. Fill the lower and upper bounds appropriately for the triple integral.

$$\int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_{\boxed{①}}^{\boxed{②}} \int_{\boxed{③}}^{\boxed{④}} \int_{\boxed{⑤}}^{\boxed{⑥}} f(x,y,z)\,dx\,dz\,dy$$

$$= \int_{\boxed{⑦}}^{\boxed{⑧}} \int_{\boxed{⑨}}^{\boxed{⑩}} \int_{\boxed{⑪}}^{\boxed{⑫}} f(x,y,z)\,dz\,dx\,dy$$

Ans: ⑤: y; ⑥: 1; ⑦: 0; ⑧: 1 ⑪: 0; ⑫: y

15.7. Triple Integrals in Cylindrical Coordinates

Recall: **(Equation (15.13))**. The conversion between the **Polar Coordinates** and the Rectangular Coordinate system reads

$(x,y)_R \leftarrow (r,\theta)_P$	$(r,\theta)_P \leftarrow (x,y)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$

(15.38)

Definition 15.52. *In the **cylindrical coordinate system**, a **point** P in the 3D space is represented as an ordered triple (r,θ,z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.*

Definition 15.53. *The conversion between the **Cylindrical Coordinates** and the Rectangular Coordinate system gives*

$(x,y,z)_R \leftarrow (r,\theta,z)_C$	$(r,\theta,z)_C \leftarrow (x,y,z)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

(15.39)

Note: The triple integral with a Cylindrical Domain E can be carried out by first separating the domain like

$$E = D \times [u_1(x,y), u_2(x,y)], \quad \text{where } D \text{ is a polar region.}$$

Problem 15.54. (a) Plot the point with the cylindrical coordinates $(2, \frac{2\pi}{3}, 1)_C$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, 7)_R$.
Solution.

Ans: (a) $(-1, \sqrt{3}, 1)_R$; (b) $(3\sqrt{2}, -\pi/4, 7)_C$.

Problem 15.55. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$
Hint. Change the triple integral into cylindrical coordinates.
Solution.

Ans: $16\pi/5$

Problem 15.56. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Solution.

Ans: $\frac{4\pi}{3}(8 - 3\sqrt{3})$

Exercises 15.7

1. Identify the surface whose equation is given.

   (a) $r^2 + 4z^2 = 4$ (b) $r = 2\cos\theta$

   Hint: (b) It can be rewritten as $r^2 = 2r\cos\theta$, which in turn reads $x^2 + y^2 = 2x$.

2. Evaluate $\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} x\,dz\,dy\,dx$, by changing the triple integral into cylindrical coordinates.

   Ans: $8/3$

3. Use cylindrical coordinates to find the **volume** of the solid E that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

   Ans: $\frac{4}{3}\pi(\sqrt{2} - 1)$

4. Use cylindrical coordinates to evaluate $\iiint_E y\,dV$, where E is the solid that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 9$, above the xy-plane, and below the plane $z = y+3$.

   Ans: 20π

15.8. Triple Integrals in Spherical Coordinates

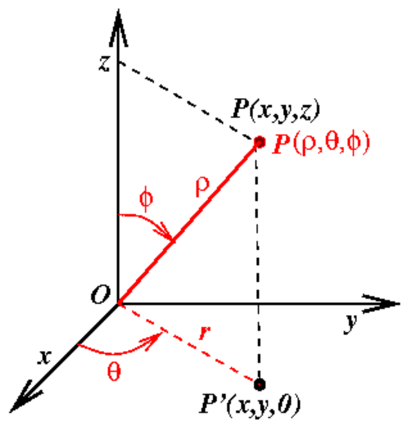


Figure 15.15: Spherical coordinates of P.

Definition 15.57. *The* ***spherical coordinates*** (ρ, θ, ϕ) *of a point* P *is shown in Figure 15.15, where* $\rho = |\overline{OP}| = \sqrt{x^2 + y^2 + z^2}$, θ *is the angle from the* x*-axis to the line segment* $\overline{OP'}$*, and* ϕ *is the angle between the positive* z*-axis and the line segment* $\overline{OP}$*.*

Note: By observing the definition, we can see the following inequalities:

$$\rho \geq 0 \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi.$$

For a convenient conversion formula, consider first

$$\boldsymbol{z = \rho \cos \phi, \quad r = \rho \sin \phi.}$$

Definition 15.58. *The conversion between the* ***Spherical Coordinates*** *and the Rectangular Coordinate system gives*

$(x, y, z)_R \leftarrow (\rho, \theta, \phi)_S$	$(\rho, \theta, \phi)_S \leftarrow (x, y, z)_R$
$\boldsymbol{x = r \cos \theta} = \rho \sin \phi \cos \theta$	$\rho^2 = x^2 + y^2 + z^2$
$\boldsymbol{y = r \sin \theta} = \rho \sin \phi \sin \theta$	$\cos \phi = \dfrac{z}{\rho}$
$z = \rho \cos \phi$	$\cos \theta = \dfrac{x}{\rho \sin \phi}$

(15.40)

Problem 15.59. (a) Plot the point with the spherical coordinates $(2, \pi/4, \pi/3)_S$ and find its rectangular coordinates.
(b) Find spherical coordinates of the point with rectangular coordinates $(0, 2\sqrt{3}, -2)_R$.
Solution.

Ans: (a) $(\sqrt{3/2}, \sqrt{3/2}, 1)_R$; (b) $(4, \pi/2, 2\pi/3)_S$

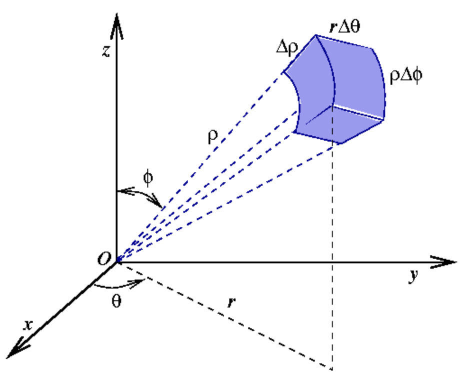


Figure 15.16: A small spherical wedge E_{ijk}, of volume $\Delta V_{ijk} \approx r\,\rho\,\Delta\rho\,\Delta\theta\,\Delta\phi$.

Triple Integral with Spherical Coordinates: In the spherical coordinate system, the counter part of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\},$$

where $a \ge 0$, $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. We divide smaller spherical wedges $\{E_{ijk}\}$ by means of equally spaced ρ_i, θ_j, ϕ_k. Figure 15.16 shows that E_{ijk} is approximately a rectangular box, of which the volume approximates

$$\Delta V_{ijk} \approx r\rho\,\Delta\rho\,\Delta\theta\,\Delta\phi = \rho^2 \sin\phi\,\Delta\rho\,\Delta\theta\,\Delta\phi. \tag{15.41}$$

Theorem 15.60. ***(Triple Integral on Spherical Wedges).***

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \times \underline{\rho^2\,\sin\phi}\,d\rho\,d\theta\,d\phi, \tag{15.42}$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}.$$

Note: *The scaling factor* $\boxed{\rho^2\,\sin\phi = r\rho}$

Problem 15.61. Evaluate $\iiint_B e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where B is the unit ball $B = \left\{(x,y,z) \mid x^2+y^2+z^2 \le 1\right\}$.

Solution.

Ans: $\frac{4}{3}\pi(e-1)$

Theorem 15.62. ***(Spherical Fubini's Theorem)***. *We can extend Theorem 15.60 to regions defined by*

$$E = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi), \alpha \le \theta \le \beta, c \le \phi \le d\},$$

in such a way:

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho\cos\phi)\, \underline{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi. \quad (15.43)$$

Problem 15.63. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Solution. Sphere: $\rho^2 = \rho\cos\phi \Rightarrow \rho = \cos\phi$.

Cone: $\rho\cos\phi = r = \rho\sin\phi \Rightarrow \cos\phi = \sin\phi$. So, $\phi = \pi/4$. Therefore,

$$V = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\phi} \underline{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

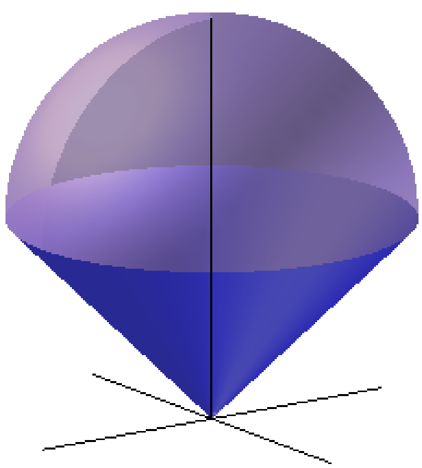


Figure 15.17

Ans: $\pi/8$

Exercises 15.8

1. Write the equation in spherical coordinates.

   (a) $x^2 + y^2 + z^2 = 1$

   (b) $z = x^2 + y^2$

   (c) $2x^2 + 2y^2 + z^2 = 4$

   (d) $z = x^2 - y^2$

   Hint: (c) $2x^2 + 2y^2 + z^2 = (x^2 + y^2 + z^2) + (x^2 + y^2)$

2. Sketch the solid whose volume is given by the integral; evaluate the integral.

   (a) $\displaystyle\int_0^{\pi/4} \int_0^{\pi} \int_0^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

   (b) $\displaystyle\int_0^{\pi/2} \int_0^{2\pi} \int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

   Ans: (b) $\pi/6$

3. Use spherical coordinates to to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

4. Use spherical coordinates to evaluate $\displaystyle\iiint_B x\, e^{\left(x^2+y^2+z^2\right)^2} \, dV$, where B is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

   Ans: $(e-1)\pi/16$

15.9. Change of Variables in Multiple Integrals

We have done **changes of variables** several times in the past. Dating as far back when we learned integration with the "u-substitution", we started using changes of variables (we made $u = g(x)$.) Indeed,

$$\int_c^d f(g(x))g'(x)\,dx = \int_{g(c)}^{g(d)} f(u)\,du. \tag{15.44}$$

Another way of expressing (15.44) is

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\frac{dx}{du}\,du, \tag{15.45}$$

where $x = x(u) : c \mapsto a,\ \ d \mapsto b$.

Example 15.64. Evaluate $\int_0^2 xe^{x^2}\,dx$.

Solution. $u = x^2 \quad\Rightarrow\quad du = 2\,x\,dx \quad\Rightarrow\quad x\,dx = \dfrac{du}{2}$; $u(0) = 0,\ u(2) = 4$. Therefore

$$\int_0^2 xe^{x^2}\,dx = \int_0^4 e^u\,\frac{du}{2} = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}(e^4 - 1).$$

Another way: $\boldsymbol{x = x(u) = \sqrt{u}} \quad\Rightarrow\quad \dfrac{dx}{du} = \dfrac{1}{2\sqrt{u}}$. Therefore

$$\int_0^2 xe^{x^2}\,dx = \int_0^4 \sqrt{u}\,e^u\,\frac{1}{2\sqrt{u}}\,du = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}(e^4 - 1).$$

Example 15.65. A change of variable is also useful in multiple integrals, as in double integrals in polar coordinates. For a polar region R, we have used the conversion:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

which is a **transformation** from the $r\theta$-plane to the xy-plane. Then,

$$\iint_R f(x,y)\,dA = \iint_Q f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta, \tag{15.46}$$

where Q is the region in the $r\theta$-plane.

Goal: The goal of this section is to write a general form for a change of variables, which turns the integral easier.

Definition 15.66. *A change of variables is a* ***transformation*** $T : Q \to R$ *(from the* uv*-plane to the* xy*-plane),* $T(u, v) = (x, y)$*, where* x *and* y *are related to* u *and* v *by the equations*

$$x = g(u, v), \quad y = h(u, v). \qquad [\textbf{or}, \ \mathbf{r}(u, v) = \langle g(u, v), h(u, v)\rangle]$$

We usually take these transformations to be C^1*-***Transformation***, meaning* g *and* h *have continuous first-order partial derivatives, and* ***one-to-one****.*

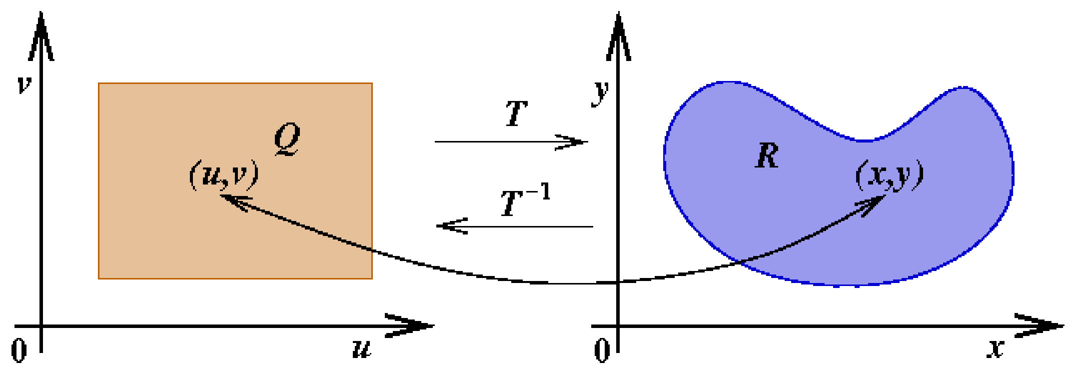


Figure 15.18: Transformation: $R = T(Q)$, the **image** of T.

Problem 15.67. A transformation is defined by $\mathbf{r}(u, v) = \langle 2u - v, u + v\rangle$. Find the image of the unit square $Q = \{(u, v) \mid 0 \le u \le 1, \ 0 \le v \le 1\}$.

Solution.

Ans: A rectangle of vertices $(0, 0)$, $(2, 1)$, $(1, 2)$, $(-1, 1)$.

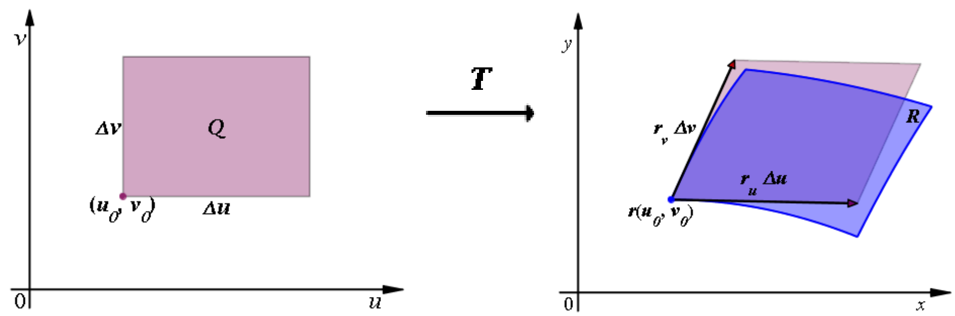


Figure 15.19: A small rectangle in the uv-plane and its image of T in the xy-plane.

Now, let's see how a change of variables affects a double integral.

- See Figure 15.19, where $T : Q \to R$ given by

$$\mathbf{r}(u,v) = \langle x, y\rangle = \langle g(u,v), h(u,v)\rangle\,. \tag{15.47}$$

- The **tangent vectors** at $\mathbf{r}(u_0, v_0)$ w.r.t the u- and v-directions are

$$\mathbf{r}_u(u_0,v_0) = \langle g_u, h_u\rangle\,(u_0,v_0), \quad \mathbf{r}_v(u_0,v_0) = \langle g_v, h_v\rangle\,(u_0,v_0).$$

- We can approximate the image region $R = T(Q)$ by a parallelogram determined by the *scaled* tangent vectors. Therefore,

$$\Delta A = A(R) \approx |(\mathbf{r}_u\Delta u) \times (\mathbf{r}_v\Delta v)| = |\mathbf{r}_u \times \mathbf{r}_v|\,\Delta u\,\Delta v. \tag{15.48}$$

- Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \det\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{bmatrix} = \det\begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix}\mathbf{k} \tag{15.49}$$

Definition 15.68. *The **Jacobian of** $T : x = g(u,v),\ y = h(u,v)$ is*

$$\frac{\partial(x,y)}{\partial(u,v)} \overset{\text{def}}{=\!=} \det\begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u\,y_v - x_v\,y_u. \tag{15.50}$$

Summary 15.69. *For a differentiable transformation $T : Q \to R$ given by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v)\rangle$,*

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right|\Delta u\,\Delta v. \tag{15.51}$$

Theorem 15.70. *Suppose that T is a C^1-**transformation** whose **Jacobian** is nonzero, and suppose that T maps a region Q in the uv-plane onto a region R in the xy-plane. Let f be a continuous function on R. Suppose also that T is an **one-to-one** transformation except perhaps along the boundary of the regions. Then*

$$\iint_R f(x,y)\,dA = \iint_Q f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv. \tag{15.52}$$

Example 15.71. (Transformation to polar coordinates). The transformation from $Q = [a,b]\times[\alpha,\beta]$ in the $r\theta$-plane to R in the xy-plane is given by

$$T:\ \ x = g(r,\theta) = r\cos\theta,\ \ y = h(r,\theta) = r\sin\theta.$$

The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix}\cos\theta & -r\sin\theta\\ \sin\theta & r\cos\theta\end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus Theorem 15.24 (p. 99) gives

$$\begin{aligned}\iint_R f(x,y)\,dA &= \iint_Q f(r\cos(\theta), r\sin(\theta))\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta\\ &= \int_\alpha^\beta\int_a^b f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta.\end{aligned} \tag{15.53}$$

Problem 15.72. Evaluate $\iint\limits_R (x+y)\,dA$, where R is the quadrilateral region with vertices given by $(0,0)$, $(3,-3)$, $(6,0)$, and $(3,3)$, using the transformation $x = u + 3v$ and $y = u - 3v$.

Solution.

Ans: 54

Problem 15.73. Evaluate the integral $\iint\limits_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1,0), (2,0), (0,-2)$, and $(0,-1)$.

Solution.

Ans: $\frac{3}{4}(e - e^{-1})$

Problem 15.74. Evaluate $\iint_R \sin(x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by $x^2 + 4y^2 = 4$.

Solution. Consider the transformation: $x = 2u,\ y = v$.

Ans: $\frac{\pi}{8}(1 - \cos 4)$

Triple Integrals

Definition 15.75. ***(Higher order Jacobian)***. *The* ***Jacobian*** *of T, given by*

$$x = g(u,v,w), \;\; y = h(u,v,w), \;\; z = k(u,v,w),$$

is the following determinant:

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}. \qquad (15.54)$$

Theorem 15.76. *Under hypotheses similar to those in Theorem 15.70, we have the following formula for triple integrals:*

$$\iiint_R f(x,y,z)\,dV = \iiint_Q f\left(x\left(u,v,w\right), y\left(u,v,w\right), z\left(u,v,w\right)\right) \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw. \qquad (15.55)$$

Self-study 15.77. Show that when dealing with spherical coordinates,

$$dV = \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi. \qquad (15.56)$$

Recall. $x = \rho\sin\phi\cos\theta, y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$.

Exercises 15.9

1. Use the given transformation to evaluate the integral.

   (a) $\displaystyle\iint_R y^2\,dA$, where R is the region bounded by $4x^2+9y^2=36$; $(x,y)=(3u,2v)$

   (b) $\displaystyle\iint_R (3x-y)\,dA$, where R is the triangular region with the three vertices $(0,0)$, $(2,1)$, and $(1,3)$; $(x,y)=(2u+v,u+3v)$

   Hint: (a) $\iint_R y^2\,dA=\iint_Q 4v^2\cdot 6\,du\,dv$, where $Q=\{(u,v)\mid u^2+v^2\le 1\}$
   Hint: (b) $\iint_R (3x-y)\,dA=\iint_Q 5u\cdot 5\,du\,dv$; figure out Q by yourself

   Ans: (a) 6π; (b) $25/6$

2. Make an appropriate change of variables to evaluate the integral

$$\iint_R \sin(x^2+4y^2)\,dA,$$

   where R is the region in the first quadrant bounded by the ellipse $x^2+4y^2=1$.

3. Make an appropriate change of variables to evaluate $\displaystyle\iint_R 2(x-y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines: $x-y=0,\ x-y=1,\ x+y=0,\ x+y=2$.

4. Make an appropriate change of variables to evaluate the integral $\displaystyle\iint_R e^{x+y}\,dA$, where R is given by the inequality $|x|+|y|\le 1$.

   Ans: $e-e^{-1}$

R.15. Review Problems for Ch. 15

1. Estimate the volume of the solid that lies below the surface $z = x + y$ and above the rectangle $R = \{(x, y) \mid 0 \le x \le 6,\ \ 0 \le y \le 4\}$. Use a Riemann sum with $m = 3$, $n = 2$, and the Midpoint Rule.

Ans: 120

2. Evaluate the double integral following the direction.

$$\iint_R (4 - x)\, dA, \qquad R = [0, 4] \times [0, 5]$$

   (a) Identify it as the volume of a solid. (You should visualize the solid.)
   (b) Evaluate the double integral, by measuring the volume.

Ans: 40

3. Evaluate the double integral:

$$\iint_R \frac{y}{1 + xy} dA, \qquad R = [0, 1] \times [0, 2].$$

Hint: Use $\int \ln x\, dx = x \ln x - x + C$, if necessary.

Ans: $-2 + 3\ln(3)$

4. Evaluate the double integral
$$\int_0^1 \int_{3y}^3 e^{x^2}\, dx\, dy$$

Ans: $-1/6 + e^9/6$

5. Evaluate the iterated integral by converting to polar coordinates.

$$\int_{-3}^{3} \int_0^{\sqrt{9-y^2}} 2\sin(x^2 + y^2)\, dx\, dy$$

Ans: $\pi(1 - \cos 9)$

6. A swimming pool is circular with <u>40–ft diameter</u>. The depth is constant along east-west lines and increases linearly from <u>2 ft</u> at the south end to <u>6 ft</u> at the north end. Find the **volume of water** in the pool, using a double integral in **polar coordinates**.

Ans: 1600π

7. Find the **surface area** of the part of the plane $z = 2x + 28y + 2030$ that lies inside the cylinder $x^2 + y^2 = 1$.

Ans: $\sqrt{789}\,\pi$

8. Fill the lower and upper bounds appropriately for the triple integral.

$$\int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_{①}^{②} \int_{③}^{④} \int_{⑤}^{⑥} f(x,y,z)\,dz\,dx\,dy$$
$$= \int_{⑦}^{⑧} \int_{⑨}^{⑩} \int_{⑪}^{⑫} f(x,y,z)\,dx\,dz\,dy$$

Ans: From ① to ⑫: $[0, 1,\ y, 1,\ 0, y;\ 0, 1,\ 0, y,\ y, 1]$

9. Use **cylindrical coordinates** to evaluate $\iiint_E x^2\,dV$, where E is the solid that lies within the cylinder $x^2 + y^2 = 1$, above the plane $z = 0$, and below the cone $z^2 = 4x^2 + 4y^2$.

Ans: $2\pi/5$

10. Use **spherical coordinates** to evaluate $\iiint_E e^{(x^2+y^2+z^2)^{3/2}}\,dV$, where E is the portion of the unit ball $x^2 + y^2 + z^2 \le 1$ that lies in the first octant.

Ans: $(e-1)\pi/6$

11. Use the transformation $x = 2u$, $y = v$ to evaluate $\iint_R (x^2 + 4y^2)\,dA$, where R is the region bounded by the ellipse $\dfrac{x^2}{4} + y^2 = 1$.

Ans: 4π

12. Evaluate $\iint_R (x+y)e^{x^2-y^2}\,dA$, where R is rectangle enclosed by the lines $x - y = 0$, $x - y = 1$, $x + y = 0$, and $x + y = 1$. ***Hint***: If you set $u = x + y$, $v = x - y$, then the transformation becomes $x = (u+v)/2$, $y = (u-v)/2$.

Ans: $e/2 - 1$

Project 2. The Volume of the Unit Ball in n-Dimensio

In this project, we will find formulas for the **volume of the unit ball** in n-dimensions (nD). From your high school, you learned volumes of unit balls for $n = 1, 2, 3$.

n	B_n	V_n
1	$\{x \mid x^2 \le 1\} = [-1, 1]$	2
2	$\{(x, y) \mid x^2 + y^2 \le 1\}$	π
3	$\{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$	$4\pi/3$

(15.57)

Define the 4D **unit ball (hypersphere)** as

$$B_4 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 \le 1\}. \tag{15.58}$$

Before finding its volume, V_4, let's try to verify $V_3 = \dfrac{4\pi}{3}$ by using a **specific integration technique**.

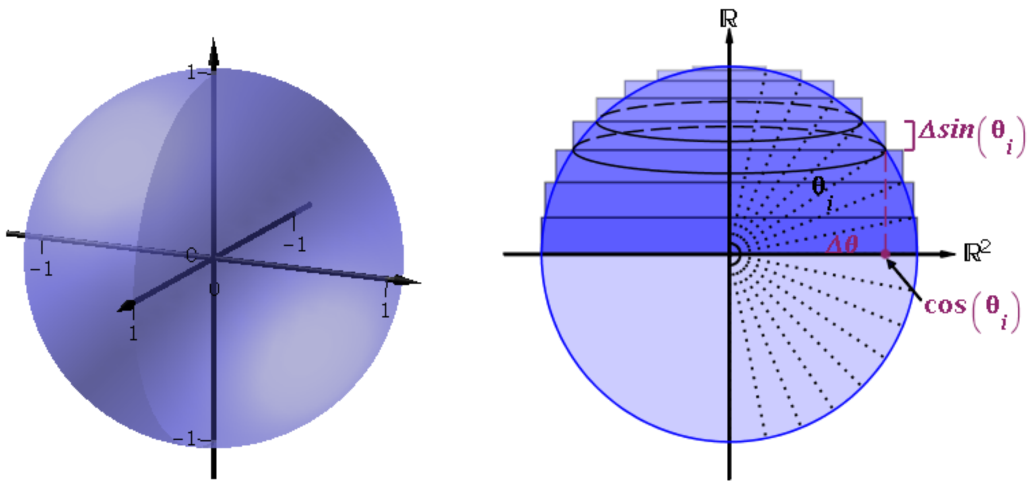


Figure 15.20: B_3 and its projection to $\mathbb{R}^2 \times \mathbb{R}$: the volume V_3 approximates the sum of the volume of circular slices having radius $\cos\theta_i$ and thickness $\Delta \sin\theta_i := \sin\theta_{i+1} - \sin\theta_i = \dfrac{\sin\theta_{i+1} - \sin\theta_i}{\Delta\theta}\,\Delta\theta \approx \cos\theta_i\,\Delta\theta$.

The computation of V_3: We first partition B_3 into horizontal circular slices. Let, for $k > 0$,

$$\Delta\theta = \frac{\pi}{2} \cdot \frac{1}{k} \text{ and } \theta_i = i\Delta\theta, \quad i = 0, 1, \cdots, k. \tag{15.59}$$

One can see from Figure 15.20 that the volume V_3 approximates the sum of the volume of **circular slices**. The i-th circular slice S_i has radius $\cos\theta_i$; its area is

$$A(S_i) = V_2 \cdot \cos^2\theta_i = \pi \cos^2\theta_i. \tag{15.60}$$

Since S_i has thickness $\Delta \sin\theta_i = \sin\theta_{i+1} - \sin\theta_i$, we have

$$V_3 \approx 2 \sum_{i=0}^{k-1} (\pi \cos^2\theta_i)\, \Delta \sin\theta_i. \tag{15.61}$$

Therefore,

$$\begin{aligned} V_3 &= \lim_{k\to\infty} 2 \sum_{i=0}^{k-1} (\pi \cos^2\theta_i)\, \Delta \sin\theta_i \\ &= 2\pi \int_0^{\pi/2} \cos^2\theta\, d(sin\theta) = 2\pi \int_0^{\pi/2} \cos^3\theta\, d\theta = 2\pi \cdot \frac{2}{3}. \end{aligned} \tag{15.62}$$

Note: Equation (15.62) can be rewritten as

$$V_3 = 2\, V_2 \int_0^{\pi/2} \cos^3\theta\, d\theta. \tag{15.63}$$

The computation of V_4: We are now ready for it! First image B_4 and its projection to $\mathbb{R}^3 \times \mathbb{R}$. With the same partitioning of the last dimension, the i-th horizontal piece S_i now becomes a **spherical slice**, rather than a circular slice, but having the same radius $\cos\theta_i$ and thickness $\Delta \sin\theta_i$. Thus, the **volume** of the i-th spherical slice reads

$$V(S_i) = V_3 \cos^3\theta_i \cdot \Delta \sin\theta_i \approx V_3 \cos^4\theta_i\, \Delta\theta. \tag{15.64}$$

Recall that $\Delta \sin\theta_i = \sin\theta_{i+1} - \sin\theta_i \approx \cos\theta_i\, \Delta\theta$. By summing up for $i = 0, 1, \cdots, k-1$, and multiplying the result by 2 (due to symmetry), we have

$$V_4 \approx 2\, V_3 \sum_{i=0}^{k-1} \cos^4\theta_i\, \Delta\theta. \tag{15.65}$$

Problem 15.78.

1. Complete a formula for V_4, by applying $k \to \infty$ to (15.65).

   Hint: Your result must be similar to (15.63).

2. Apply the above arguments recursively to find formulas for V_n, $n \geq 2$.
3. Use a computer algebra system (e.g., Maple) to evaluate exact values of V_n, for $n = 1, 2, \cdots, 20$.
4. Plot $\{(n, V_n) \mid n = 1, 2, \cdots, 20\}$.

   Hint: You may use Maple-code 15.79 and your plot must look like Figure 15.21.

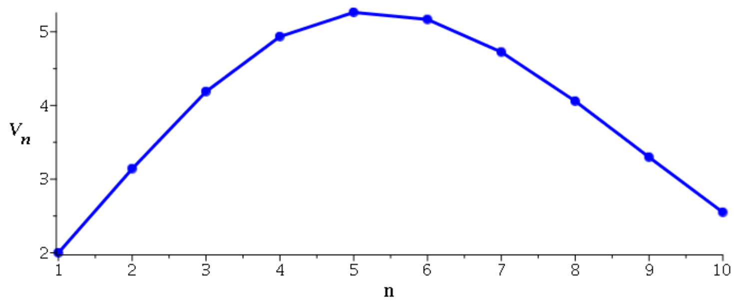


Figure 15.21: A plot for V_n, where $\max(V) = V_5 = \dfrac{8\pi^2}{15} \approx 5.263789$.

Maple-code 15.79. Assume you have a formula for V_n of the form

$$V_n = V_{n-1}\, g(n). \tag{15.66}$$

Then you may implement a Maple code:

Maple Script for the Computation of Vn and Plotting

```
with(plots): with(plottools):
with(VectorCalculus): with(Student[MultivariateCalculus]):

m := 20:
V := Vector(m):
V[1] := 2:
for n from 2 to m do V[n] := V[n-1]*g(n); end do:
max[index](V); max(V); evalf(%);

X := [seq(n, n = 1..m)]:
pp := pointplot(Vector(X), Vector(V), color = blue, symbol = solidcircle, symbolsize = 12):
pl := plot(Vector(X), Vector(V), color = blue, thickness = 3):
display(pp, pl, scaling = constrained, labels = ["n", V__n], labelfont = ["times", "bold", 13])
```

Figure 15.21 is constructed using the above code, with `m := 10:` and `g(n)` defined appropriately. □

Report. Submit hard copies of your experiences.

- Derive a formula for V_n of the form in (15.66).
- Implement a code to evaluate V_n, $n = 1, 2, \cdots, 20$, exactly.
- Plot the results.
- Collect all your work, in order.
- Attach a "summary" or "conclusion" page at the beginning of report.

You may work in a small group; however, you must report individually.

CHAPTER **16**

Vector Calculus

In this chapter, we study the calculus of vector fields. In particular, you will learn

Subjects	Applications
Line integral	Work done by a force vector field in moving an object along a curve
Surface integral	The rate of fluid flow across a surface
Fundamental theorem of calculus, in 2/3-D	Green's theorem, Stokes's theorem, and Divergence theorem

Contents of Chapter 16

This chapter corresponds to Chapter 16 in STEWART, ***Calculus*** (8th Ed.), 2015.

16.1. Vector Fields

16.1.1. Definitions

Definition 16.1. *If D is a region in $\mathbb{R}^2$, a* ***(2D) vector field*** *on D is a function $\boldsymbol{F}$ that assigns to each point $(x, y) \in D$ a two-dimensional vector $\boldsymbol{F}(x, y)$. If D is a solid region in $\mathbb{R}^3$, a* ***(3D) vector field*** *on D is a function $\boldsymbol{F}$ that assigns to each point $(x, y, z) \in D$ a three-dimensional vector $\boldsymbol{F}(x, y, z)$.*

Expressions for vector fields:

$$\begin{aligned} \boldsymbol{F}(x,y) &= \langle P(x,y),\, Q(x,y)\rangle \\ &= P(x,y)\,\mathbf{i} + Q(x,y)\,\mathbf{j}, \\ \boldsymbol{F}(x,y,z) &= \langle P(x,y,z),\, Q(x,y,z),\, R(x,y,z)\rangle \\ &= P(x,y,z)\,\mathbf{i} + Q(x,y,z)\,\mathbf{j} + R(x,y,z)\,\mathbf{k}. \end{aligned}$$

Example 16.2. $\boldsymbol{F}(x, y) = \langle x,\, x - y\rangle$ is a vector field in $\mathbb{R}^2$. $\boldsymbol{G}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$. Let's sketch $\boldsymbol{F}$.

(x, y)	$\boldsymbol{F}(x, y) = \langle x,\, x - y\rangle$
$(0, 0)$	
$(1, 0)$	
$(1, 1)$	
$(0, 1)$	

Problem 16.3. Let $\boldsymbol{F}(x, y) = \langle -y,\, x\rangle$. Describe $\boldsymbol{F}$ by sketching some of the vectors $\boldsymbol{F}(x, y)$.

Solution.

(x, y)	$\boldsymbol{F}(x, y) = \langle -y,\, x\rangle$
$(1, 0)$	
$(0, 1)$	
$(-1, 0)$	
$(0, -1)$	

Note:

- $\mathbf{x} \cdot \boldsymbol{F}(\mathbf{x}) = \langle x,\, y\rangle \cdot \langle -y,\, x\rangle = -xy + xy = 0.$
  Thus, $\boldsymbol{F}(\mathbf{x}) = \langle -y,\, x\rangle$ is perpendicular to the position vector $\mathbf{x}$.
- $|\boldsymbol{F}(\mathbf{x})| = \sqrt{y^2 + x^2} = |\mathbf{x}|$.
  Therefore, $\boldsymbol{F}(\mathbf{x})$ has the same magnitude as $\mathbf{x}$.

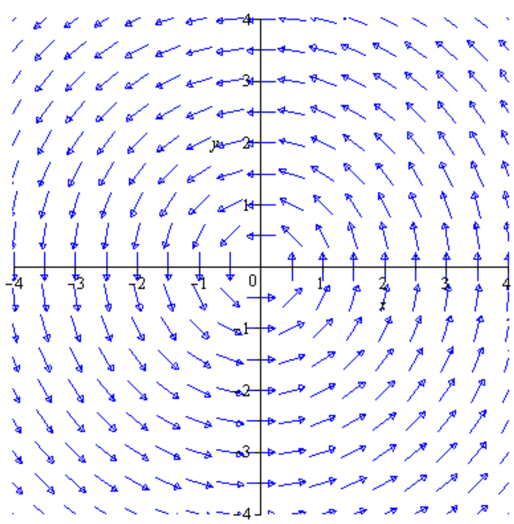


Figure 16.1: The vector field $\boldsymbol{F} = \langle -y, x\rangle$, showing directions only.

Vector fields in $\mathbb{R}^3$

Problem 16.4. Sketch the vector field on $\mathbb{R}^3$ given by $\boldsymbol{F}(x, y, z) = z\,\mathbf{k} = \langle 0,\, 0,\, z\rangle$.

Example 16.5.

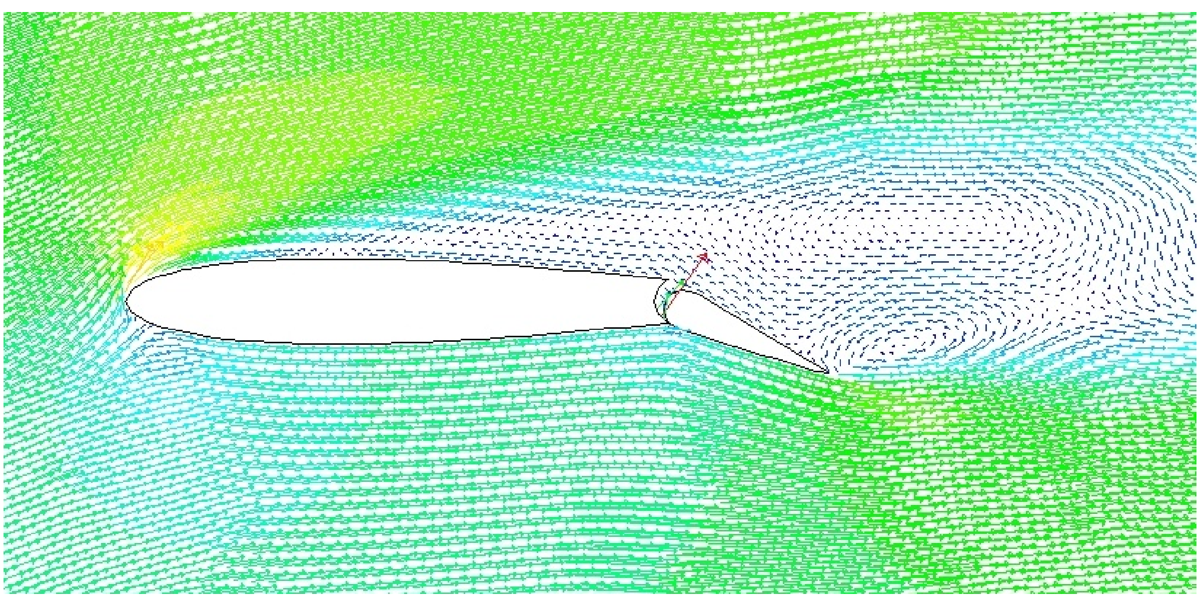


Figure 16.2: Airfoil simulation, showing the velocity field.

16.1.2. Gradient fields and potential functions

- Suppose that $f(x,y)$ is a differentiable function on D. Earlier we defined the **gradient** ∇f of f:

$$\nabla f(x,y) = \langle f_x(x,y),\ f_y(x,y)\rangle \ = \ f_x\,\mathbf{i} + f_y\,\mathbf{j}.$$

We now see that ∇f is a two-dimensional vector field on D.

- Similarly, if $f(x,y,z)$ is a differentiable function on a solid $D \subset \mathbb{R}^3$, then $\nabla f(x,y,z)$ is a three-dimensional vector field on D.

From now on, we will refer to the gradient of a function f as the **gradient vector field** of f.

Problem 16.6. Find the gradient vector field of

$$f(x,y) = x^2y - y^3.$$

Solution.

Ans: $\langle 2xy, x^2 - 3y^2\rangle$

Definition 16.7. *A vector field* $\boldsymbol{F}$ *is* ***conservative*** *if there is a differentiable function* f *such that*

$$\nabla f = \boldsymbol{F}.$$

The function f *is called a* ***potential function*** *of* $\boldsymbol{F}$*, or simply* ***potential****.*

Claim 16.8. Gradient fields are, always, conservative.

Problem 16.9. (Continuation of Problem 16.6). Let $\boldsymbol{F}(x,y) = \langle 2xy,\, x^2 - 3y^2 \rangle$. Then $\boldsymbol{F}$ is conservative.

Solution. Let's try to find f such that $\nabla f = \boldsymbol{F}$.

Ans: $f(x,y) = x^2y - y^3 + K$

Note: Not every vector field is conservative, and it is not difficult to give an example of a vector field that is **nonconservative**.

Example 16.10. Show that the vector field $\boldsymbol{F}(x,y) = (x^2 + y)\,\mathbf{i} + y^3\,\mathbf{j}$ is not conservative.

Proof. **Assume that $\boldsymbol{F}$ is conservative**. Then, there exists f such that $\nabla f = \langle f_x, f_y \rangle = \boldsymbol{F}$:

$$f_x = x^2 + y, \qquad f_y = y^3.$$

Then

$$f_{xy} = 1 \quad \text{and} \quad f_{yx} = 0. \tag{16.1}$$

Since both mixed partials are constants, they are continuous everywhere. Thus, by the Clairaut's theorem, we must have

$$f_{xy} = f_{yx}.$$

However, in (16.1), they are not equal. Contradiction! □

We will study properties of **conservative vector fields** in Section 16.3 below, in detail.

Problem 16.11. At time $t = 1$, a particle is located at $(1, 3)$. When it moves in a velocity field $\mathbf{v}(x, y) = \langle xy - 2,\ y^2 - 10\rangle$, find its approximate location at $t = 1.05$.

Solution. ***Clue***: $\mathbf{r}(t) \approx \mathbf{r}(t_0) + \mathbf{r}'(t_0) \cdot (t - t_0)$, where $\mathbf{r}'$ is the velocity vector.

Ans: $\langle 1.05, 2.95\rangle$

Exercises 16.1

1. Match the vector fields F with the plots labeled (I)–(IV). Give reasons for your choices.

   (a) $\boldsymbol{F} = \langle e^x, 5y\rangle$ (c) $\boldsymbol{F} = \langle x+y, y\rangle$
   (b) $\boldsymbol{F} = \langle \sin(x+y), x\rangle$ (d) $\boldsymbol{F} = \langle x, -y\rangle$

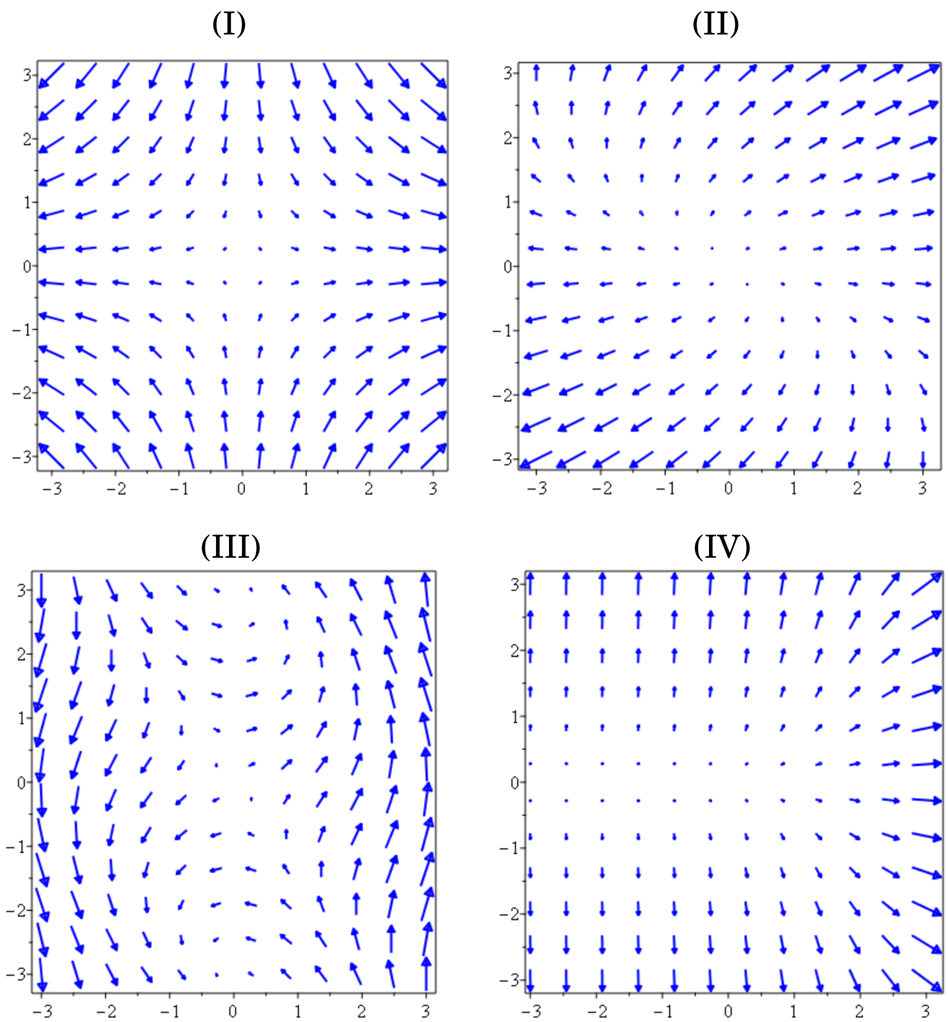


Figure 16.3: Maple `fieldplot`.

Hint: Let's see Figure (III), for example; arrows are directing up for $x > 0$ and down for $x < 0$, which implies that the second component of $\boldsymbol{F}$ is closely related to x. Now, what can you say about Figure (IV)? Arrows never look the west direction, which implies that the first component of $\boldsymbol{F}$ is nonnegative.

2. CAS Use a CAS (`fieldplot` in Maple and `PlotVectorField` in Mathematica) to plot

$$\boldsymbol{F}(x, y) = (y^3 - xy^2)\,\mathbf{i} + (2xy - 2x^2)\,\mathbf{j}.$$

   Explain the appearance by finding the set of points (x, y) such that $\boldsymbol{F}(x, y) = 0$. (Attach the figure.)

3. Find the gradient vector field ∇f and sketch it.

   (a) $f(x,y) = \dfrac{(x-y)^2}{2}$

   (b) $f(x,y) = \dfrac{x^3 - y^3}{3}$

4. Match the functions f with their gradient vector fields plotted with labels (I)–(IV). Give reasons for your choices.

   (a) $f(x,y) = xe^y$

   (b) $f(x,y) = x^2 + y^2$

   (c) $f(x,y) = x(x-2y)$

   (d) $f(x,y) = \cos(x^2+y^2)$

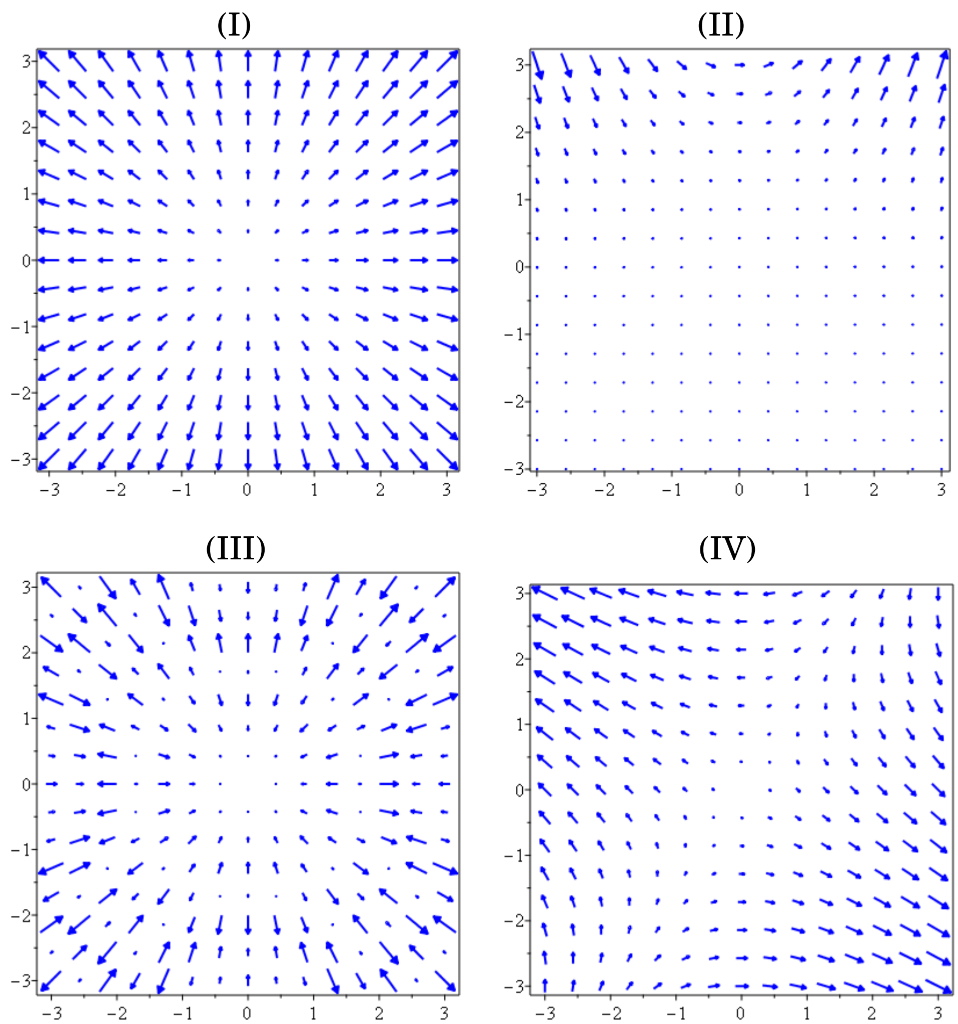


Figure 16.4: Maple `fieldplot` for ∇f.

16.2. Line Integrals

Recall: In single-variable calculus, if a force $f(x)$ is applied to an object to **move it along a straight line** from $x = a$ to $x = b$, then the amount of work done is given by the integral

$$W = \int_a^b f(x)\,dx \quad \Big(= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x\Big). \tag{16.2}$$

Up to this point, our intervals of integration were always either bijective function or a closed interval $[a, b]$. In this section, we will be integrating over a **parametrized curve** instead of a nice interval as before.

Goal: To integrate functions along a curve, as opposed to along an interval.

Definition 16.12. *A* ***plane curve*** C *is given by the* ***vector equation***

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle, \quad a \le t \le b, \tag{16.3}$$

or equivalently, by the ***parametric equations***

$$x = g(t), \;\; y = h(t), \quad a \le t \le b. \tag{16.4}$$

Recall: You have learned

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{\Big(\frac{\Delta x_i}{\Delta t}\Big)^2 + \Big(\frac{\Delta y_i}{\Delta t}\Big)^2}\,\Delta t$$

and therefore

$$\begin{aligned} ds &= \lim_{n\to\infty} \Delta s_i = \sqrt{\Big(\frac{dx}{dt}\Big)^2 + \Big(\frac{dy}{dt}\Big)^2}\,dt \\ &= \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = |\mathbf{r}'(t)|\,dt. \end{aligned}$$

Thus the **arc length** of C can be computed as

$$L = \int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

16.2.1. Line integrals for scalar functions in the plane

Now, suppose that a force is applied to **move an object along a path** traced by a curve C. If the amount of force is given by $f(x, y)$, then the **amount of work** done must be given by the integral

$$W = \int_C f(x, y)\, ds, \tag{16.5}$$

where s is the arc length element, i.e., $ds = \sqrt{dx^2 + dy^2}$.

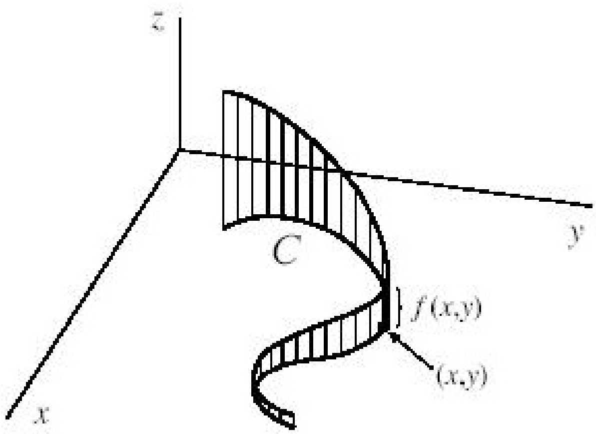


Figure 16.5: A function defined on a curve C.

Assumption. The curve C is smooth, i.e., $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$.

Definition 16.13. *If f is defined on a smooth curve C given by (16.3), then **line integral of f along C** is*

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i, \tag{16.6}$$

if this limit exists. Here $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$.

The line integral defined in (16.6) can be rewritten as

$$\begin{aligned} \int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\,\sqrt{(x'(t))^2+(y'(t))^2}\,dt \\ &= \int_a^b f(x(t),y(t))\,|\mathbf{r}'(t)|\,dt. \end{aligned} \tag{16.7}$$

Problem 16.14. Evaluate $\displaystyle\int_C (2+x^2y)ds$, where C is upper half of the unit circle $x^2+y^2=1$.

Solution. ***Clue***: Find the parametric equation for C and then follow the formula (16.7).

Ans: $2\pi+\frac{2}{3}$

Definition 16.15. *C is a* ***piecewise smooth curve*** *if it is a union of a finite number of smooth curves $C_1, C_2, \cdots, C_n$. That is,*

$$C = C_1 \cup C_2 \cup \cdots \cup C_n.$$

In the case, we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds. \quad (16.8)$$

Problem 16.16. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution. ***Clue***: Begin with parametric representation of C_1 and C_2. For example, $C_1 : x = t,\ y = t^2,\ 0 \le t \le 1$ and $C_2 : x = 1,\ y = t,\ 1 \le t \le 2$.

Ans: $\frac{1}{6}(5\sqrt{5} - 1) + 2$

Application to Physics: To compute the **mass of a wire** that is shaped like a plane curve C, where the density of the wire is given by a function $\rho(x, y)$ defined at each point (x, y) on C, we can evaluate the line integral

$$m = \int_C \rho(x, y)\, ds. \tag{16.9}$$

Thus **the center of mass of the wire** is the point $(\overline{x}, \overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x, y)\, ds, \quad \overline{y} = \frac{1}{m}\int_C y\rho(x, y)\, ds. \tag{16.10}$$

Problem 16.17. A wire takes the shape of the semicircle, $x^2+y^2 = 1, \;\; y \ge 0$, and its density is proportional to the distance from the line $y = 1$. Find the center of mass of the wire.

Solution. ***Clue***: First parametrize the wire and use $\rho(x, y) = k(1 - y)$.

Ans: $(\overline{x}, \overline{y}) = \left(0, \frac{4-\pi}{2(\pi-2)} \approx 0.38\right)$, where $m = k(\pi - 2)$

Definition 16.18. ***Line integrals of f along C with respect to x and y are defined as***

$$\begin{aligned} \int_C f(x,y)\,dx &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta x_i, \\ \int_C f(x,y)\,dy &= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta y_i. \end{aligned} \tag{16.11}$$

The line integrals can be evaluated by expressing everything in terms of t:

$$x = x(t),\ \ y = y(t),\ \ dx = x'(t)dt,\ \ dy = y'(t)dt.$$

$$\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(x(t),y(t))x'(t)\,dt, \\ \int_C f(x,y)\,dy &= \int_a^b f(x(t),y(t))y'(t)\,dt. \end{aligned} \tag{16.12}$$

Note: It frequently happens that line integral with respect x and y occur together. When this happens, it is customary to abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy.$$

Let $\boldsymbol{F}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$ and $\mathbf{r} = \langle x,y\rangle = \langle x(t),y(t)\rangle$ represent the curve C. Then, since $d\mathbf{r} = \langle dx, dy\rangle$, we can rewrite the above as

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C \boldsymbol{F}\cdot d\mathbf{r}, \tag{16.13}$$

which is a line integral of vector fields. We will consider it in detail in § 16.2.3 below (p. 166).

Problem 16.19. Evaluate $\displaystyle\int_C y^2\,dx + x\,dy$, where

$$\begin{cases} \text{(a)} \quad C = C_1: & \text{the line segment from } (-5,-3) \text{ to } (0,2) \\ \text{(b)} \quad C = C_2: & \text{the arc of } x = 4 - y^2 \text{ from } (-5,-3) \text{ to } (0,2) \end{cases}$$

Solution. ***Clue***: $C_1 : \mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1,\ 0 \le t \le 1$ and $C_2 : x = 4 - t^2,\ y = t,\ -3 \le t \le 2$.

Ans: (a) $-\frac{5}{6}$ (b) $40\frac{5}{6}$

Orientation of curves: It is important to note that the value of line integrals with respect to x or y (or z, in 3-D) depends on the **orientation of** C, unlike line integrals with respect to the arc length s. If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well. We **denote by** $-C$ **the curve with its orientation reversed**. We then have

$$\int_{-C} P\,dx = -\int_C P\,dx, \quad \int_{-C} Q\,dy = -\int_C Q\,dy. \tag{16.14}$$

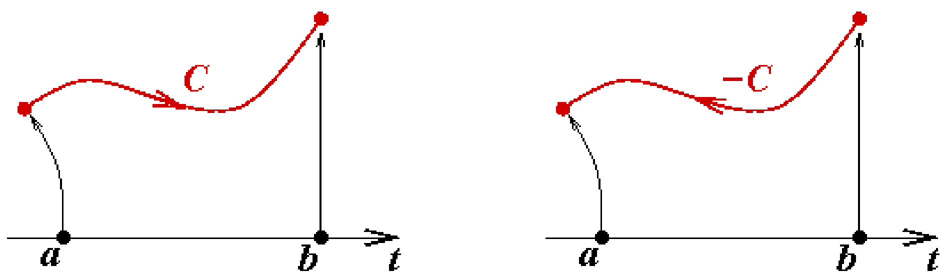


Figure 16.6: Curve C and its reversed curve $-C$.

Note: For line integrals with respect to the arc length s,

$$\int_{-C} f\,ds = \int_C f\,ds. \tag{16.15}$$

Problem 16.20. (Variant of Problem 16.19(a)): The reversed curve $-C_1$ is the line segment from $(0, 2)$ to $(-5, -3)$:

$$\mathbf{r}(t) = (1-t)\langle 0, 2\rangle + t\langle -5, -3\rangle = \langle -5t, -5t+2\rangle, \ \ 0 \le t \le 1.$$

Thus we must have $\displaystyle\int_{-C_1} y^2\,dx + x\,dy = \frac{5}{6}$.

Solution.

16.2.2. Line integrals in space

First, the definition for the line integral (with respect to arc length) can be generalized as follows.

Definition 16.21. *Suppose that C is a smooth space curve given by*

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$$

Then the ***line integral of** f **along** C *is defined in a similar manner as in Definition 16.13:*

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i. \tag{16.16}$$

It can be evaluated using a formula similar to (16.7):

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt \\ &= \int_a^b f(x(t),y(t),z(t))\,\sqrt{(x')^2+(y')^2+(z')^2}\,dt.\end{aligned} \tag{16.17}$$

Note:

- When $f(x,y,z) \equiv 1$,
$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L: \ \text{ arc length}$$
- When $\boldsymbol{F} = \langle P, Q, R\rangle$,
$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

Problem 16.22. Evaluate $\int_C y \sin z \, ds$, where C is the **circular helix** given by $\mathbf{r}(t) = \langle \cos t, \sin t, t \rangle$, $0 \le t \le 2\pi$.

Solution. ***Hint***: You may use one of formulas: $\sin^2 t = (1 - \cos 2t)/2$, $\cos^2 t = (1 + \cos 2t)/2$.

Ans: $\sqrt{2}\pi$

Problem 16.23. Evaluate $\int_C z \, dx + x \, dy + y \, dz$, where C is given by $x = t^2, \; y = t^3, \; z = t^2, \quad 0 \le t \le 1.$

Solution.

Ans: $\frac{3}{2}$

16.2.3. Line integrals of vector fields

Recall: In Calculus III, we have found that the **work** done by a constant force $\boldsymbol{F}$ in moving an object from a point to another point Q in the space is

$$W = \boldsymbol{F} \cdot \boldsymbol{D}, \tag{16.18}$$

where $\boldsymbol{D} = \overrightarrow{PQ}$, the displacement vector. □

In general: Let C be a smooth space curve given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle, \quad a \le t \le b.$$

Then the **work** done by a force $\boldsymbol{F}$ in moving an object along the curve C is

$$W = \lim_{n\to\infty} \sum_{i=1}^{n} \boldsymbol{F}(x_i^*, y_i^*, z_i^*) \cdot [\boldsymbol{T}(x_i^*, y_i^*, z_i^*)\, \Delta s_i] = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds, \tag{16.19}$$

where $\mathbf{r}(t_i) = (x_i, y_i, z_i)$, $\Delta s_i = |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$, and $\boldsymbol{T}$ is the unit tangential vector

$$\boldsymbol{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}. \tag{16.20}$$

Since $ds = |\mathbf{r}'(t)|\, dt$, we have

$$W = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_a^b \boldsymbol{F} \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} |\mathbf{r}'(t)|\, dt = \int_a^b \boldsymbol{F} \cdot \mathbf{r}'(t)\, dt = \int_C \boldsymbol{F} \cdot d\mathbf{r}. \tag{16.21}$$

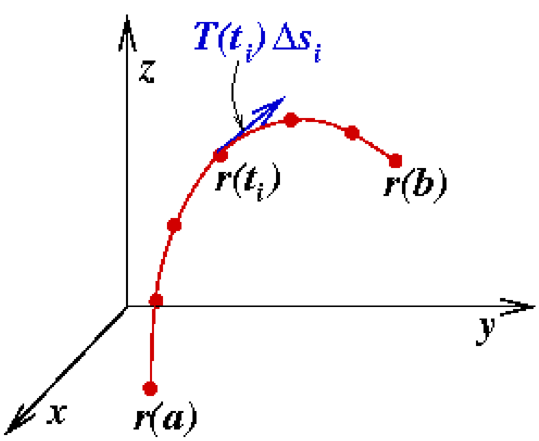


Figure 16.7

Definition 16.24. *Let* $\boldsymbol{F}$ *be a continuous vector field defined on a smooth curve* C *given by* $\mathbf{r}(t), a \le t \le b$*. Then the* ***line integral of F along*** C *is*

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} \overset{\text{def}}{=\!=} \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{16.22}$$

We say that work is the line integral with respect to arc length of the ***tangential component*** *of force.*

Note: Although $\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds$ and integrals with respect to arc length are unchanged when orientation is reversed, it is still true that

$$\int_{-C} \boldsymbol{F} \cdot d\mathbf{r} = -\int_C \boldsymbol{F} \cdot d\mathbf{r}.$$

Why?

Problem 16.25. Evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$, where $\boldsymbol{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \;\; 0 \le t \le 1$.

Solution.

Ans: $\frac{27}{28}$

Theorem 16.26. ***(Equivalent to Definition 16.24, p. 167).***
Let $\boldsymbol{F} = \langle P, Q, R\rangle$. Then

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz. \tag{16.23}$$

Problem 16.27. Let $\boldsymbol{F}(x, y) = \left\langle \dfrac{x}{\sqrt{x^2+y^2}}, \dfrac{y}{\sqrt{x^2+y^2}} \right\rangle$ and C the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$.

(a) Use a graph of $\boldsymbol{F}$ and C to guess whether $\int_c \boldsymbol{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

(b) Evaluate the integral.

Solution. ***Hint***: (b) $C : \mathbf{r}(t) = \langle t, 1 + t^2\rangle\,, \;\; -1 \le t \le 1$; use Eqn. (16.22).

Ans: 0

Exercises 16.2

1. Evaluate the line integral, using the formula $\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$.

   (a) $\displaystyle\int_C x^2 y\,ds$, where C is given by $\mathbf{r}(t) =< \cos 2t, \sin 2t >$, $0 \le t \le \pi/4$

   (b) $\displaystyle\int_C 2xye^{xyz}\,ds$, where C is the line segment from $(0,0,0)$ to $(2,1,2)$

   Ans: (a) $1/3$; (b) $e^4 - 1$

2. Let $\boldsymbol{F}$ be the vector field shown in the Figure 16.8.

   (a) If C_1 is the horizontal line segment from $P(3,2)$ to $Q(-3,2)$, determine whether $\displaystyle\int_{C_1} \boldsymbol{F}\cdot d\mathbf{r}$ is positive, negative, or zero.

   (b) Let C_2 be the clockwise-oriented circle of radius 3 centered at the origin. Determine whether $\displaystyle\int_{C_2} \boldsymbol{F}\cdot d\mathbf{r}$ is positive, negative, or zero.

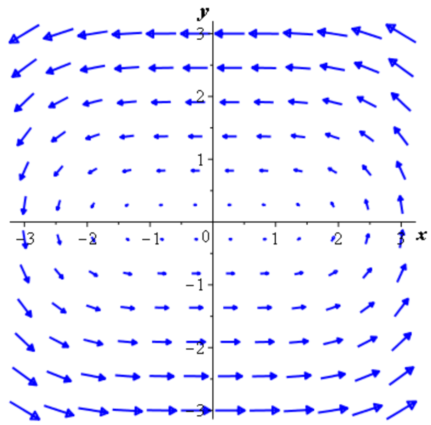


Figure 16.8

3. Use (16.22) to evaluate the line integral $\displaystyle\int_C \boldsymbol{F}\cdot d\mathbf{r}$, where C is parameterized by $\mathbf{r}(t)$.

   (a) $\boldsymbol{F}(x,y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$,
   $\mathbf{r}(t) = (t^3 - 2t)\,\mathbf{i} + (t^3 + 2t)\,\mathbf{j}, \quad 0 \le t \le 1$

   (b) $\boldsymbol{F}(x,y,z) = \langle -y, x, xy\rangle$,
   $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle, \quad 0 \le t \le \pi$

   Ans: (a) -9; (b) π

4. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $y \ge 0$. If the linear density of the wire is $\rho(x,y) = ky$, find the **mass** and **center of mass** of the wire. ***Hint***: $C : \mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle, \;\; 0 \le t \le \pi$

   Ans: $8k$, $(0, \pi/2)$

16.3. The Fundamental Theorem for Line Integrals

Recall: The Part 2 of **Fundamental Theorem of Calculus** (FTC2) is

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{16.24}$$

Goal: It would be nice to get a generalization of the FTC2 (16.24) to line integrals.

16.3.1. Conservative vector fields

Theorem 16.28. *Suppose that $\boldsymbol{F}$ is continuous, and is a* ***conservative vector field****; that is, $\boldsymbol{F} = \nabla f$ for some f. Then*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{16.25}$$

Proof. By the Chain rule and the FTC2,

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_a^b \nabla f(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt &= \int_a^b \frac{d}{dt}[(f\circ\mathbf{r})(t)]\,dt \\ &= (f\circ\mathbf{r})(t)|_a^b = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).\end{aligned}$$

□

Theorem 16.28 is the **Fundamental Theorem for Line Integrals**, which is a generalization of the FTC2. The function f is called a **potential function** of $\boldsymbol{F}$, or simply **potential**.

Problem 16.29. Let $\boldsymbol{F}(x, y) = \left\langle 3 + 2xy^2, 2x^2y - 4 \right\rangle$.

(a) Find a function f such that $\nabla f = \boldsymbol{F}$.

(b) Evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$, where $C:\ \mathbf{r}(t) = \langle \cos t, 2\sin t \rangle\,,\ \ 0 \le t \le \pi$.

Solution.

Ans: (a) $f(x, y) = 3x + x^2y^2 - 4y + K$ (b) -6

Problem 16.30. (Revisit of Problem 16.27). Let $\boldsymbol{F}(x, y) = \dfrac{x}{\sqrt{x^2 + y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2 + y^2}}\,\mathbf{j}$ and C the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$. Find a potential of $\boldsymbol{F}$ and evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$.

Solution.

Ans: $f(x, y) = \sqrt{x^2 + y^2}$ and $\int_C \boldsymbol{F} \cdot d\mathbf{r} = 0$.

16.3.2. Independence of path

Definition 16.31. *We say the line integral* $\int_C \boldsymbol{F} \cdot d\mathbf{r}$ *is* ***independent of path*** *if*

$$\int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r},$$

for any two paths C_1 *and* C_2 *that have the same initial and terminal points.*

Observation 16.32. In general, $\int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} \neq \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r}$. (See Problem 16.19, p. 162.) However, Theorem 16.28 says that when $\boldsymbol{F} = \nabla f$,

$$\int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \int_{C_2} \nabla f \cdot d\mathbf{r} = \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r}.$$

Thus line integrals of **conservative** fields are **independent of path**.

Definition 16.33. *A curve* C *is* ***closed*** *if its terminal point coincides with its initial point, that is,* $\mathrm{r}(b) = \mathrm{r}(a)$. *A* ***simple curve*** *is a curve that does not intersect itself.*

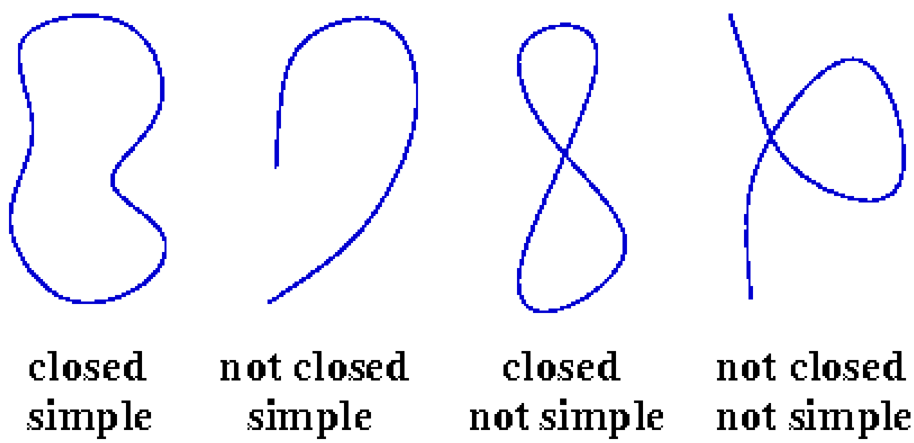


Figure 16.9

Theorem 16.34. *$\int_C \boldsymbol{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \boldsymbol{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.*

Proof. ($\Rightarrow$) For a closed curve C, choose two points A and B to decompose C into two parts: $C = C_1 \cup C_2$. Then

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} + \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} - \int_{-C_2} \boldsymbol{F} \cdot d\mathbf{r} = 0,$$

because C_1 and $-C_2$ have the same initial and terminal points.
($\Leftarrow$) Let C_1 and C_2 have the same initial and terminal points. Then

$$0 = \int_{C_1 \cup (-C_2)} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} + \int_{-C_2} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} - \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r},$$

where the first equality comes from the assumption. □

Pictorial definitions

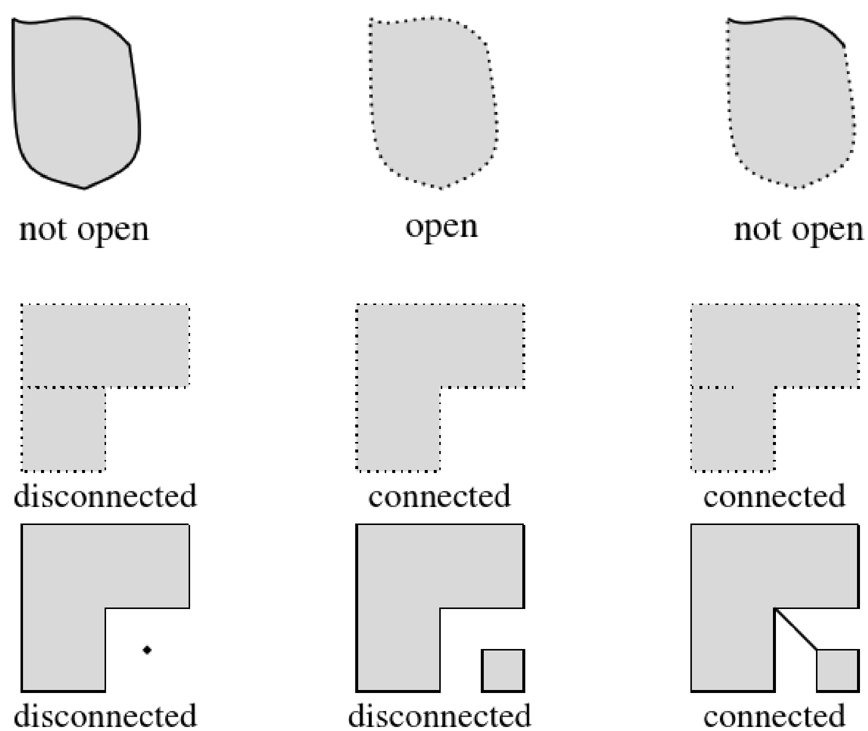


Figure 16.10: Pictorial definitions for D.

Definition 16.35. *A set D is said to be* ***open*** *if every point P in D has a disk with center P that is contained wholly and solely in D.* **Note.** *D cannot contain any boundary points.*

Definition 16.36. *A set D is said to be* ***connected*** *if for every two points P and Q in D, there exists a path which connects P to Q.*

Theorem 16.37. *Suppose that the line integral of a vector field* $\boldsymbol{F}$ *is* ***independent of path*** *within an* ***open connected region*** *D, then* $\boldsymbol{F}$ *is a* ***conservative*** *vector field on D.*

Proof. (sketch). Choose an arbitrary point $(a, b) \in D$ and define

$$f(x, y) = \int_{(a,b)}^{(x,y)} \boldsymbol{F} \cdot d\mathbf{r}.$$

Since this line integral is independent of path, we can define $f(x, y)$ using any path between (a, b) and (x, y). By choosing a path that ends with a horizontal line segment from (x_1, y) to (x, y) contained entirely in D, $x_1 < x$, we can show that

$$\partial f/\partial x(x, y) = \partial/\partial x \Big[\int_{(a,b)}^{(x_1,y)} \boldsymbol{F} \cdot d\mathbf{r} + \int_{(x_1,y)}^{(x,y)} \boldsymbol{F} \cdot d\mathbf{r} \Big] = 0 + \partial/\partial x \int_{x_1}^{x} \boldsymbol{F} \cdot \langle dx, 0\rangle = P.$$

Similarly, we can prove that $\partial f/\partial y(x, y) = Q$. □

It follows from Observation 16.32 and Theorem 16.37:

Corollary 16.38. *In an open connected region,* $\boldsymbol{F}$ *is* ***conservative*** *if and only if its line integral is* ***independent of path****.*

Theorem 16.39. ***(Clairaut's Theorem for conservative vector fields).*** *If $\boldsymbol{F}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on D, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{16.26}$$

throughout the domain D.

Quesiton. Does (16.26) imply conservativeness of $\boldsymbol{F}$?

Ans: No, in general. But, almost!

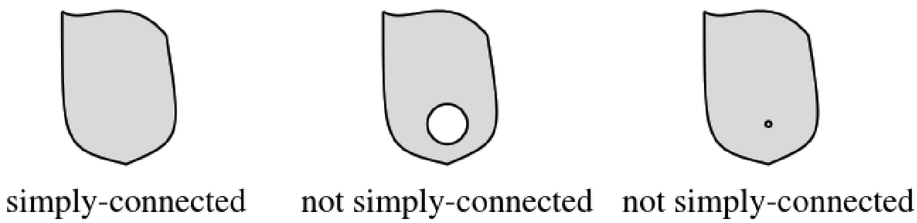


Figure 16.11: Simply-connectedness of D.

Definition 16.40. *D is a **simply-connected region** if it is connected and every simple closed curve contains only points in D.*

Theorem 16.41. *Let $\boldsymbol{F} = \langle P, Q\rangle$ be a vector field on an open simply-connected region D. If P and Q have **continuous first-order partial derivatives** throughout D,*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{16.27}$$

*then $\boldsymbol{F}$ is **conservative**.*

Note: Theorem 16.41 is a special case of Green's Theorem which we will see in Section 16.4.

Self-study **16.42.** Determine whether or not the vector field $\boldsymbol{F}(x,y) = \langle 3+2xy, x^2+x-3y^2 \rangle$ is conservative.

Solution. ***Hint***: Check if $P_y = Q_x$ is satisfied.

Ans: no

Problem **16.43.** Determine whether or not the vector field $\boldsymbol{F}(x,y) = \langle e^y + y\cos x,\; xe^y + \sin x \rangle$ is conservative.

Solution.

Ans: yes

16.3.3. Potential functions

Recall: When $\boldsymbol{F}$ is conservative, we know from (16.25) on p.170 that

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)), \qquad (16.28)$$

which is easy to evaluate when the potential f is known.

Problem 16.44. Given $\boldsymbol{F}(x,y) = \langle e^y + y\cos x,\ xe^y + \sin x\rangle$,

(a) Find a potential.

(b) Evaluate $\int_C \boldsymbol{F}\cdot d\mathbf{r}$, where C is parameterized as

$$\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t\rangle,\ \ 0 \le t \le \pi.$$

Solution.

Ans: (a) $f(x,y) = xe^y + y\sin x + K$ (b) $-e^{\pi} - 1$

Problem 16.45. Let $\boldsymbol{F}(x, y, z) = \langle y^2,\, 2xy + e^{3z},\, 3ye^{3z} \rangle$. Find f such that $\nabla f = \boldsymbol{F}$.

Solution.

Ans: $f = xy^2 + ye^{3z} + K$

Problem 16.46. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a conservative vector field, where P, Q, R have continuous first-order partial derivatives. Then,

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y. \tag{16.29}$$

Solution. ***Hint***: Use Clairout's theorem.

□

Problem 16.47. Show that $\int_C y\,dx + x\,dy + yz\,dz$ is not independent of path.

Solution. ***Hint***: Use (16.29) to check if it is conservative.

Exercises 16.3

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.

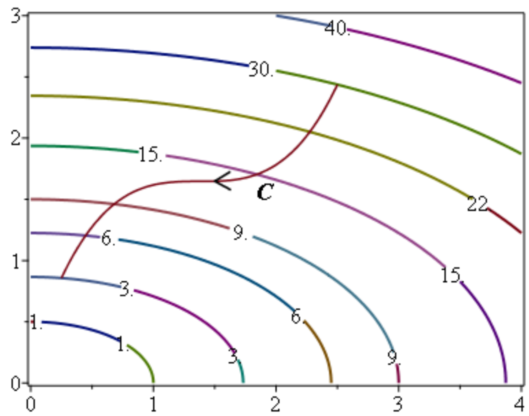


Figure 16.12

2. Determine whether the vector field $\boldsymbol{F}$ is conservative or not. If it is, find its potential.

   (a) $\boldsymbol{F}(x,y) = \langle x+y, x-y\rangle$
   (b) $\boldsymbol{F}(x,y) = \langle 2xy, x^2+2xy\rangle$
   (c) $\boldsymbol{F}(x,y) = \langle 2xy^4, x^2y^3\rangle$
   (d) $\boldsymbol{F}(x,y) = \langle ye^x, e^x - 2y\rangle$

3. (i) Find the potential of $\boldsymbol{F}$ and (ii) use part (i) to evaluate $\int_C \boldsymbol{F}\cdot d\mathbf{r}$ along the given curve C.

   (a) $\boldsymbol{F}(x,y) = \langle e^y,\ xe^y + \sin y\rangle$, $C:\mathbf{r}(t) = \langle -\cos t, e^t \sin t\rangle,\ \ 0 \le t \le \pi$
   (b) $\boldsymbol{F}(x,y,z) = \langle 2y+z, 2x+z, x+y\rangle$, C is the line segment from $(1,0,0)$ to $(2,2,2)$
   (c) $\boldsymbol{F}(x,y,z) = \langle \sin z,\ -\sin y,\ x\cos z\rangle$, $C:\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle,\ \ 0 \le t \le \pi/2$

   Ans: (a) 2; (b) 16; (c) $\cos(1) - 1$

4. Show that the line integral is independent of path and evaluate the integral.

   (a) $\displaystyle\int_C x\,dx - y\,dy$, C is any path from $(0,1)$ to $(3,0)$
   (b) $\displaystyle\int_C (\sin y - ye^{-x})\,dx + (e^{-x} + x\cos y)\,dy$, C is any path from $(1,0)$ to $(0,\pi)$

   Ans: (a) 5; (b) π

5. The figure below depicts two vector fields, one of which is conservative. Which one is it? Why is the other one not conservative?

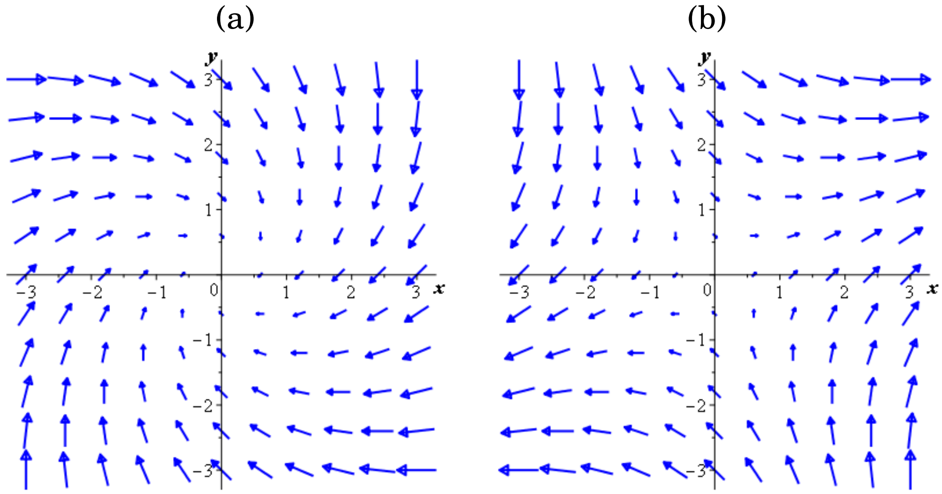


Figure 16.13: Two vector fields, one of which is conservative.

16.4. Green's Theorem

Green's Theorem gives the **relationship** between a **line integral** around a simple closed curve C and a **double integral** over the plane region D bounded by C.

Definition 16.48. *The* ***positive orientation*** *of a simple closed curve C refers to a single counterclockwise traversal of C (with keeping the domain on the left). The other directional orientation is called the* ***negative orientation***.

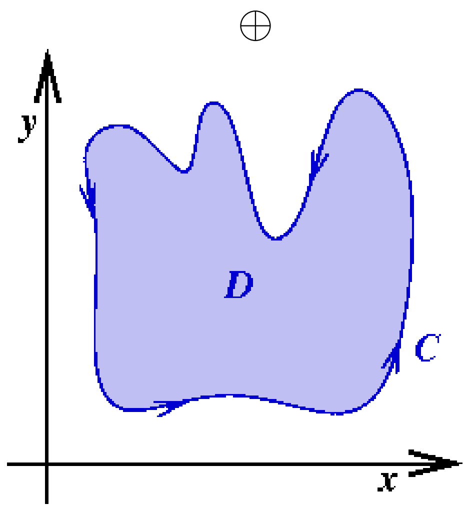


Figure 16.14: ⊕-orientation and ⊖-orientation of a simple closed curve C.

Theorem 16.49. ***(Green's Theorem)***. *Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\mathbf{F} = \langle P, Q\rangle$ has* ***continuous partial derivatives*** *on an open region including D, then*

$$\oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{16.30}$$

Note: The proof of Green's Theorem on simple regions is based on the following identities

$$\oint_C P dx = -\iint_D \frac{\partial P}{\partial y}\, dA, \quad \oint_C Q dy = \iint_D \frac{\partial Q}{\partial x}\, dA. \tag{16.31}$$

Notation 16.50. We denote the line integral calculated by using the **positive orientation** of the closed curve C by

$$\oint_C P\, dx + Q\, dy, \quad \oint_C P\, dx + Q\, dy, \text{ or } \oint_C P\, dx + Q\, dy.$$

We denote line integrals calculated by using the **negative orientation** of the closed curve C by

$$\oint_C P\, dx + Q\, dy.$$

Problem 16.51. Evaluate $\oint_C x^4 dx + xy dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution. Although the given line integral could be evaluated by the methods of Section 6.2, we would use Green's Theorem.

Ans: $\frac{1}{6}$

Problem 16.52. Evaluate $\oint_C \boldsymbol{F} \cdot d\mathbf{r}$, where $\boldsymbol{F} = \langle y - \cos y, x \sin y\rangle$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.

Solution. ***Hint***: Check the orientation of the curve.

Ans: 4π

16.4.1. Application to area computation

Recall

$$A(D) = \iint_D 1\, dA.$$

If we choose P and Q such that

$$\partial Q/\partial x - \partial P/\partial y = 1, \tag{16.32}$$

then the area of D can be computed as

$$A(D) = \iint_D 1\, dA = \oint_C P dx + Q dy. \tag{16.33}$$

The following choices are common:

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = x \end{cases} \quad \begin{cases} P(x,y) = -y \\ Q(x,y) = 0 \end{cases} \quad \begin{cases} P(x,y) = -\frac{y}{2} \\ Q(x,y) = \frac{x}{2} \end{cases} \tag{16.34}$$

Then, Green's Theorem give the following formulas for the area of D:

$$A(D) = \oint_C x\, dy = -\oint_C y\, dx = \frac{1}{2}\oint_C x\, dy - y\, dx \tag{16.35}$$

Problem 16.53. Find the area enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, an ellipse.

Solution. ***Clue***: The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, $0 \le t \le 2\pi$. ***Hint***: You may use $\sin^2 x = \frac{1-\cos 2x}{2}$ or $\cos^2 x = \frac{1+\cos 2x}{2}$.

Ans: $ab\pi$

Problem 16.54. Use a formula in (16.35) to find the area of the shaded region in Figure 16.15.

Solution. ***Hint***: For the slanted edge $(C_3) : x = t,\ y = 3 - t,\ 1 \le t \le 3$.

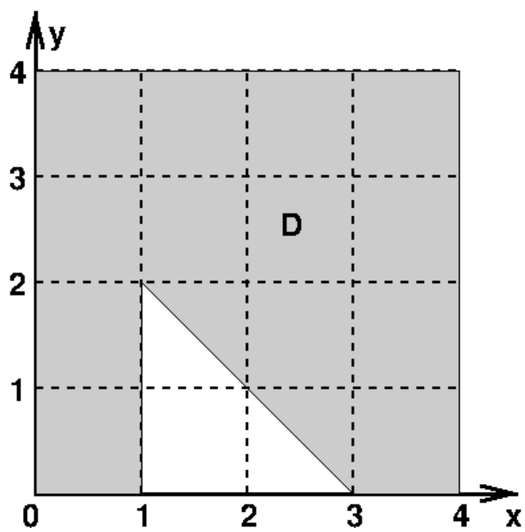


Figure 16.15

Ans: 14

Problem 16.55. Using the identity (an application of Green's Theorem)

$$A(D) = \iint_D dA = \oint_{\partial D} x\,dy,$$

we will try to show that the area of D (the shaded region) is 6.

- First, observe that the line integrals on vertical and horizontal line segments of the figure are all zero.
- Thus the area must be the same as the line integral on the slant side, the line segment from $P(4,0)$ to $Q(2,2)$, which we denote by C_2.

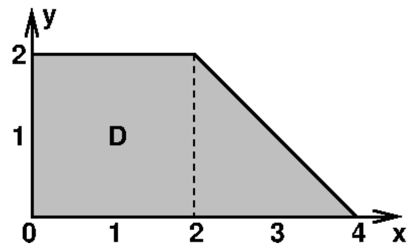


Figure 16.16

(a) Evaluate $\int_{C_2} x\,dy$, where C_2 is parametrized by

$$\mathbf{r}(t) = (1-t)P + t\,Q, \;\; 0 \le t \le 1.$$

(b) Evaluate $\int_{C_2} x\,dy$, where C_2 is parametrized by

$$\mathbf{r}(t) = \langle t,\, 4-t\rangle, \text{ with } t \text{ moving } 4 \searrow 2.$$

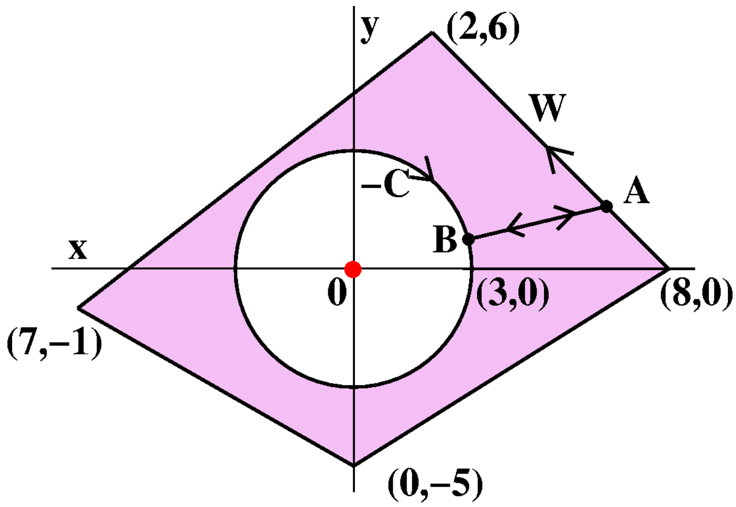
(c) Find "**the mid value of x**" and "**the change in y**", on C_2. Multiply the results to see if it is the same as the output in (a) and (b).[1]

Solution.

[1]The method introduces an effective algorithm for the computation of area. See Project 3, p.230.

16.4.2. Generalization of Green's Theorem

Although Green's Theorem is proved only for the case where D is simple, we can now extend it to the case where D is either **a finite union of simple regions** or **of holes**.



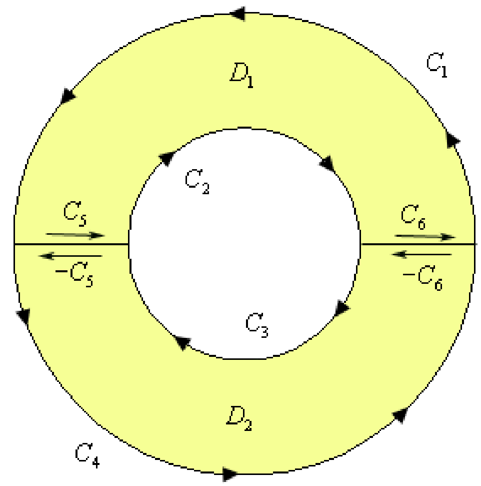


Figure 16.17: Regions having holes.

For example: For the right figure above,

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \oint_{\partial D_1} P\,dx + Q\,dy + \oint_{\partial D_2} P\,dx + Q\,dy.\end{aligned} \tag{16.36}$$

Along the common boundary, the opposite directional line integral will be canceled. Thus

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy. \tag{16.37}$$

Theorem 16.56. ***(Generalized Green's Theorem)***. *Let D be either a finite union of simply-connected regions or of holes, of which* ***the boundary is finite and oriented****. If $\mathbf{F} = \langle P, Q\rangle$ has* ***continuous partial derivatives*** *on an open region including D, then*

$$\oint_{\partial D} P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA, \tag{16.38}$$

where ∂D is the boundary of D ***positively oriented****.*

Problem 16.57. Evaluate $\oint_C (1-y^3)dx+(x^3+e^{y^2})dy$, where C is the boundary of the region between the circles $x^2+y^2=4$ and $x^2+y^2=9$, having the positive orientation.

Solution.

Ans: $\frac{195\pi}{2}$

Example 16.58. *Let* $\boldsymbol{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$*. Show that* $\oint_C \boldsymbol{F} \cdot d\mathbf{r} = 2\pi$ *for any positively oriented simple closed curve* C *that encloses the origin.*

Warning: You CANNOT use Green's Theorem for this problem. Why?

Solution. ***Clue***: Choose $C' : x^2 + y^2 = a^2$, for small a. Then,

$$\oint_{\partial D} \boldsymbol{F} \cdot d\mathbf{r} = \oint_C \boldsymbol{F} \cdot d\mathbf{r} + \oint_{-C'} \boldsymbol{F} \cdot d\mathbf{r} = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \Big)\, dA,$$

where D is the region bounded by C and $-C'$. However,

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y^2 - x^2}{(x^2+y^2)^2} - \frac{y^2 - x^2}{(x^2+y^2)^2} = 0. \tag{16.39}$$

Thus we have

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} = \oint_{C'} \boldsymbol{F} \cdot d\mathbf{r} \tag{16.40}$$

By introducing parametric representation of $C' : \mathbf{r}(t) = \langle a\cos t, a\sin t \rangle,\ 0 \le t \le 2\pi$, we can conclude

$$\oint_{C'} \boldsymbol{F} \cdot d\mathbf{r} = \int_0^{2\pi} \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \int_0^{2\pi} \frac{a^2 \sin^2 t + a^2 \cos^2 t}{a^2}\, dt = 2\pi. \quad \square$$

Problem 16.59. Let $\boldsymbol{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle$, the same as in the above example. Show that $\oint_C \boldsymbol{F} \cdot d\mathbf{r} = 0$ for any simple closed path C that does not pass through or enclose the origin.

Now, you CAN use Green's Theorem. Why?

Solution. ***Clue***: See if $\boldsymbol{F}$ is conservative, i.e., $Q_x = P_y$, checking conditions of Theorem 16.41 (p. 175) or Green's Theorem.

Let's try to solve another problem before closing the section.

Problem 16.60. Evaluate $\oint_c y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannual region D in the upper half-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution.

Ans: $14/3$

Summary 16.61. *Green's Theorem can be summarized as follows.*

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{16.41}$$

is applicable when

1. *The boundary of D is **finite and oriented**.*
2. *The vector field $\boldsymbol{F} = \langle P, Q\rangle$ has **continuous partial derivatives** over the whole region D. (It is about **quality** of the vector field.)*

Exercises 16.4

1. Evaluate the line integral $\oint_C y^2\,dx + 3xy\,dy$, where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(2,2)$:

   (a) directly

   (b) using Green's theorem

   Hint: For (a), you should parametrize each of three line segments.
   For example: $C_3 : \mathbf{r}(t) = \langle t, t \rangle$, $t = 2 \searrow 0$.

   Ans: $4/3$

2. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

   (a) $\int_C (2y + \ln(1+x^2))dx + (6x + y^2)dy$, where C is the triangle with vertices $(0,0)$, $(3,0)$, and $(1,1)$

   (b) $\int_C (x^2 - y^3 + y)dx + (x^3 + x - y^2)dy$, where C is the circle $x^2 + y^2 = 4$

   Ans: (b) 24π

3. Use Green's Theorem to evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

   (a) $\boldsymbol{F}(x,y) =< y^3 \cos x, x + 3y^2 \sin x >$, C is the triangle from $(0,0)$ to $(8,0)$ to $(4,4)$ to $(0,0)$

   (b) $\boldsymbol{F}(x,y) =< 5y - 2030x^2 + \sin y,\ y^2 + x\cos y >$, C consists of the three line segments: from the origin to $(0,2)$, then to $(2,0)$, and then back down to the origin

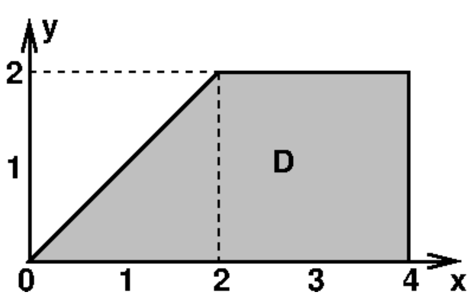
   (c) $\boldsymbol{F}(x,y) =< y + y^2 - \cos y, x \sin y >$, C is the circle $x^2 + y^2 = 4$ oriented clockwise

   Ans: (a) 16; (c) 4π

4. Use the identity (an application of Green's Theorem)

$$A(D) = \iint_D dA = \int_{\partial D} x\,dy$$

   to show that the area of D (the shaded region) is 6. You should compute the line integral for each line segment of the boundary, first introducing an appropriate parametrization.



16.5. Curl and Divergence

16.5.1. Curl

We now define the curl of a vector field, which helps us represent rotations of different sorts in physics and such fields. It can be used, for instance, to represent the velocity field in fluid flow.

Definition 16.62. *Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist. Then the* **curl** *of* $\boldsymbol{F}$ *is the vector field on $\mathbb{R}^3$ defined by*

$$\textbf{\textit{curl}}\ \boldsymbol{F} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle. \tag{16.42}$$

Definition 16.63. *Define the* ***vector differential operator*** ∇ *(**"del"**) as*

$$\nabla = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle = \langle \partial_x,\ \partial_y,\ \partial_z \rangle.$$

Then

$$\begin{aligned} \nabla \times \boldsymbol{F} &= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix}\right) \\ &= \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle \\ &= \textbf{\textit{curl}}\ \boldsymbol{F} \end{aligned} \tag{16.43}$$

So, the easiest way to remember Definition 16.62 is

$$\textbf{\textit{curl}}\ \boldsymbol{F} = \nabla \times \boldsymbol{F}. \tag{16.44}$$

Note: If $\boldsymbol{F}$ represents the velocity field in fluid flow, then the particles in the fluid tend to rotate about the axis that points in the direction of $\nabla \times \boldsymbol{F}$; the magnitude $|\nabla \times \boldsymbol{F}|$ measures how quickly the fluid rotates.

Quesiton. Why do **tornado** evolve? What is ***the change in the air*** after a tornado? *Answer*: Energy consumption

Remark 16.64. *If* $\boldsymbol{F}$ *is* ***conservative*** *and has* ***continuous partial derivatives****, then*

$$\textbf{\textit{curl}}\ \boldsymbol{F} = 0. \tag{16.45}$$

(See also Problem 16.46 on p.179.)

Theorem 16.65. *If* f *is a function of three variables that has continuous second-order partial derivatives, then*

$$\textbf{\textit{curl}}\,(\nabla f) = \nabla \times (\nabla f) = 0. \tag{16.46}$$

Proof. Use Clairout's Theorem.

Problem 16.66. Show that the vector field $\boldsymbol{F} = \langle xz, xyz, -y^2\rangle$ is not conservative.

Solution. ***Clue***: Check if $\textbf{\textit{curl}}\,\boldsymbol{F} \neq 0$.

Theorem 16.67. *If $\boldsymbol{F}$ is a vector field whose component functions have* ***continuous partial derivatives*** *on a* ***simply-connected*** *domain and* ***curl*** $\boldsymbol{F} = 0$*, then $\boldsymbol{F}$ is* ***conservative****.*

Note: The above theorem is a 3D version of Theorem 16.41, p. 175.

Problem 16.68. Let $\boldsymbol{F} = \langle y^2z^3,\, 2xyz^3,\, 3xy^2z^2 \rangle$.

(a) Show that $\boldsymbol{F}$ is conservative.

(b) Find f such that $\boldsymbol{F} = \nabla f$.

Solution.

Ans: (a) ***curl*** $\boldsymbol{F} = 0$, (b) $f(x, y) = xy^2z^3 + K$

16.5.2. Divergence

Definition 16.69. *Let* $\boldsymbol{F} = \langle P, Q, R\rangle$ *be a vector field on* $\mathbb{R}^3$ *and its partial derivatives exist. Then the* ***divergence*** *of* $\boldsymbol{F}$ *is defined as*

$$\mathbf{div}\,\boldsymbol{F} \equiv \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \boldsymbol{F}.$$

Theorem 16.70. *Let* $\boldsymbol{F} = \langle P, Q, R\rangle$ *whose components have continuous second-order partial derivatives. Then*

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0. \tag{16.47}$$

Note: The above theorem is analogous to $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Problem 16.71. Show that $\boldsymbol{F} = \langle xz, xyz, -y^2\rangle$ cannot be the curl of another vector field.

Solution. ***Clue***: Check if $\nabla \cdot \boldsymbol{F} = 0$

Remark 16.72. The reason for the name **divergence** can be understood in the context of fluid flow. If $\boldsymbol{F}$ is the velocity of a fluid, the $\boldsymbol{div}\,\boldsymbol{F}$ represents the **net change rate of the mass per unit volume**. Thus, if $\boldsymbol{div}\,\boldsymbol{F} = 0$, then $\boldsymbol{F}$ is said to be **incompressible**. Another differential operator occurs when we compute the divergence of a gradient vector field ∇f:

$$\boldsymbol{div}\,(\nabla f) = \nabla \cdot (\nabla f) = \nabla^2 f = \Delta f.$$

The operator $\nabla^2 = \nabla \cdot \nabla = \Delta$ is called the **Laplace operator**, which is also applicable to vector fields like

$$\Delta \boldsymbol{F} = \Delta \langle P,\, Q,\, R\rangle = \langle \Delta P,\, \Delta Q,\, \Delta R\rangle.$$

16.5.3. Vector forms of Green's Theorem

Recall: Green's Theorem (p. 182): Let $\boldsymbol{F} = \langle P,\, Q\rangle$. Then

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} \equiv \oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{16.48}$$

Now, regard $\boldsymbol{F}$ as a vector field in $\mathbb{R}^3$ with the 3rd component 0. Then

$$\nabla \times \boldsymbol{F} = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{bmatrix}\right) = \langle 0,\, 0,\, Q_x - P_y\rangle.$$

So we can rewrite the equation in Green's Theorem as

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} \equiv \oint_C \boldsymbol{F} \cdot \boldsymbol{T} ds = \iint_D (\nabla \times \boldsymbol{F}) \cdot \mathbf{k}\, dA, \tag{16.49}$$

which expresses the line integral of the **tangential component** of $\boldsymbol{F}$ along C as the double integral of the **vertical component** of $\boldsymbol{curl}\,\boldsymbol{F}$ over the region D enclosed by C.

Line integral of the normal component of $\boldsymbol{F}$

Example 16.73. Let $\boldsymbol{F} = \langle P, Q\rangle$. What is $\oint_C \boldsymbol{F}\cdot \mathbf{n}\, ds$?

Solution. Let $\mathbf{r} = \langle x(t), y(t)\rangle$ define the curve C. Then

$$\boldsymbol{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{\langle x', y'\rangle}{|\mathbf{r}'|} \quad \text{and} \quad \mathbf{n} = \frac{\langle y', -x'\rangle}{|\mathbf{r}'|}, \tag{16.50}$$

where $\mathbf{n}$ is the **outward unit normal vector**, 90° clockwise rotation of $\boldsymbol{T}$. Thus we have

$$\begin{aligned} \boldsymbol{F}\cdot \mathbf{n}\, ds &= \langle P, Q\rangle \cdot \frac{\langle y', -x'\rangle}{|\mathbf{r}'|}\, |\mathbf{r}'|\, dt \\ &= (P\, y' - Q\, x')\, dt \\ &= -Q\, dx + P\, dy. \end{aligned}$$

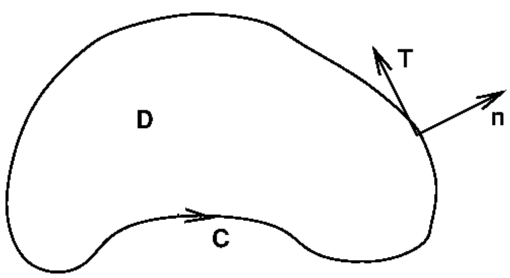


Figure 16.18

It follows from Green's Theorem that

$$\begin{aligned} \oint_C \boldsymbol{F}\cdot \mathbf{n}\, ds &= \oint_C -Q\, dx + P\, dy = \iint_D \big(P_x - (-Q)_y\big)\, dA \\ &= \iint_D \big(P_x + Q_y\big)\, dA = \iint_D \nabla\cdot \boldsymbol{F}\, dA. \end{aligned} \tag{16.51}$$

when P and Q have continuous partial derivatives over D. □

Exercises 16.5

1. Find (i) the curl and (ii) the divergence of the vector field.

   (a) $\boldsymbol{F}(x,y,z) = x^2yz\,\mathbf{j} + y^2z^2\,\mathbf{k}$ (b) $\boldsymbol{F}(x,y,z) = \langle x\sin y,\ y\sin z,\ z\sin x\rangle$

   Ans: (b) $\nabla\times\boldsymbol{F} = -\langle y\cos z, z\cos x, x\cos y\rangle$, $\nabla\cdot\boldsymbol{F} = \sin x + \sin y + \sin z$

2. The vector field $\boldsymbol{F}$ is shown in the xy-plane and looks the same in all other horizontal planes. (That is, $\boldsymbol{F}$ is independent of z and its third component is 0.)

   (a) Is ***div F*** positive, negative, or zero? Explain.
   (b) Determine whether ***curl F*** $= \mathbf{0}$. If not, in which direction does it point?
   (c) Use Theorem 16.67 to conclude if $\boldsymbol{F}$ is conservative.

   Hint: **The vector field in (I)**: You may express it as $\boldsymbol{F} = \langle P(x), 0, 0\rangle$, where P is a decreasing function of x only. Thus *div* $\boldsymbol{F} < 0$. **The vector field in (II)**: Let $\boldsymbol{F} = \langle P(x,y), Q(x,y), 0\rangle$. Then ***div F*** $= P_x + Q_y$ and ***curl F*** $= \langle 0, 0, Q_x - P_y\rangle$. For example, $P_y < 0$ in (II), because the horizontal components of the arrows (P) become smaller as y increases. What can you say about P_x, Q_y, and Q_x?

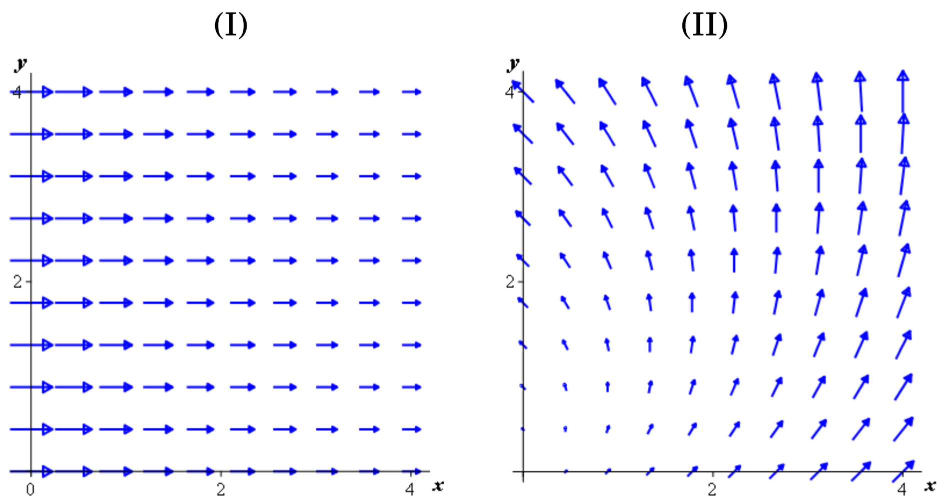


Figure 16.19

3. Determine whether or not $\boldsymbol{F}$ is conservative. If it is conservative, find its potential.

   (a) $\boldsymbol{F} = \langle yz^4, xz^4 + 2y, 4xyz^3\rangle$
   (b) $\boldsymbol{F} = \langle \sin z, 1, x\cos z\rangle$

   Ans: (b) $f = y + x\sin z + K$

16.6. Parametric Surfaces and Their Areas

16.6.1. Parametric surfaces

Goal: This section will aim to describe surfaces by a function $\mathbf{r}(u, v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, in a similar fashion that we described vector functions by $\mathbf{r}(t)$ earlier.

Definition 16.74. *A* ***parametric surface*** *is the set of points $\{(x, y, z)\}$ in $\mathbb{R}^3$ such that the components are expressed by a vector function of the form*

$$\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle, \quad (u, v) \in D \subset \mathbb{R}^2.$$

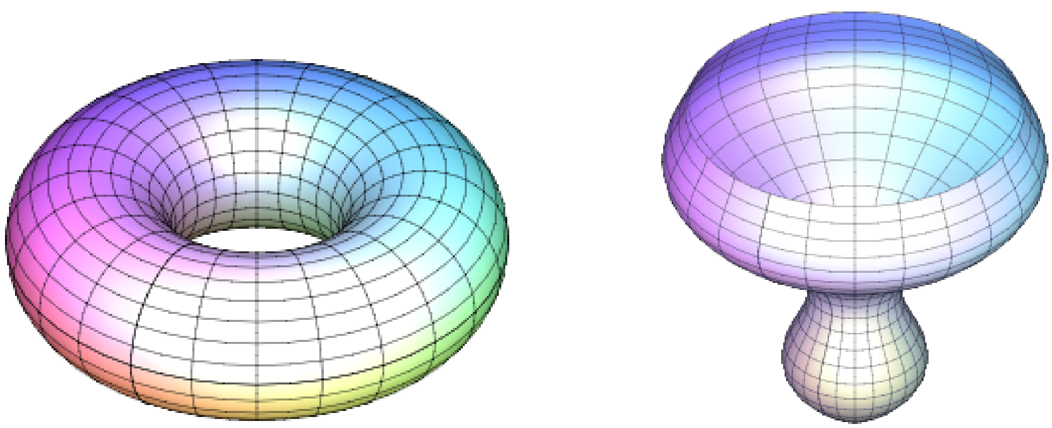


Figure 16.20: Examples of parametric surfaces.

Maple Script

```
with(plots): with(plottools):

plot3d([(4+2*cos(p))*cos(t), (4+2*cos(p))*sin(t), 2*sin(p)], p = 0..2*Pi, t = 0..2*Pi,
    axes = none, lightmodel = light1, scaling = constrained, orientation = [30,55]);

r := z/2+sin(z):
plot3d([r, t, z], t = 0..2*Pi, z =0..10, coords = cylindrical,
    axes = none, lightmodel = light1, scaling = constrained, orientation = [30,55]);
```

Problem 16.75. Identify and sketch $\mathbf{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u\rangle$, when $(u,v) \in D \equiv [0,2\pi] \times [0,5]$.

Solution. ***Clue***: $x^2 + z^2 = 4$.

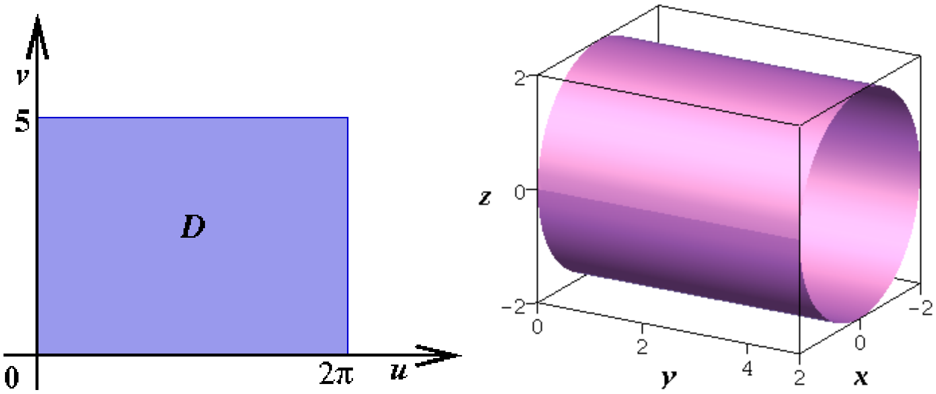


Figure 16.21

Self-study 16.76. Sketch $\mathbf{r}(s,t) = \langle s\cos 3t,\ s\sin 3t, s^2\rangle$, when $(s,t) \in [0,2] \times [0,2\pi]$. Discuss what the effect of the "3" is.

Quesiton. Given a surface, what is a parametric representation of it?

Problem 16.77. Find a parametric representation of the plane which passes $P_0(1,1,1)$ and contains $\mathbf{a} = \langle 1,2,0\rangle$ and $\mathbf{b} = \langle 2,0,-3\rangle$.

Solution. ***Clue***: $\mathbf{r}(u,v) = P_0 + u\,\mathbf{a} + v\,\mathbf{b}$.

Problem 16.78. Find a parametric representation of $x^2+y^2+z^2=a^2$.

Solution. ***Clue***: Use the spherical coordinates; the parameters are $(\theta,\,\phi)$.

Ans: $\mathbf{r}(\theta,\phi) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$, D?

Problem 16.79. Find a parametric representation of the cylinder $x^2 + y^2 = 4$, $0 \le z \le 1$.

Solution. ***Hint***: Use cylindrical coordinates $(r = 2, \theta, z)$.

Problem 16.80. Find a vector representation of the elliptic paraboloid $z = x^2 + 2y^2$.

Solution. ***Hint***: Let x, y be parameters.

Ans: $\mathbf{r}(x, y) = \langle x, y, x^2 + 2y^2 \rangle$

In general, for $z = f(x, y)$,

$$\mathbf{r}(x, y) = \langle x,\ y,\ f(x, y)\rangle \tag{16.52}$$

is considered as a parametric representation of the surface.

Note: Parametric representations are not unique.

Problem 16.81. Find a parametric representation of $z = 2\sqrt{x^2 + y^2}$.

Clue: A representation is as in (16.52), while another one can be formulated using (r, θ) as with polar coordinates. Also, recall that when polar coordinates are considered, $x = r\cos\theta$, $y = r\sin\theta$.

Solution. ①

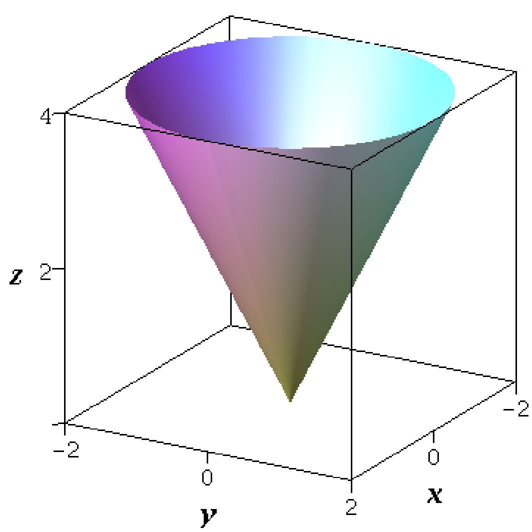


Figure 16.22

②

Surfaces of Revolution

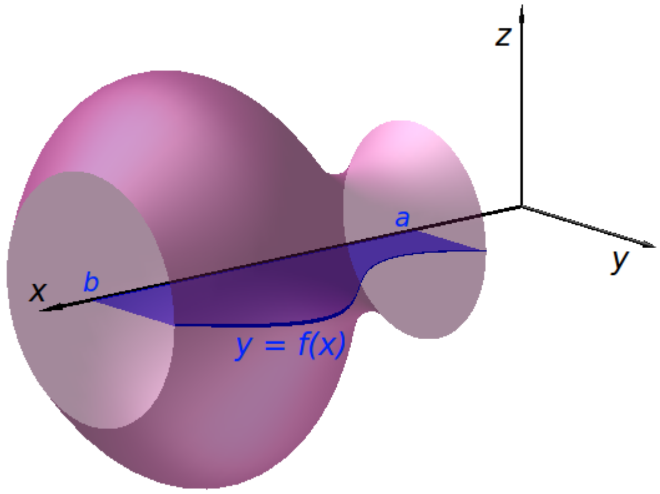


Figure 16.23: Surface of revolution

Let S be the surface obtained by rotating

$$y = f(x), \quad a \le x \le b,$$

about the x-axis (where $f(x) \ge 0$). Then, S can be represented as

$$\begin{aligned} \mathbf{r}(x, \theta) &= \langle x,\ f(x)\cos\theta,\ f(x)\sin\theta\rangle, \\ &\qquad (x, \theta) \in [a, b] \times [0, 2\pi]. \end{aligned} \tag{16.53}$$

Problem 16.82. Find parametric equations for the surface generated by rotating the curve $y = \sin(x)$, $0 \le x \le 2\pi$, about the x-axis.

Solution.

16.6.2. Tangent planes

Recall: The **plane** passing $\mathbf{x}_0 = (x_0, y_0, z_0)$ and having a normal vector $\mathbf{v} = \langle a, b, c \rangle$ can be formulated as

$$\mathbf{v} \cdot (\mathbf{x} - \mathbf{x}_0) = 0,$$

or equivalently

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{16.54}$$

Now, we will find the tangent plane to a parametric surface S traced out by

$$\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v) \rangle$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

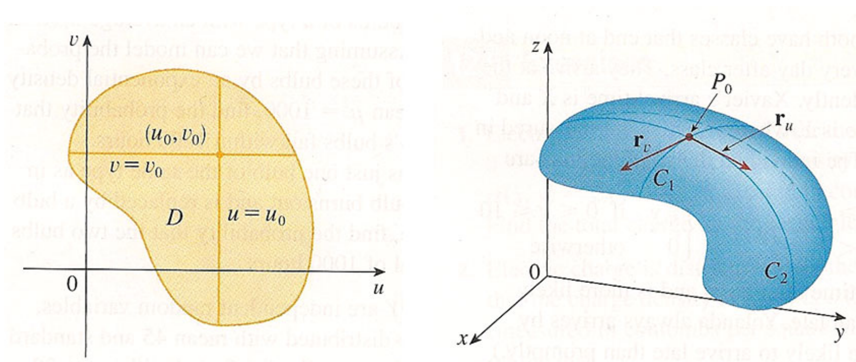


Figure 16.24

What we need: a normal vector, which can be determined by

$$\mathbf{r}_u \times \mathbf{r}_v.$$

Definition 16.83.

1. *A surface S represented by $\mathbf{r}$ is **smooth** if $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ over the whole domain.*
2. *A **tangent plane** is the plane containing $\mathbf{r}_u$ and $\mathbf{r}_v$ and having a normal vector $\mathbf{r}_u \times \mathbf{r}_v$.*

Problem 16.84. Find the tangent plane to

$$S : x = u^2,\ y = v^2,\ z = u + 2v; \quad \textbf{at } (1, 1, 3)$$

Solution.

Ans: $-2(x-1) - 4(y-1) + 4(z-3) = 0$

16.6.3. Surface area

Let $\mathbf{r} : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$. Then the **surface area** of S is

$$A(S) = \iint_S dS. \tag{16.55}$$

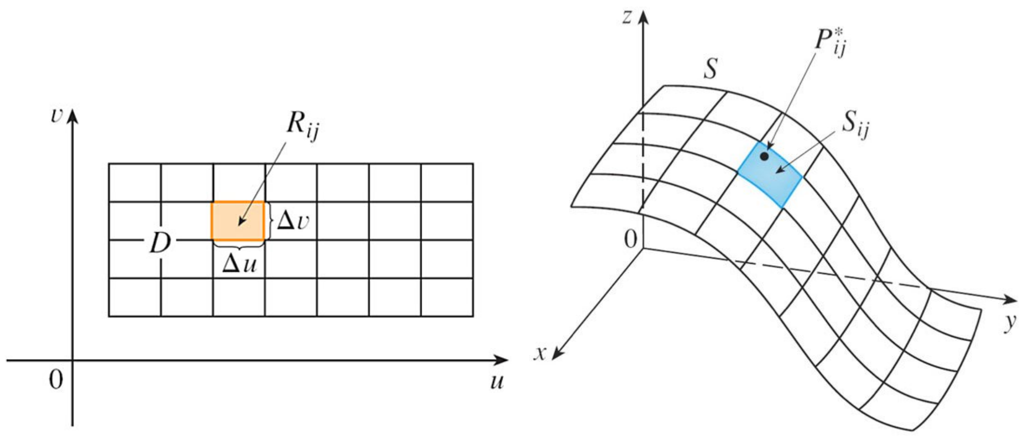


Figure 16.25: $\mathbf{r} : R_{ij} \mapsto S_{ij}$.

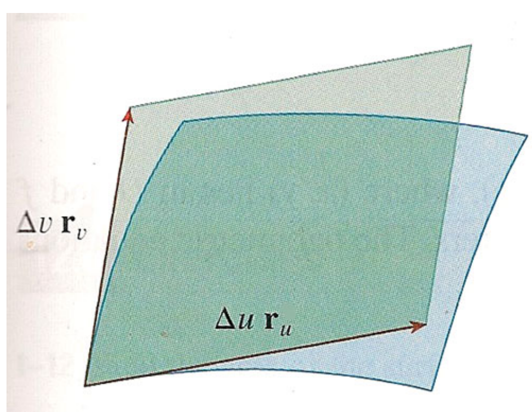


Figure 16.26: Approximating a patch by a parallelogram.

The area of the patch S_{ij} can be approximated by

$$\begin{aligned} \Delta S_{ij} &\approx A(\text{parallelogram}) \\ &= |(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v \end{aligned} \tag{16.56}$$

Definition 16.85. *If a smooth surface S is represented by*

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle, \quad (u,v) \in D,$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the ***surface area*** *of S is*

$$A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v|\, dA. \tag{16.57}$$

That is, $dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA$.

Problem 16.86. Find the area of the surface given by parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 1$.

Solution.

Ans: 1

Change of Variables vs. $\Delta S \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$

Recall: **(Summary 15.69 in § 15.9, p. 134)**. For a differentiable transformation $T : Q \subset \mathbb{R}^2 \to R \subset \mathbb{R}^2$ given by $\mathbf{r}(u, v) = \langle x(u, v), y(u, v)\rangle$,

$$\Delta A \approx \left|\frac{\partial(x, y)}{\partial(u, v)}\right| \Delta u\, \Delta v, \tag{16.58}$$

where $\partial(x, y)/\partial(u, v)$ is the **Jacobian** of T defined as

$$\frac{\partial(x, y)}{\partial(u, v)} = \text{det} \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u\, y_v - x_v\, y_u. \tag{16.59}$$

Now, consider R as a flat region embedded in $\mathbb{R}^3$. Define

$$\widetilde{R} = R \times \{0\} \subset \mathbb{R}^3.$$

Then, $\widetilde{T} : Q \to \widetilde{R}$ is represented by $\widetilde{\mathbf{r}}(u, v) = \langle x(u, v), y(u, v), 0\rangle$;

$$\widetilde{\mathbf{r}}_u \times \widetilde{\mathbf{r}}_v = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{bmatrix}\right) = \langle 0, 0, x_u\, y_v - x_v\, y_u\rangle\,. \tag{16.60}$$

Therefore

$$|\widetilde{\mathbf{r}}_u \times \widetilde{\mathbf{r}}_v| = \left|\frac{\partial(x, y)}{\partial(u, v)}\right|. \tag{16.61}$$

Eqn. (16.58) is a special case of $\Delta S \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v$.

Summary 16.87. *Let* $\mathbf{r} : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ *be a* ***parametric representation*** *of the surface* S. *Then*

1. *The map* $\mathbf{r}$ *can be viewed as a* ***change of variables***.
2. *The quantity* $|\mathbf{r}_u \times \mathbf{r}_v|$ *is simply the* ***scaling factor*** *for* $\mathbf{r}$.

Surface Area of the Graph of a Function

As a special case, consider the surface S made by the graph of

$$z = g(x,y), \;\; (x,y) \in D.$$

Then the surface S can be represented by

$$\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle.$$

Since

$$\mathbf{r}_x = \langle 1, 0, g_x\rangle \;\text{ and }\; \mathbf{r}_y = \langle 0, 1, g_y\rangle,$$

we obtain

$$\mathbf{r}_x \times \mathbf{r}_y = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{bmatrix}\right) = \langle -g_x, \, -g_y, \, 1\rangle \qquad \textbf{(16.62)}$$

Thus we conclude the following.

Let S be made by the graph of $z = g(x,y), \;\; (x,y) \in D$. Then the surface area of S is

$$A(S) = \iint_D \sqrt{g_x^2 + g_y^2 + 1}\, dA. \qquad \textbf{(16.63)}$$

Problem 16.88. Find the area of the part of paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution. (See Problem 15.41 on p. 114.)

Ans: $\frac{\pi}{6}(37\sqrt{37} - 1)$

Exercises 16.6

1. Identify the surface with the vector equation.

   (a) $\mathbf{r}(u,v) = \langle u-3, u+v, 4u+3v-2\rangle$
   (b) $\mathbf{r}(s,t) = \langle 2\cos t, s, 2\sin t\rangle, \quad 0 \le t \le \pi$

2. Match the parametric equations with the graphs labeled (I)–(III) and give reasons for your choices. Determine which families of grid curves on the surface have u constant and which have v constant.

   (a) $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$
   (b) $\mathbf{r}(u,v) = \langle v, 2\cos u, 2\sin u\rangle$
   (c) $\mathbf{r}(u,v) = \langle v\sin u, v\cos u, \cos v\sin v\rangle$

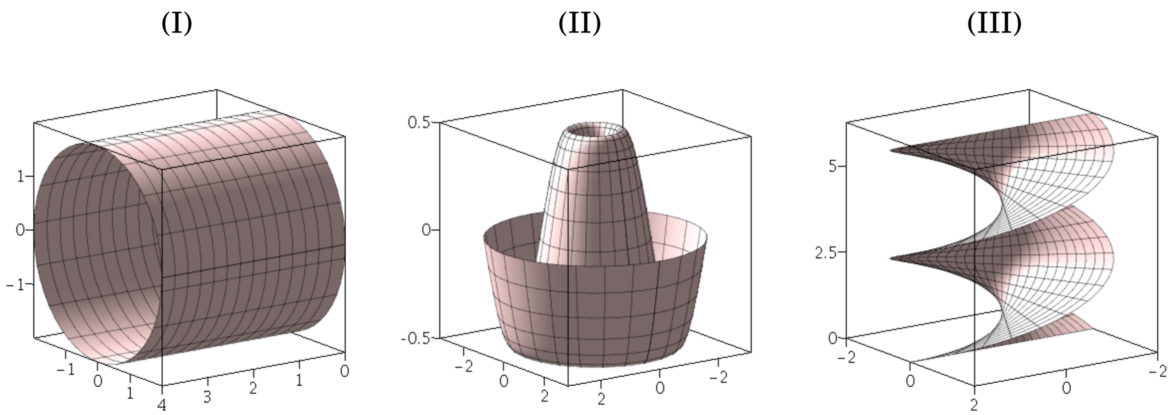


Figure 16.27

3. Find the parametric representation for the surface.

   (a) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.
   (b) The part of the plane $y + z = 1$ that lies inside the cylinder $x^2 + z^2 = 1$. (See Figure 16.28.)

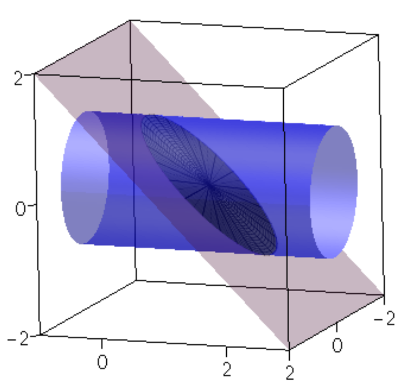


Figure 16.28

Hint: For (a), use the spherical coordinates (with $\rho = 2$) to specify the values of ϕ appropriately. Of course, $0 \le \theta \le 2\pi$. For (b), use the polar coordinates for the region in the xz-plane; that is, $x = r\cos\theta$, $z = r\sin\theta$. Then, you may set $y = 1 - z$. You have to specify the domain, values of r and θ, appropriately.

4. Find an equation of the tangent plane to the given surface at the specific point.

   (a) $\mathbf{r}(x, y) = \langle x, y, x^2 - y^2 \rangle, \quad (2, 1, 3)$
   (b) $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v \rangle, \quad (u, v) = (1, \pi/2)$

   Ans: (b) $\mathbf{r}_u \times \mathbf{r}_v(1, \pi/2) = \langle 1, 0, 1 \rangle \Rightarrow 1 \cdot (x - 0) + 0 \cdot (y - 1) + 1 \cdot (z - \pi/2) = x + z - \pi/2 = 0$

5. Find the area of the surface.

   (a) The part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 6$
   (b) The surface parametrized by $\mathbf{r}(u, v) = \langle u^2, uv, \dfrac{v^2}{2} \rangle$, defined on $\{(u, v) \mid u^2 + v^2 \leq 1\}$

   Ans: (a) $\frac{62\pi}{3}$; (b) $3\pi/4$

16.7. Surface Integrals

This section deals with surface integrals of the form

$$\iint_S f(x,y,z)\,dS \quad \text{or} \quad \iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$$

16.7.1. Surface integrals of scalar functions

Suppose that the surface S has a parametric representation

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v)\in D.$$

Then, from the previous section, we have

$$dS = |r_u \times r_v|\,dA$$

Thus we can reach at the formula

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u\times r_v|\,dA. \tag{16.64}$$

Remark 16.89.

- *When* $z = g(x,y)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$. *Thus the formula (16.64) reads*

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA. \tag{16.65}$$

- **Similarity**: *For line integrals,* $\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$

Problem 16.90. Compute the surface integral $\iint_S xy\,dS$, where S is the triangular region with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$.

Solution. ***Clue***: The surface S (triangular region) can be expressed by $\dfrac{x}{1}+\dfrac{y}{2}+\dfrac{z}{2}=1$. Thus $z=2-2x-y$. Now, what is D?

Ans: $\frac{1}{\sqrt{6}}$

Problem 16.91. Evaluate $\iint\limits_S z\,dS$, where S is the surface whose side S_1 is given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2\le 1$ in the plane $z=0$, and whose top S_3 is the disk $x^2+y^2\le 1$ in the plane $z=1$.

Solution. ***Clue***: $S_1 : x=\cos\theta,\ y=\sin\theta,\ z=z;\ (\theta,z)\in D\equiv[0,2\pi]\times[0,1]$. Then $|\mathbf{r}_\theta\times\mathbf{r}_z|=1$.

Ans: $\pi+0+\pi=2\pi$

16.7.2. Surface integrals of vector fields

Oriented Surfaces

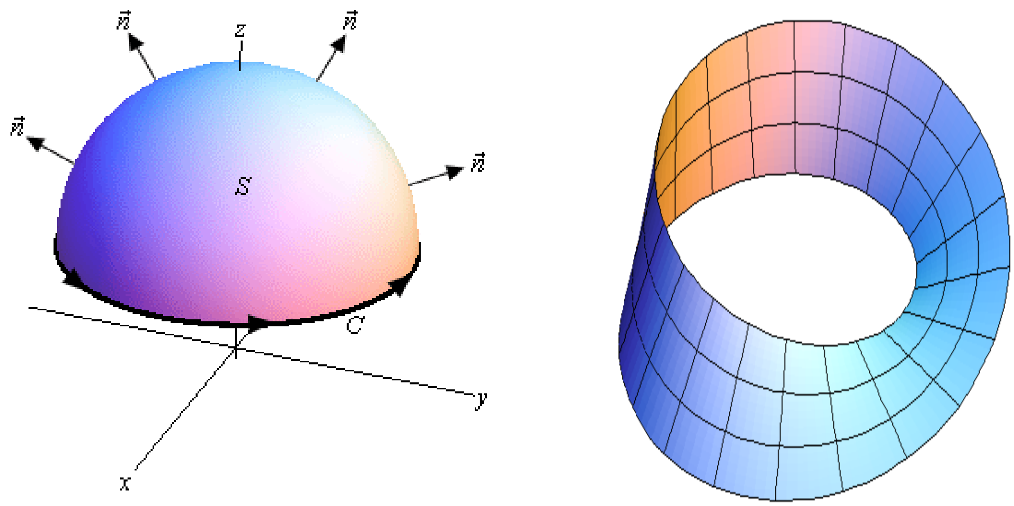


Figure 16.29: Oriented surface and Möbius strip.

Definition 16.92. *Let the surface S have a vector representation* r*.*

- *A unit normal vector* n *is defined as*

$$\mathrm{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \tag{16.66}$$

- *The surface S is called an* ***oriented surface*** *if the (chosen) unit normal vector* n ***varies continuously*** *over S.*
  (A counter example: Möbius strip.)
- *For closed surfaces, the* ***positive orientation*** *is the one outward.*

Is it confusing? Then, consider this:

Definition 16.93. *A surface S is called* ***orientable*** *if it has* ***two separate sides****.*

A Historic View, for Surface Integrals of Vector Fields

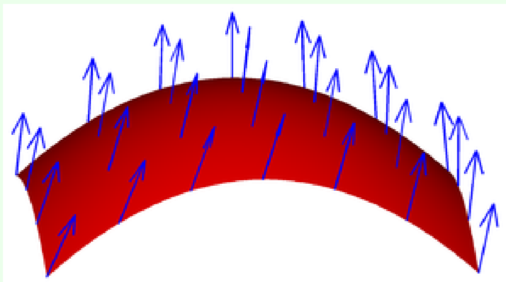


Figure 16.30: A vector field on a surface.

Suppose that S is an oriented surface. Imagine we have a fluid flowing through S, such that $\mathbf{v}(\mathbf{x})$ determines the velocity of the fluid at $\mathbf{x}$. The **flux** is defined as the quantity of fluid flowing through S per unit time.

The illustration implies that if the vector field is tangent to S at each point, then the flux is zero because the fluid just flows in parallel to S, and neither in nor out.

Thus, if $\mathbf{v}$ has both a tangential and a normal component, then **only the normal component contributes to the flux**. Based on this reasoning, to find the flux, we need to take the dot product of $\mathbf{v}$ with the unit surface normal $\mathbf{n}$ to S, which will give us a scalar field to be integrated over S appropriately.

Definition 16.94. *Let* $\boldsymbol{F}$ *be a continuous vector field defined on an oriented surface* S *with unit normal vector* $\mathbf{n}$*. The* ***surface integral of*** $\boldsymbol{F}$ ***over*** S *is*

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \overset{\text{def}}{=\!=} \iint_S \boldsymbol{F}\cdot \mathbf{n}\, dS. \tag{16.67}$$

This integral is also called the ***flux*** *of* $\boldsymbol{F}$ *across* S*.*

For the computation of the flux, the right side of (16.67), you may utilize

$$\mathbf{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|} \quad \text{and} \quad dS = |\mathbf{r}_u \times \mathbf{r}_v|\, dA, \tag{16.68}$$

when S is parametrized by $\mathbf{r} : D \to S$.

Surface Integrals of Vector Fields. Let r be a parametric representation of S, from $D \subset \mathbb{R}^2$. The flux **across the surface** S can be measured by

$$\begin{aligned}\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} &\stackrel{\text{def}}{=\!=} \iint_S \boldsymbol{F} \cdot \mathbf{n}\, dS \\ &= \iint_D \boldsymbol{F}(\mathbf{r}) \cdot \Big(\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}\Big)\, |\mathbf{r}_u \times \mathbf{r}_v|\, dA \\ &= \iint_D \boldsymbol{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\, dA.\end{aligned} \tag{16.69}$$

Note that $\boldsymbol{F} \cdot \mathbf{n}$ and $\boldsymbol{F}(\mathbf{r}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)$ are scalar functions.

Remark 16.95. Line integrals of vector fields is defined to measure quantities **along the curve**. That is,

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\mathbf{r} &\stackrel{\text{def}}{=\!=} \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds \\ &= \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt = \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt,\end{aligned} \tag{16.70}$$

where C is parametrized by $\mathbf{r} : [a, b] \to C$.

Problem 16.96. Find the flux of $\boldsymbol{F} = \langle x, y, 1\rangle$ across a upward helicoid: $\mathbf{r}(u, v) = \langle u \cos v, u \sin v, v\rangle$, $0 \le u \le 2$, $0 \le v \le \pi$.

Solution. ***Hint***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.

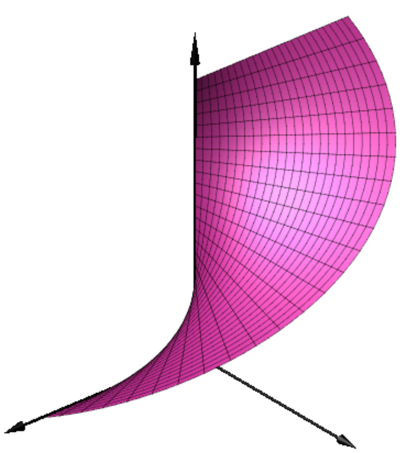


Figure 16.31

Ans: 2π

Example 16.97. *Find the flux of* $\boldsymbol{F} = \langle z, y, x\rangle$ *across the unit sphere* $x^2 + y^2 + z^2 = 1$.

Solution. First, consider a **vector representation** of the surface:

$$\mathbf{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle, \;\; 0 \le \phi \le \pi,\, 0 \le \theta \le 2\pi.$$

Then,

$$\begin{aligned} \boldsymbol{F}(\mathbf{r}) &= \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta\rangle, \\ \mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\theta\rangle, \end{aligned}$$

from which we have

$$\boldsymbol{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta.$$

Thus

$$\begin{aligned} \text{Flux} &= \iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA \\ &= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\, d\phi d\theta \\ &= \int_0^{2\pi}\int_0^{\pi} (\sin^3\phi\sin^2\theta)\, d\phi d\theta \\ &= \int_0^{2\pi} \sin^2\theta d\theta \int_0^{\pi} \sin^3\phi\, d\phi = \pi \cdot \frac{4}{3}. \qquad \square \end{aligned}$$

Note: The answer of the previous example is actually the volume of the unit sphere. In Section 16.9, we will study the so-called **Divergence Theorem** (formulated for closed surfaces)

$$\iint_{\partial E} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \nabla \cdot \boldsymbol{F}\, dV$$

The above example can be solved easily using the Divergence Theorem; see Problem 16.105, p. 227.

Surfaces defined by $z = g(x, y)$:

- A vector representation: $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$.
- Normal vector: $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$.
- Thus, when $\boldsymbol{F} = \langle P,\ Q,\ R\rangle$,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \;=\; \iint_D \boldsymbol{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y)\, dA = \iint_D (-P\, g_x - Q\, g_y + R)\, dA. \tag{16.71}$$

Problem 16.98. Evaluate $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle y, x, z\rangle$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution. ***Hint***: For S_1 (the upper part), use the formula in (16.71). For S_2 (the bottom: $z = 0$), you may try to get $\boldsymbol{F}\cdot\mathbf{n}$, where $\mathbf{n} = -\mathbf{k}$.

Ans: $\frac{\pi}{2} + 0 = \frac{\pi}{2}$

Formula 16.99. *Let* $\boldsymbol{F} =< P, Q, R >$.

- $\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA$

  $\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\,dA$, *when* S *is given by* $z = g(x,y)$
- $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_S \boldsymbol{F} \cdot \mathbf{n}\,dS = \iint_D \boldsymbol{F} \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA$

  $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D (-Pg_x - Qg_y + R)\,dA$, *when* S *is given by* $z = g(x,y)$
- **Note**: *When* S *is given by* $z = g(x,y)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$

Exercises 16.7

1. Evaluate the surface integral $\iint_S f(x,y,z)\,dS$.

   (a) $f(x,y,z) = x$, S is the helicoid given by the vector equation $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi/2$ (***Hint***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.)

   (b) $f(x,y,z) = (x^2 + y^2)z$, S is the hemisphere $x^2 + y^2 + z^2 = 1$, $z \ge 0$

   Ans: (a) $(2\sqrt{2} - 1)/3$; (b) $\pi/2$

2. Evaluate the surface integral $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$.

   (a) $\boldsymbol{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$, S is the part of the paraboloid $z = x^2 + y^2$, $z \le 1$

   (b) $\boldsymbol{F}(x,y,z) = \langle z, x - z, y\rangle$, S is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented downward

   (c) $\boldsymbol{F} = \langle y, -x, z\rangle$, S is the upward helicoid parametrized by $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 2$, $0 \le v \le \pi$ (***Hint***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.)

   Ans: (a) 0; (b) $-1/3$; (c) $2\pi + \pi^2$

3. CAS Use a CAS to find the integral, either $\iint_S f(x,y,z)\,dS$ or $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$. First try to find the exact value; if the CAS does not work properly for the exact value, then try to estimate the integral correct four decimal places.

   (a) $f(x,y,z) = 2x^2 + 2y^2 + z^2$, S is the surface $z = x\cos y$, $0 \le x \le 1$, $0 \le y \le 1$

   (b) $\boldsymbol{F}(x,y,z) = \langle x^2+y^2, y^2+z^2, x^2\rangle$, S is the part of the cylinder $x^2+z^2 = 1$ that lies above the xy-plane and between the planes $y = 0$ and $y = 1$, with upward orientation ***Hint***: You may use $\mathbf{r}(\theta, y) = \langle \cos\theta, y, \sin\theta\rangle$, for a representation of S.

   Ans: (b) $2/3$

16.8. Stokes's Theorem

Stokes' Theorem is a high-dimensional version of Green's Theorem studied in § 16.4.

Recall: **(Green's Theorem, p. 182).** Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q \rangle$ have **continuous partial derivatives** on an open region including D, then

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} \stackrel{\text{def}}{=\!=} \oint_C P\,dx + Q\,dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot \mathbf{k}\,dA. \tag{16.72}$$

(For the last equality, see (16.49) on p.196.)

Theorem 16.100. (Stokes's Theorem) *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\boldsymbol{F} = \langle P, Q, R \rangle$ be a vector field whose components have **continuous partial derivatives** on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \boldsymbol{F} \cdot d\mathbf{r} = \iint_S (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S} \tag{16.73}$$

Remark 16.101.

- *See Figure 16.29(left) on p. 216, for an oriented surface of which the boundary has positive orientation.*
- ***Computation** of the surface integral: for $\mathbf{r} : D \to S$,*

$$\iint_S (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S} \stackrel{\text{def}}{=\!=} \iint_S (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot \mathbf{n}\,dS = \iint_D (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA. \tag{16.74}$$

- *Green's Theorem is a special case in which S is flat and lies on the xy-plane ($\mathbf{n} = \mathbf{k}$). Compare the last terms in (16.72) and (16.74).*

Problem 16.102. Evaluate $\int_C \boldsymbol{F} \cdot d\mathrm{r}$, where $\boldsymbol{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution. ***Clue***: You may start with the computation of $\nabla \times \boldsymbol{F}$ and consider a vector representation for S: $z = g(x, y) = 2 - y$. Then use the formula (16.74).

Ans: π

Problem 16.103. Use Stokes's Theorem to compute the surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle xz, yz, xy \rangle$ and S is the part of sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2=1$ and above the xy-plane.

Solution. ***Hint***: $\displaystyle\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\mathbf{r} = \int_a^b \boldsymbol{F} \cdot \mathbf{r}'(t)\, dt.$ A vector representation of C is $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle, \ \ 0 \le t \le 2\pi.$

Ans: 0

Exercises 16.8

1. A hemisphere H and a part P of a paraboloid are shown in the figure below. Let $\boldsymbol{F}$ be ba vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Which of the following is true? Give reasons for your choice.

   A. $\displaystyle\iint_H (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} < \iint_P (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S}$

   B. $\displaystyle\iint_H (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} = \iint_P (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S}$

   C. $\displaystyle\iint_H (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} > \iint_P (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S}$

   D. cannot compare

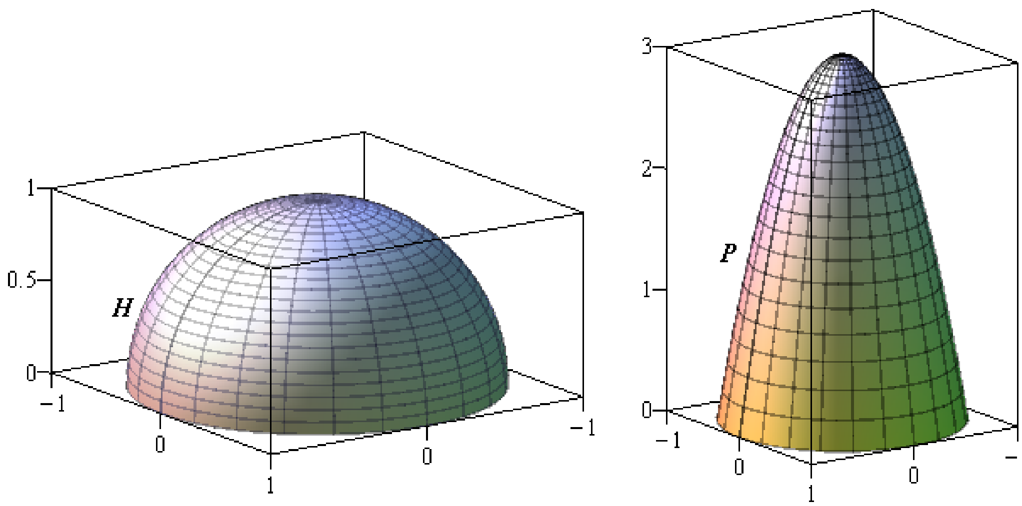


Figure 16.32

2. Use Stokes's Theorem to evaluate $\displaystyle\iint_S \textbf{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y) =< -y, x, x^2+y^2 >$ and S is the part of the sphere $x^2+y^2+z^2=8$ that lies inside the cone $z=\sqrt{x^2+y^2}$, oriented upward. (***Clue***: The boundary of S can be parametrized as $\mathbf{r}(t)=\langle 2\cos t, 2\sin t, 2\rangle$, $0\le t\le 2\pi$.) ***Hint***: Use the formula given in the hint of Problem 16.103.

   Ans: 16π

3. Use Stokes's Theorem to evaluate $\displaystyle\int_C \boldsymbol{F}\cdot d\mathbf{r}$. For each case, let C be oriented counter-clockwise when viewed from above.

   (a) $\boldsymbol{F}(x,y,z) =< z^2+x, x^2+y, y^2+z >$, C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

   (b) $\boldsymbol{F}(x,y,z) =< x, y, z-x >$, C is the curve of intersection of the plane $2y+z=2$ and the cylinder $x^2+y^2=1$

Hint: $\int_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S \textbf{curl}\,\boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \textbf{curl}\,\boldsymbol{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)dS$. (a) $\textbf{curl}\,\boldsymbol{F} = \langle 2y, 2z, 2x\rangle$ and $\mathbf{r}_x\times\mathbf{r}_y=\langle 1,1,1\rangle$. Figure out yourself what S, D, and $\mathbf{r}$ are.

Ans: (a) 1; (b) 2π

16.9. The Divergence Theorem

Recall: Let $\boldsymbol{F} = \langle P, Q\rangle$. In § 16.5.3, we considered vector forms of Green's Theorem including

$$\oint_C \boldsymbol{F}\cdot\mathbf{n}\,ds = \iint_D \nabla\cdot\boldsymbol{F}\,dA. \tag{16.75}$$

(See (16.51), p. 197.)

The Divergence Theorem is a generalization of the above.

Theorem 16.104. **(Divergence Theorem)** *Let E be a simple solid region and S be the boundary surface of E, given with positive (outward) orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ have **continuous partial derivatives** on an open region that contains E. Then*

$$\oiint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV. \tag{16.76}$$

Note: Let a surface S is parametrized by r. Then, from § 16.7.2 (p. 216), we know

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \overset{\text{def}}{=\!=} \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS = \iint_D \boldsymbol{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA, \tag{16.77}$$

whether or not S is closed. □

Note: The Divergence Theorem is developed mainly for closed surfaces; however, it can be applied for **unclosed surfaces** as in Review Problem R.16.10, p. 235.

Problem 16.105. (Revisit of Example 16.97, p. 219) Find the flux of $\boldsymbol{F} = \langle z, y, x \rangle$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution.

Ans: $\frac{4}{3}\pi$, the volume of the unit sphere

Problem 16.106. Find the flux of $\boldsymbol{F}$ across S, where

$$\boldsymbol{F}(x, y, z) = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2 z)\,\mathbf{k}$$

and S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solution.

Ans: $\frac{32}{3}\pi$

Problem 16.107. Use the Divergence Theorem to evaluate $\iint_S (x^2 + 2y^2 + z\,e^x)\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution. ***Hint***: Find n and express the integrand as $\boldsymbol{F} \cdot \mathbf{n}$; then try to use the Divergence Theorem.

Ans: 4π

Problem 16.108. Assume that S and E satisfy the conditions of the Divergence Theorem and functions have all required continuous partial derivatives, first or second-order. Prove the following.

1. $\iint_S \mathbf{a} \cdot \mathbf{n}\,dS = 0$, where a is a constant vector.

2. $V(E) = \dfrac{1}{3}\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \langle x, y, z\rangle$.

3. $\iint_S \textit{\textbf{curl}}\,\boldsymbol{F} \cdot d\boldsymbol{S} = 0$.

Exercises 16.9

1. Verify the Divergence Theorem is true for the vector field $\boldsymbol{F}$ defined on the region E.

   $\boldsymbol{F}(x,y,z) = \langle 2x, yz, xy\rangle,\ E = [0,1]\times[0,1]\times[0,1]$, the unit cube
   Clue: For the computation of $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, you should evaluate it on each of the six sides.

2. Use the Divergence Theorem to evaluate the total flux $\displaystyle\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$.

   (a) $\boldsymbol{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$, S is the boundary of of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$

   (b) $\boldsymbol{F}(x,y,z) = (x + y^2 + \cos z)\,\mathbf{i} + [\sin(\pi z) + xe^{-z}]\,\mathbf{j} + z\,\mathbf{k}$, S is a part of the cylinder $x^2 + y^2 = 1$ that lies between $z = 0$ and $z = 1$

   (c) $\boldsymbol{F}(x,y,z) = \langle x^2y^2, xye^z, xy^2z - xe^z\rangle$, S is the boundary of the box bounded by the coordinate planes and the planes $x = 1$, $y = 3$, and $z = 4$

   Ans: (b) 2π; (c) 54

3. As a variant of Problem 16.107, let's consider the following problem:
   Evaluate $\displaystyle\iint_S (x^2 + 2y^2 + 3z^2 + z\,e^x)\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 4$.

   Ans: 128π

Project 3. The Area of Heart

In this project, we will use the identity (an application of *Green's Theorem*)

$$A(D) = \iint_D dA = \oint_{\partial D} x\,dy \tag{16.78}$$

to compute the area of a closed curve saved in a data file. Also we will explore a **mid-point formula** for line integrals.

A dataset

- Download a heart data and save it in `heart-data.txt`:
  https://skim.math.msstate.edu/LectureNotes/heart-data.txt.
  It includes data points of the form $\{(x_i, y_i)\}$, representing a closed curve starting and ending at $(0, 0)$, positively oriented.
- When you draw a figure for it, using e.g. `heart.m` below, you will see a heart as in Figure 16.33.

heart.m

```
DATA = readmatrix('heart-data.txt');
X = DATA(:,1); Y = DATA(:,2);
figure, plot(X,Y,'r-','linewidth',2);
```

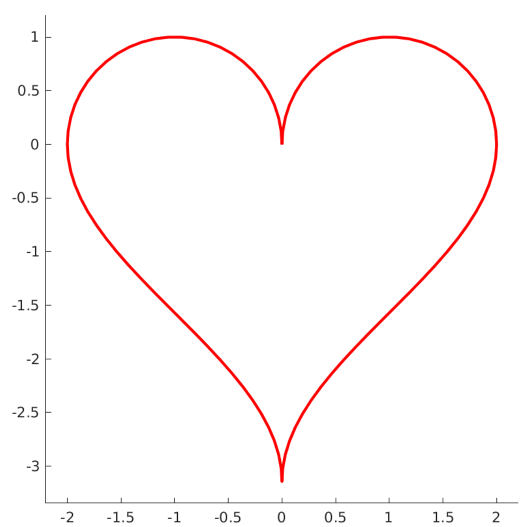


Figure 16.33: A plot of the closed curve in `heart-data.txt`.

We will explore a **mid-point approximation** for line integrals.

What to do

- *First, download a heart data and save it in* `heart-data.txt`:
  https://skim.math.msstate.edu/LectureNotes/heart-data.txt.
- *In the following, let* $(\widehat{x}_i, \widehat{y}_i)$ *be the mid-point of* (x_i, y_i) *and* (x_{i+1}, y_{i+1}) :

$$\widehat{x}_i = \frac{x_i + x_{i+1}}{2} \quad \textit{and} \quad \widehat{y}_i = \frac{y_i + y_{i+1}}{2}.$$

1. Draw a figure for the dataset.
2. Implement a computer program for the computation of the area:

$$A(\heartsuit) = \oint_{\partial\heartsuit} x\,dy = \sum_{i=1}^{m-1} \int_{C_i} x\,dy \approx \sum_{i=1}^{m-1} \widehat{x}_i \cdot (y_{i+1} - y_i), \tag{16.79}$$

   where m denotes the number of points in the data file and C_i is the line segment connecting (x_i, y_i) and (x_{i+1}, y_{i+1}).

   Note: The approximation in (16.79) results in the exact value. Why?

3. Implement a program for an **approximation** of the line integral:

$$\oint_{\partial\heartsuit} (x+y)\,dx + (x-y)\,dy \approx \sum_{i=1}^{m-1} (\widehat{x}_i + \widehat{y}_i)(x_{i+1} - x_i) + (\widehat{x}_i - \widehat{y}_i)(y_{i+1} - y_i). \tag{16.80}$$

   (a) In general, the mid-point formula for line integrals may not result in the exact value. However, the approximation in (16.80) can produce the exact value for the vector field $\boldsymbol{F} = \langle x+y, x-y \rangle$. Why?
   (b) Can you predict how large the integral must be? Why?

Let's consider a very basic for coding.

Computer implementation, in a nutshell

- In computer implementation, one of the major issues is how to deal with 'loop', which is a recursive execution of operations.
- A loop can start with an initialization.
- For example, let's try to add the square of integers from 1 to 10. Then, you may implement a code as in `square_sum.m` below.

square_sum.m

```
n = 10;
sum = 0;
for i = 1:n
    sum = sum + i^2;
end
```

Report. Upload a file including your experiences:

- Plot the given data.
- Implement a code for each of (16.79) and (16.80).
- Collect all your work, in order, including the plot, the code, the results (the area and the estimation of line integral).
- Attach a "summary" or "conclusion" page at the beginning of report.

R.16. Review Problems for Ch. 16

1. Evaluate the line integral $\int_C x^2 y\, ds$, where C is given by $\mathbf{r}(t) =< \cos t, \sin t >$, $0 \le t \le \pi/2$. ***Formula***: $\int_C f(x,y)\, ds = \int_a^b f(\mathbf{r}(t))\, |\mathbf{r}'(t)|\, dt$

   Ans: $1/3$

2. Let $\boldsymbol{F}(x,y) = x^2y^3\mathbf{i} + x^3y^2\mathbf{j}$.

   (a) Is $\boldsymbol{F}$ conservative? Why?
   (b) Find a function f such that $\nabla f = \boldsymbol{F}$.
   (c) Evaluate the line integral $\int_C \boldsymbol{F} \cdot d\mathbf{r}$, where C is parameterized as

   $$\mathbf{r}(t) = \left\langle t^3 - 2t, t^3 + 2t \right\rangle, \quad 0 \le t \le 1.$$

   Formula: $\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt$. *When $\boldsymbol{F}$ is conservative,* $\int_C \boldsymbol{F} \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

   Ans: (a) Yes. (b) $f = x^3y^3/3 + K$. (c) -9.

3. Let $\boldsymbol{F}(x,y) = 2xe^{-y}\,\mathbf{i} + (2y - x^2e^{-y})\,\mathbf{j}$ and C is any path from $(1,0)$ to $(2,1)$.

   (a) Show that the integral $\int_C \boldsymbol{F} \cdot d\mathbf{r}$ is **independent of path**.
   (b) Evaluate the integral $\int_C \boldsymbol{F} \cdot d\mathbf{r}$.

   Ans: (a) $Q_x = P_y = -2xe^{-y}$; (b) $f(x,y) = x^2e^{-y} + y^2 + K$. $f(2,1) - f(1,0) = 4/e$

4. Use **Green's Theorem** to evaluate the line integral $\int_C (y+e^{\sqrt{x}})dx+(2x+3\cos y^2)dy$, where C is the triangle with vertices from $(0,0)$ to $(0,4)$ to $(2,0)$ to $(0,0)$. ***Formula***: $\oint_C P dx + Q dy = \iint_D (Q_x - P_y)\, dA$.

   Ans: -4

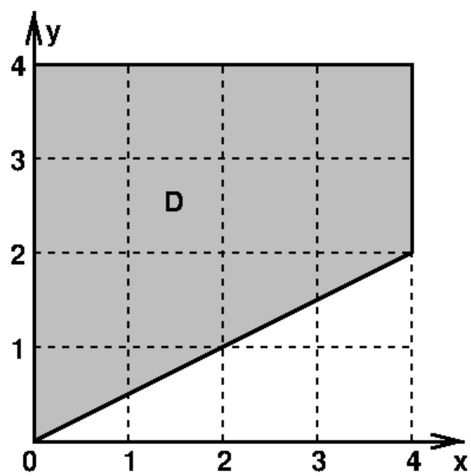
5. Is there a vector field $\boldsymbol{G}$ on $\mathbb{R}^3$ such that curl $\boldsymbol{G} =< x\sin y,\ \cos y,\ z - xy >$? Verify your answer.

   Ans: No, because $\nabla \cdot (\nabla \times G) = 1 \ne 0$.

6. Use the identity (an application of Green's Theorem)

$$A(D) = \iint_D dA = \int_{\partial D} x\,dy$$

to show that the area of D (the shaded region) is $16 - \frac{4\cdot 2}{2} = 12$. You have to compute **the line integral for each of four line segments** of the boundary. For the slant line segment, in particular, you should introduce an appropriate parameterization for the line integral.



Answer: For the slant line segment $(C_1) : x = t,\ y = t/2,\ 0 \le t \le 4$. So, $\int_{C_1} x dy = \int_0^4 t\,\frac{1}{2}\,dt = 4$. For the right vertical line segment (C_2): $\int_{C_2} x dy = \int_{C_2} 4dy = 8$. For the others, the line integral is zero.

7. Evaluate the **surface integral** $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + 2z\,\mathbf{k}$ and S is a part of the paraboloid $z = x^2 + y^2$, $z \le 1$.

Formula: $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS = \iint_D \boldsymbol{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA$. *When* $\boldsymbol{F} =< P, Q, R >$ *and the surface is given by* $z = g(x,y)$, $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D(-Pg_x - Qg_y + R)\,dA$.

Ans: $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D(-Pg_x - Qg_y + R)\,dA = \iint_D(-2x^2 - 2y^2 + 2z)\,dA = \iint_D 0\,dA = 0.$

8. Use **Stokes's Theorem** to evaluate $\int_C \boldsymbol{F}\cdot d\mathbf{r}$, where

$$\boldsymbol{F}(x,y,z) =< x + y^2, y + z^2, z + x^2 >$$

and C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. (Orient C to be counterclockwise when viewed from above.)

Hint: Let S be the part of plane $x+y+z=1$ defined over the triangle of vertices $(0,0)$, $(1,0)$, and $(0,1)$. Then the curve C is the boundary of S. ***Formula***: $\int_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S (\nabla\times\boldsymbol{F})\cdot d\boldsymbol{S}$. *For the surface integral, you may use the last equality in* ***Formula*** *of Problem 7.*

Answer: $\nabla\times\boldsymbol{F} =< -2z,-2x,-2y >$ and $\mathbf{r}_x\times\mathbf{r}_y =< 1,1,1 >$. $G = \iint_D (-2x-2y-2z)\,dA = \iint_D (-2)\,dA = -1$.

9. Use the **Divergence Theorem** to evaluate the flux of $\boldsymbol{F}$ across S, where

$$\boldsymbol{F}(x,y,z) = 12z\,\mathbf{i} + 4y\,\mathbf{j} - 5x\,\mathbf{k}$$

and S is the boundary of of the solid region E enclosed by the paraboloid $z = 1-x^2-y^2$ and the plane $z=0$. ***Formula***: *Divergence Theorem:* $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV$

Ans: 2π

10. **(Unclosed Surface)**. Use the **Divergence Theorem** to evaluate the flux of $\boldsymbol{F}$ across S, where

$$\boldsymbol{F}(x,y,z) = 3018y\,\mathbf{i} + (5x+3y)\,\mathbf{j} + (z-1)\,\mathbf{k}$$

and S is a part of the paraboloid $z = 1-x^2-y^2$, $z\geq 0$.

Hint: Note that S is not a closed surface. First compute integrals over S_1 and $S_2 = S\cup S_1$, where S_1 is the disk $x^2+y^2\leq 1$, $z=0$, oriented downward. ***Formula***: $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS$. *Divergence Theorem:* $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV$.

Answer: $\iint_{S_2} \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV = \iiint_E 4\,dV = 4\iint_{S_1}\int_0^{1-x^2-y^2} dz\,dA = 2\pi$. And $\iint_{S_1} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{S_1} \boldsymbol{F}\cdot\mathbf{n}\,dS = \iint_{S_1}(1-z)\,dS = \pi$, because $\mathbf{n} =< 0,0,-1 >$ and $z=0$ on S_1. Thus $2\pi-\pi=\pi$.

F.1. Formulas for Chapter 16

Line Integrals

Formula 16.109. **(16.17)** *If f is defined on a smooth curve C given by a vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$, $a \le t \le b$, then* ***line integral of*** *f* ***along*** *C is*

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t),z(t))\,\sqrt{(x')^2+(y')^2+(z')^2}\,dt. \quad (16.81)$$

Formula 16.110. **(16.22)** *Let $\boldsymbol{F}$ is a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a \le t \le b$. Then the line integral of $\boldsymbol{F}$ along C is*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \int_C \boldsymbol{F}\cdot \boldsymbol{T}\,ds = \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt. \quad (16.82)$$

The Fundamental Theorem for Line Integrals

Formula 16.111. **(16.25)** *Suppose that $\boldsymbol{F}$ is continuous, and is a* ***conservative*** *vector field; that is, $\boldsymbol{F} = \nabla f$ for some scalar-valued function f. Then*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad (16.83)$$

Note: If $\boldsymbol{F} = \langle P, Q\rangle$ satisfies $P_y = Q_x$ over an open simply-connected domain, then $\boldsymbol{F}$ is conservative.

Green's Theorem

Formula 16.112. **(16.30)** *Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ have* ***continuous partial derivatives*** *on an open region including D, then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \oint_C Pdx + Qdy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \quad (16.84)$$

Surface Integrals

Formula 16.113. **(16.64)** *Suppose the surface S is defined by a vector function $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle,\ \ (u,v)\in D$. Then*

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA. \quad (16.85)$$

Formula 16.114. **(16.65)** *When $z = g(x,y)$, $\mathbf{r}_{\boldsymbol{x}} \times \mathbf{r}_{\boldsymbol{y}} = \langle -\boldsymbol{g_x}, -\boldsymbol{g_y}, 1\rangle$. Thus the formula (16.85) reads*

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA. \quad (16.86)$$

Surface Integrals of Vector Fields

Formula 16.115. (16.70) *Let* $\boldsymbol{F}$ *be a continuous vector field defined on an oriented surface* S *with unit normal vector* $\mathbf{n}$. *The* ***surface integral of*** $\boldsymbol{F} = \langle P, Q, R\rangle$ ***over*** S *is*

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\mathbf{r} &\overset{\text{def}}{=} \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds \\ &= \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,|\mathbf{r}'(t)|\,dt = \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt,\end{aligned} \tag{16.87}$$

Formula 16.116. (16.71) *When the surface* S *is defined by* $z = g(x,y)$, $\mathbf{r}_x\times\mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$ *and*

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dA = \iint_D (-P\,g_x - Q\,g_y + R)\,dA. \tag{16.88}$$

Stokes' Theorem

Formula 16.117. (16.74) *Let* S *be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve* C *with positive orientation. Let* $\boldsymbol{F} = \langle P, Q, R\rangle$ *be a vector field whose components have* ***continuous partial derivatives****. Then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \overset{\text{def}}{=} \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot\mathbf{n}\,dS = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA. \tag{16.89}$$

The Divergence Theorem

Formula 16.118. (16.76) *Let* E *be a simple solid region and* S *be the boundary surface of* E, *given with positive (outward) orientation. Let* $\boldsymbol{F} = \langle P, Q, R\rangle$ *have* ***continuous partial derivatives*** *on an open region that contains* E. *Then*

$$\oiint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV. \tag{16.90}$$

Chapter 17

Optimization Methods

Optimization is the branch of research-and-development that aims to solve the problem of finding the elements which maximize or minimize a given real-valued function, while respecting constraints. Many problems in engineering and machine learning can be cast as optimization problems, which explains the growing importance of the field. An **optimization problem** is the problem of finding **the best solution** from all **feasible solutions**.

In this chapter, we will discuss details about two of common optimization methods:

- Method of Euler-Lagrange equations (variational calculus), and
- Gradient descent method.

Contents of Chapter 17

17.1. Variational Calculus: Euler-Lagrange Equations

Consider the following minimization problem

$$\min_u \int_\Omega |\nabla u|\, d\mathbf{x} \quad \text{subj. to} \quad \|v_0 - u\| = \sigma, \tag{17.1}$$

where $\|v_0 - u\| = \Big(\int_\Omega (v_0-u)^2 d\mathbf{x}\Big)^{1/2}$ and

v_0 : given observed data (image)
u : a desired image to be restored
σ : the **standard deviation** of noise $\eta = v_0 - u$

Introducing a **Lagrange multiplier** λ, the problem (17.1) can be written equivalently as

$$\begin{aligned} \min_u \mathcal{J}(u), \quad \mathcal{J}(u) &\stackrel{\text{def}}{=\!=} \int_\Omega |\nabla u|\, d\mathbf{x} + \frac{\lambda}{2}\|v_0-u\|^2 \\ &= \int_\Omega \Big[|\nabla u| + \frac{\lambda}{2}(v_0-u)^2\Big] d\mathbf{x}. \end{aligned} \tag{17.2}$$

Method of Lagrange multipliers

Recall: Earlier in § 14.8, we considered a problem of the form

$$\min_{\mathbf{x}} f(\mathbf{x}) \quad \text{subj.to} \quad g(\mathbf{x}) = c. \tag{17.3}$$

The problem could be solved by finding $(\mathbf{x}, \lambda)$ such that

$$\boxed{\nabla f(\mathbf{x}) = \lambda \nabla g(\mathbf{x}) \ \text{ and } \ g(\mathbf{x}) = c}. \tag{17.4}$$

(See Strategy 14.85, p. 65.) The first equation of (17.4) can be written as

$$\nabla\,[f(\mathbf{x}) + \lambda g(\mathbf{x})] \;=\; 0, \tag{17.5}$$

which (assuming λ found) is a necessary condition for

$$\min_{\mathbf{x}} \,[f(\mathbf{x}) + \lambda g(\mathbf{x})]\,. \tag{17.6}$$

17.1.1. Total variation

Definition 17.1. *Let f be a differentiable function defined on $\Omega \subset \mathbb{R}^d$. Then the **total variation** (TV) of f over Ω is defined as*

$$TV(f) = \int_\Omega |\nabla f| \, d\mathbf{x}. \tag{17.7}$$

Remark 17.2. Thus the problem in (17.1),

$$\min_u \int_\Omega |\nabla u| \, d\mathbf{x} \quad \text{subj. to} \quad \|v_0 - u\| = \sigma, \tag{17.8}$$

is the problem of finding a solution that minimizes the TV, given the constraint $\|v_0 - u\| = \sigma$. It has been widely used in the field of **image processing**, particularly in **mathematical denoising**.

Example 17.3. Let f be defined on an interval $[a, b]$. Then it follows from (17.7) that

$$TV(f) = \int_a^b |f'(x)| \, dx. \tag{17.9}$$

Problem 17.4. Find the TV of $f(x) = 2x^3 - 3x^2 + 4$ over $[-1, 2]$.

Solution.

Ans: 11

17.1.2. Calculus of variation

For simplicity, we will derive the **Euler-Lagrange equation** for problems in one variable. Consider

$$\min_u \mathcal{J}(u), \quad \mathcal{J}(u) = \int_{x_1}^{x_2} f(x,u,u')dx. \tag{17.10}$$

The Question:

What is the function u that satisfies

$$u(x_1) = u_1 \quad \text{and} \quad u(x_2) = u_2 \tag{17.11}$$

and renders $\mathcal{J}$ in (17.10) a minimum?

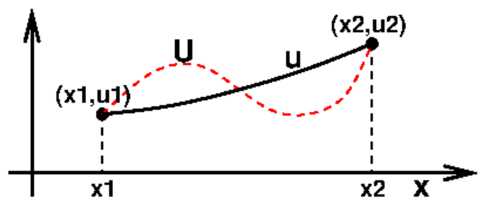


Figure 17.1: The minimizer u and its variation U as a comparison function.

An Answer:

1. Let u be the minimizing function of $\mathcal{J}$ in (17.10).
2. Consider the one-parameter family of ***comparison functions***

   $$U(x) = u(x) + \varepsilon\,\eta(x), \tag{17.12}$$

   where η is an arbitrary differentiable function such that

   $$\eta(x_1) = \eta(x_2) = 0.$$

3. See Figure 17.1. Note

   $$U'(x) = u'(x) + \varepsilon\eta'(x). \tag{17.13}$$

4. Consider

   $$\mathcal{J}(\varepsilon) = \int_{x_1}^{x_2} f(x,U,U')\,dx = \int_{x_1}^{x_2} f(x, u+\varepsilon\eta, u'+\varepsilon\eta')\,dx.$$

   Then, since $U = u$ (the minimizer) when $\varepsilon = 0$, we have

   $$\mathcal{J}'(0) = 0. \tag{17.14}$$

Implication of $\mathcal{J}'(0) = 0$

Differentiating $\mathcal{J}$ with respect to ε reads

$$\frac{d\mathcal{J}}{d\varepsilon} = \mathcal{J}'(\varepsilon) = \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial U}\frac{\partial U}{\partial \varepsilon} + \frac{\partial f}{\partial U'}\frac{\partial U'}{\partial \varepsilon}\Big)dx = \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial U}\,\eta + \frac{\partial f}{\partial U'}\,\eta'\Big)dx. \tag{17.15}$$

Thus,

$$\mathcal{J}'(0) = \int_{x_1}^{x_2} \Big(\frac{\partial f}{\partial u}\,\eta + \frac{\partial f}{\partial u'}\,\eta'\Big)dx = 0. \tag{17.16}$$

Now, apply **integration by parts** (for the second part) to get

$$\begin{aligned} \mathcal{J}'(0) &= \int_{x_1}^{x_2} \frac{\partial f}{\partial u}\,\eta\,dx \; + \frac{\partial f}{\partial u'}\,\eta\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2} \frac{d}{dx}\Big(\frac{\partial f}{\partial u'}\Big)\eta\,dx \\ &= \int_{x_1}^{x_2} \Big[\frac{\partial f}{\partial u} - \frac{d}{dx}\Big(\frac{\partial f}{\partial u'}\Big)\Big]\,\eta\,dx = 0. \end{aligned} \tag{17.17}$$

Since the above holds for arbitrary η, we must have

$$\frac{\partial f}{\partial u} - \frac{d}{dx}\Big(\frac{\partial f}{\partial u'}\Big) = 0, \tag{17.18}$$

which is called the **Euler-Lagrange equation** of $\mathcal{J}$.

Problem 17.5. Let $f(x) = |x|^\alpha$. Find $\dfrac{d\,f}{dx}$. What is $\dfrac{d\,f}{dx}$ when $\alpha = 1$?

Solution. ***Hint***: Begin with $\ln f = \alpha \ln |x|$. Or, define $f_\varepsilon(x) = \Big(\sqrt{x^2+\varepsilon^2}\Big)^\alpha$ and find $\lim\limits_{\varepsilon\to 0} f'_\varepsilon(x)$.

Ans: $\frac{d\,|x|^\alpha}{dx} = \frac{\alpha}{x}|x|^\alpha$; $\frac{d\,|x|}{dx} = \frac{|x|}{x} = \frac{x}{|x|}$

Problem 17.6. In 1D, the objective function $\mathcal{J}$ in (17.2) reads

$$\mathcal{J}(u) = \int_a^b \left[|u_x| + \frac{\lambda}{2}(v_0 - u)^2\right] dx, \tag{17.19}$$

where $u_x = du/dx$. Find its Euler-Lagrange equation.
Solution.

Ans: $-\lambda(v_0 - u) - \left(\frac{u_x}{|u_x|}\right)_x = 0$, or $-\left(\frac{u_x}{|u_x|}\right)_x = \lambda(v_0 - u)$.

Self-study 17.7. Find the Euler-Lagrange equation for

$$\mathcal{J}(u) = \int_a^b \left[(u_x)^2 + \lambda(v_0 - u)^2\right] dx. \tag{17.20}$$

Solution.

Ans: $-u_{xx} = \lambda(v_0 - u)$.

Remark 17.8. In 2D, the objective function (17.2) becomes

$$\mathcal{J}(u) = \int_\Omega \Big[|\nabla u| + \frac{\lambda}{2}(v_0 - u)^2\Big] d\mathbf{x} = \int_\Omega \Big[\sqrt{u_x^2 + u_y^2} + \frac{\lambda}{2}(v_0 - u)^2\Big] d\mathbf{x}; \tag{17.21}$$

its Euler-Lagrange equation reads

$$-\nabla \cdot \Big(\frac{\nabla u}{|\nabla u|}\Big) = \lambda\,(v_0 - u), \tag{17.22}$$

where the left-hand side is called the **curvature**. For a convenient simulation of (17.22), we can parametrize the **energy descent direction** by an artificial time t:

$$\frac{\partial u}{\partial t} - \nabla \cdot \Big(\frac{\nabla u}{|\nabla u|}\Big) = \lambda\,(v_0 - u), \quad u(\mathbf{x}, t = 0) = v_0(\mathbf{x}). \tag{17.23}$$

which is called the ***total variation (TV) model*** [6] in the literature of **image processing**. The stationary solution of (17.23) would show a smaller TV value than v_0.

Example 17.9. The TV model has a tendency to converge to a piecewise constant image, which is call a ***staircasing***.

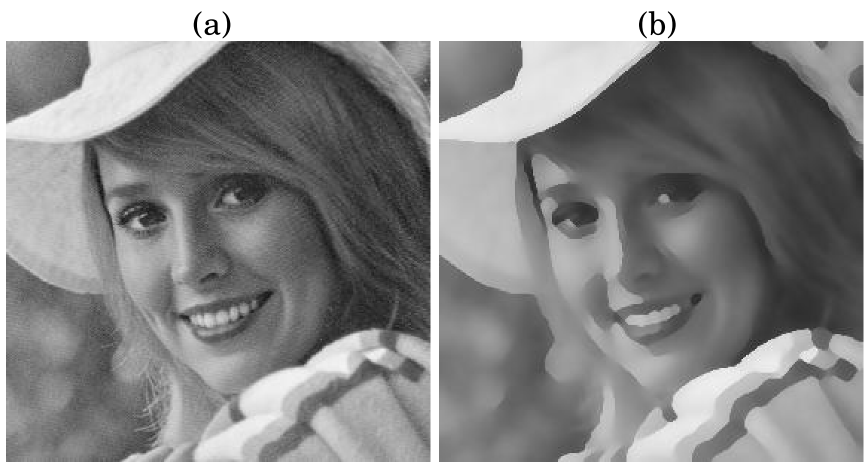


Figure 17.2: Staircasing of the TV model: (a) The original Elaine and (b) its TV result.

Example 17.10. To overcome the staircasing effect, the TV model can be modified as

$$\frac{\partial u}{\partial t} - |\nabla u|\, \nabla \cdot \left(\frac{\nabla u}{|\nabla u|}\right) = \lambda\, |\nabla u|\, (v_0 - u), \tag{17.24}$$

where the Euler-Lagrange equation is scaled by $|\nabla u|$ before applying time parametrization. The above is call the ***improved TV (ITV) model*** [4].

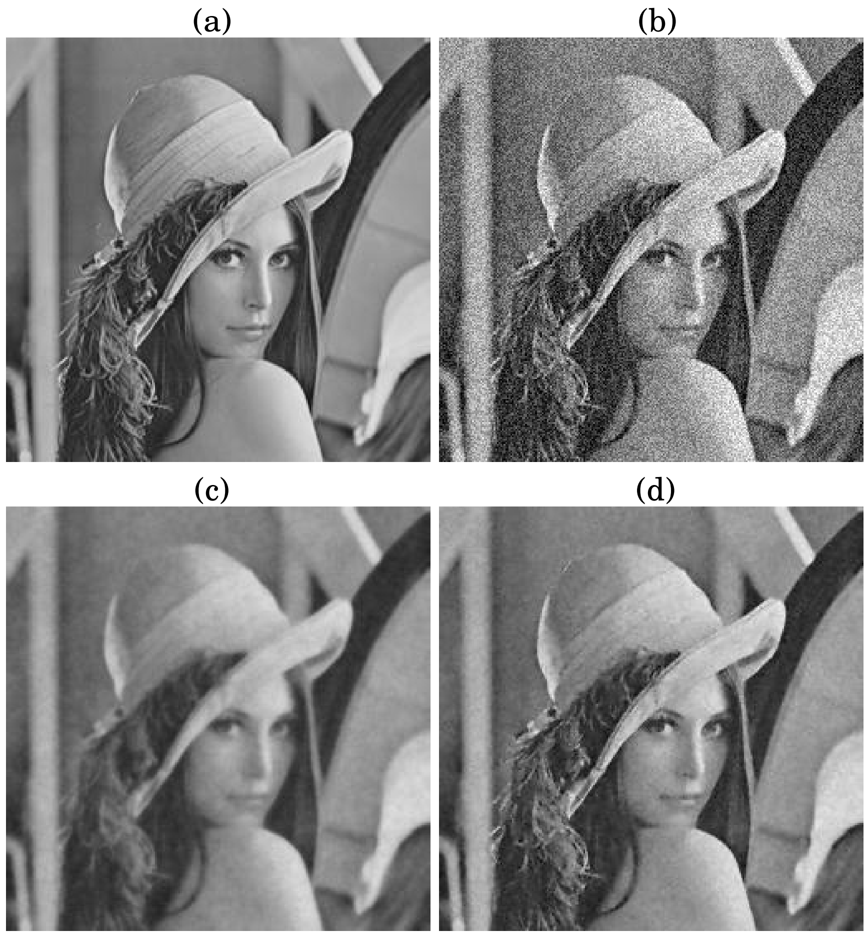


Figure 17.3: Lena: (a) The original image, (b) a noisy image, (c) a restored image by ITV, and (d) a restored image by ITV-END. The ***PSNR*** (peak signal-to-noise ratio) measures 22.8, 27.0, and 30.3 respectively for (b), (c), and (d).

Note: ***END*** stands for ***equalized net diffusion***, which is invented by Kim [2] as another scaling operation incorporated with ITV, in order to preserve fine features of the image more effectively.

17.2. Gradient Descent Method

The first method is one of the oldest methods in optimization: **gradient descent method**, a.k.a **steepest descent method**. The method was suggested by Augustin-Louis Cauchy in 1847 [3]. He was a French mathematician and physicist who made pioneering contributions to mathematical analysis. Today, it is used to solve problems with thousands of variables comfortably.

Figure 17.4: Augustin-Louis Cauchy

Optimization Problem

Let $\Omega \subset \mathbb{R}^d$, $d \geq 1$. Given a real-valued function $f : \Omega \to \mathbb{R}$, the general problem of finding the value that minimizes f is formulated as follows.

$$\min_{\mathbf{x} \in \Omega} f(\mathbf{x}). \tag{17.25}$$

In this context, f is the **objective function**. $\Omega \subset \mathbb{R}^d$ is the **domain** of the function (also known as the **constraint set**).

Problem 17.11. (Revisit of Problem 14.79). Find all local extrema of $f(x, y) = x^4 + y^4 - 4xy + 1$. What is the global minimum, $\min_{(x,y)\in\mathbb{R}^2} f(x, y)$?

Solution.

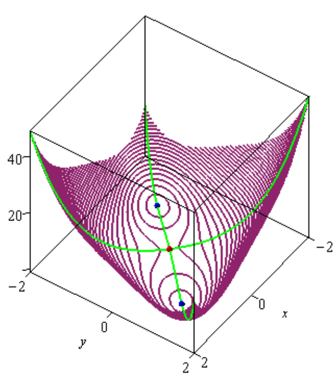


Figure 17.5

Ans: local min at: $(\pm 1, \pm 1)$, saddle point: $(0, 0)$; global min $= -1$

Example 17.12. (Rosenbrock function). For example, the **Rosenbrock function** in the two-dimensional (2D) space is defined as[1]

$$f(x,y) = (1-x)^2 + 100\,(y-x^2)^2. \tag{17.26}$$

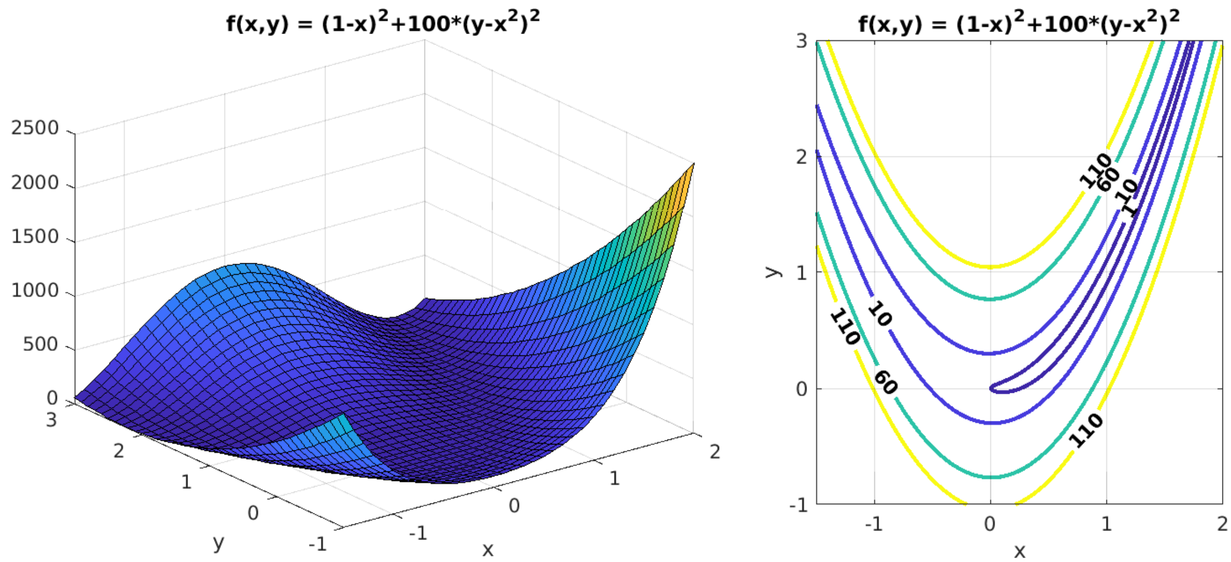


Figure 17.6: Plots of the Rosenbrock function $f(x,y) = (1-x)^2 + 100\,(y-x^2)^2$.

Rosenbrock function

Note: The Rosenbrock function is commonly used when evaluating the performance of optimization algorithms, due to the following reasons.

- Its minimizer `x = np.array([1.,1.])` is found in curved valley, and so minimizing the function is non-trivial, and
- ***Python***: The Rosenbrock function is included in the `scipy.optimize` package (as `rosen`), as well as its gradient (`rosen_der`) and its Hessian (`rosen_hess`).

[1]The Rosenbrock function in 3D is given as $f(x,y,z) = [(1-x)^2 + 100\,(y-x^2)^2] + [(1-y)^2 + 100\,(z-y^2)^2]$, which has exactly one minimum at $(1,1,1)$. Similarly, one can define the Rosenbrock function in general N-dimensional spaces, for $N \ge 4$, by adding one more component for each enlarged dimension. That is, $f(\mathbf{x}) = \sum_{i=1}^{N-1} \big[(1-x_i)^2 + 100(x_{i+1}-x_i^2)^2\big]$, where $\mathbf{x} = [x_1, x_2, \cdots, x_N] \in \mathbb{R}^N$. See Wikipedia (https://en.wikipedia.org/wiki/Rosenbrock_function) for details.

Recall: The ***gradient*** $\boldsymbol{\nabla f}$ is a vector (a direction to move), which is
- pointing in the **direction of greatest increase** of the function, and
- **zero** $(\boldsymbol{\nabla f = 0})$ at local maxima or local minima.

The goal of the ***gradient descent*** (GD) method is to address directly the process of minimizing the function f, using the fact that $-\nabla f(\mathrm{x})$ is the direction of ***steepest descent*** of f at x. Given an initial point x_0, we move it to the direction of $-\nabla f(\mathrm{x}_0)$ so as to get a smaller function value. That is,

$$\mathbf{x}_1 = \mathbf{x}_0 - \gamma\,\nabla f(\mathbf{x}_0) \Rightarrow f(\mathbf{x}_1) < f(\mathbf{x}_0).$$

We repeat this process till reaching at a desirable minimum. Thus the method is formulated as follows.

Gradient descent method

Algorithm 17.13. *Given an initial point* x_0*, find iterates* x_{n+1} *recursively using*

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \gamma\,\nabla f(\mathbf{x}_n), \tag{17.27}$$

for some $\gamma > 0$*. The parameter* γ *is called the* ***step length*** *or* ***learning rate****.* □

To understand the basics of GD method thoroughly, we consider the method for solving unconstrained minimization problems defined in 1D.

17.2.1. The gradient descent method in 1D

Consider the minimization problem in 1D:

$$\min_x\ f(x), \quad x \in S, \tag{17.28}$$

where S is a closed interval in $\mathbb{R}$. Then its gradient descent method reads

$$x_{n+1} = x_n - \gamma\, f'(x_n). \tag{17.29}$$

Picking the step length γ: Assume that the step length was chosen to be independent of n, although one can play with other choices as well. ***The question is how to select γ in order to make the best gain of the method***. To turn the right-hand side of (17.29) into a more manageable form, we invoke Taylor's Theorem:[2]

$$f(x+t) = f(x) + t\, f'(x) + \int_x^{x+t} (x+t-s)\, f''(s)\, ds. \tag{17.30}$$

Assuming that $|f''(s)| \le L$, we have

$$f(x+t) \le f(x) + t\, f'(x) + \frac{t^2}{2}\, L.$$

Now, letting $x = x_n$ and $t = -\gamma\, f'(x_n)$ reads

$$\begin{aligned} f(x_{n+1}) &= f(x_n - \gamma\, f'(x_n)) \\ &\le f(x_n) - \gamma\, f'(x_n)\, f'(x_n) + \frac{1}{2} L\, [\gamma\, f'(x_n)]^2 \\ &= f(x_n) - [f'(x_n)]^2 \Big(\gamma - \frac{L}{2}\gamma^2\Big). \end{aligned} \tag{17.31}$$

The gain (learning) from the method occurs when

$$\gamma - \frac{L}{2}\gamma^2 > 0 \quad \Rightarrow \quad 0 < \gamma < \frac{2}{L}, \tag{17.32}$$

and it will be best when $\gamma - \frac{L}{2}\gamma^2$ is maximal. This happens at the point

$$\boxed{\gamma = \frac{1}{L}} \tag{17.33}$$

It follows from (17.31) and (17.33) that

$$f(x_{n+1}) \le f(x_n) - \frac{1}{2L}[f'(x_n)]^2. \tag{17.34}$$

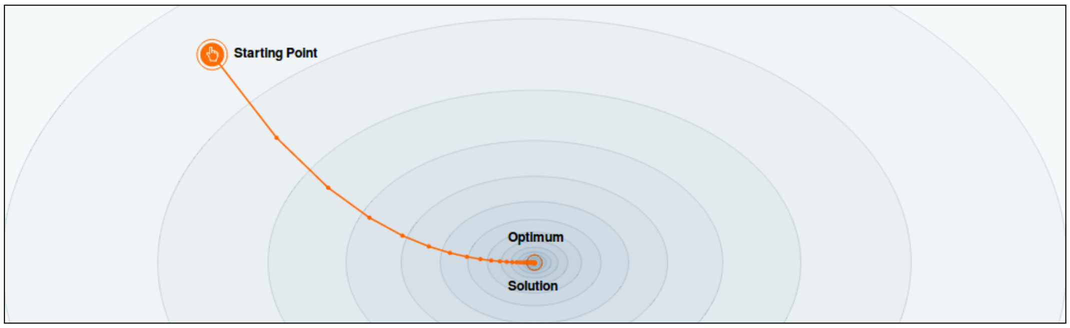
Theorem 17.14. ***(Convergence of GD method)**. If f is bounded from below and the level sets of f are bounded, there is a point $\widehat{x}$ such that*

$$\lim_{n\to\infty} x_n = \widehat{x}, \quad f'(\widehat{x}) = 0. \tag{17.35}$$

[2]**Taylor's Theorem, with integral remainder**: Suppose $f \in C^{n+1}[a,b]$ and $x_0 \in [a,b]$. Then, for every $x \in [a,b]$, $f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + R_n(x)$, $R_n(x) = \frac{1}{n!}\int_{x_0}^{x} (x-s)^n\, f^{(n+1)}(s)\, ds$.

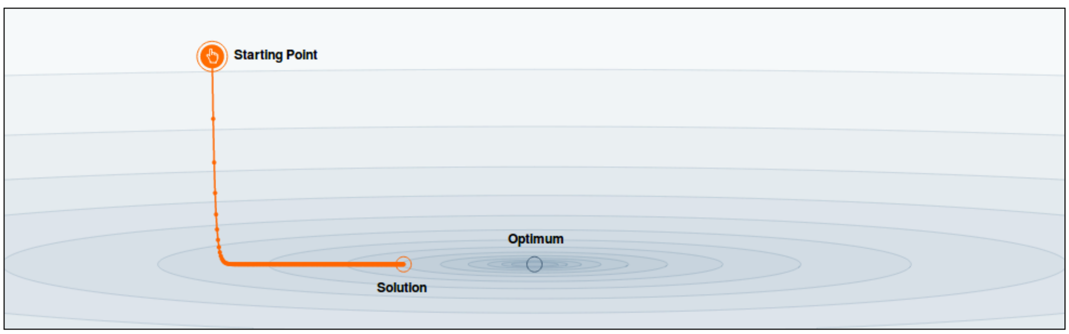
17.2.2. Examples

Here we examine the convergence of gradient descent on three examples: a well-conditioned quadratic, an badly-conditioned quadratic, and a non-convex function, as shown by Dr. Fabian Pedregosa, UC Berkeley.



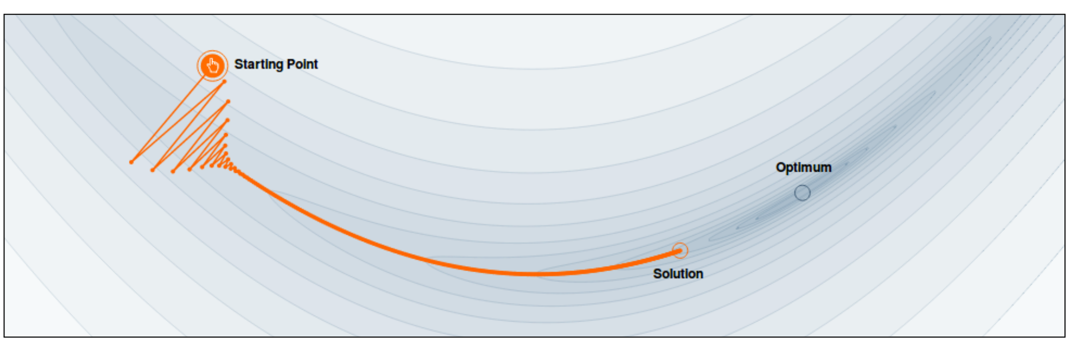
$\gamma = 0.2$

Figure 17.7: On a well-conditioned quadratic function, the gradient descent converges in a few iterations to the optimum



$\gamma = 0.02$

Figure 17.8: On a badly-conditioned quadratic function, the gradient descent converges and takes many more iterations to converge than on the above well-conditioned problem. This is ***partially*** because gradient descent requires a ***much smaller step length*** on this problem to converge.



$\gamma = 0.02$

Figure 17.9: Gradient descent also converges on a badly-conditioned non-convex problem. Convergence is slow in this case.

17.2.3. The choice of step length and line search

The convergence of the gradient descent method can be extremely sensible to the choice of step length. It often requires to choose the step length **adaptively**: the step length would better be chosen small in regions of large variability of the gradient, while in regions with small variability we would like to take it large.

Backtracking line search procedures allow to select a step length depending on the current iterate and the gradient. In this procedure, we select an initial (optimistic) step length γ_n and evaluate the following inequality (known as **sufficient decrease condition**):

$$f(\mathbf{x}_n - \gamma_n \nabla f(\mathbf{x}_n)) \le f(\mathbf{x}_n) - \frac{\gamma_n}{2} \|\nabla f(\mathbf{x}_n)\|^2. \tag{17.36}$$

If this inequality is verified, the current step length is kept. If not, the step length is divided by 2 (or any number larger than 1) repeatedly until (17.36) is verified. To get a better understanding, refer to (17.34) on p. 250, with (17.33).

GD, with Backtracking Line Search

The GD algorithm with backtracking line search becomes

(17.37)

```
input: initial guess x_0, step length γ > 0;
for n = 0, 1, 2, ··· do
    initial step length estimate γ_n;
    while (TRUE) do
        if f(x_n − γ_n ∇f(x_n)) ≤ f(x_n) − (γ_n/2) ||∇f(x_n)||^2
            break;
        else γ_n = γ_n/2;
    end while
    x_{n+1} = x_n − γ ∇f(x_n);
end for
return x_{n+1};
```

The following examples show the convergence of gradient descent with the aforementioned backtracking line search strategy for the step length.

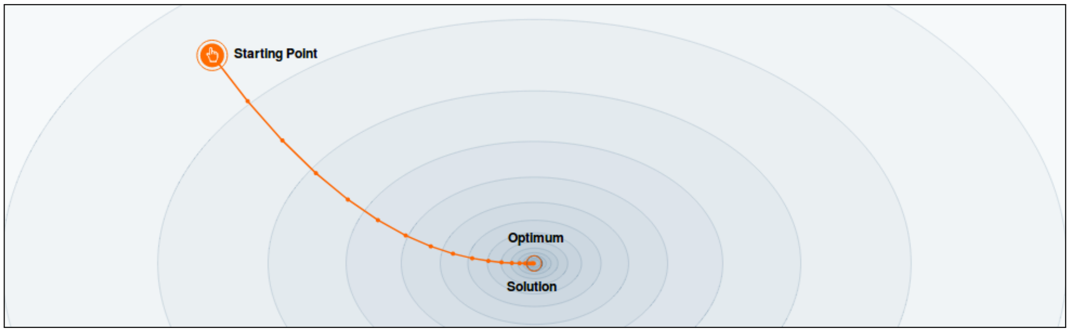


Figure 17.10: On a well-conditioned quadratic function, the gradient descent converges in a few iterations to the optimum. Adding the backtracking line search strategy for the step length does not change much in this case.

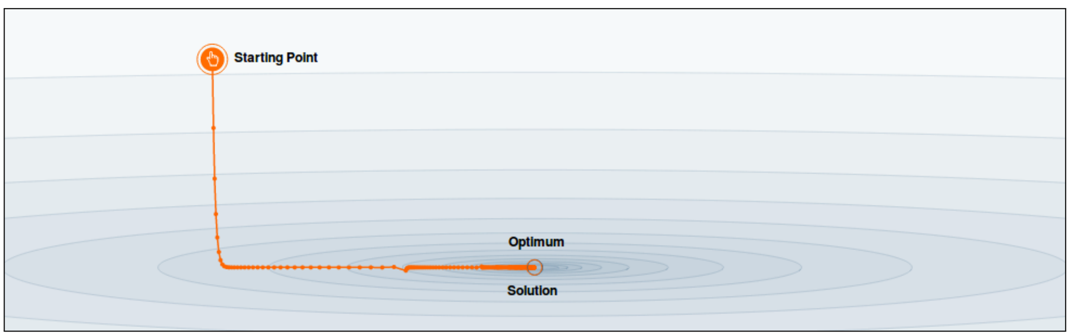


Figure 17.11: In this example we can clearly see the effect of the backtracking line search strategy: once the algorithm in a region of low curvature, it can take larger step lengths. The final result is a much improved convergence compared with the fixed step-length equivalent.

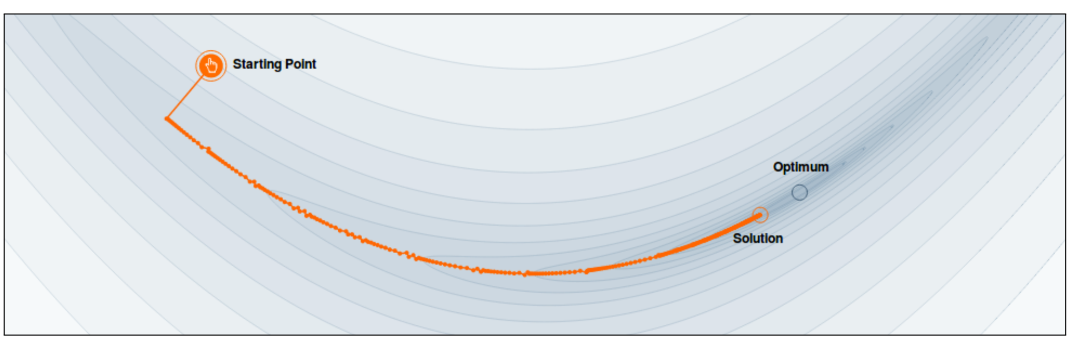


Figure 17.12: The backtracking line search also improves convergence on non-convex problems.

17.2.4. Optimizing optimization

Multiple Local Minima Problem

Remark 17.15. Although you can choose the step length **smartly**, there is no guarantee for your algorithm to converge to the desired solution (the global minimum), particularly when the objective involves multiple local minima.

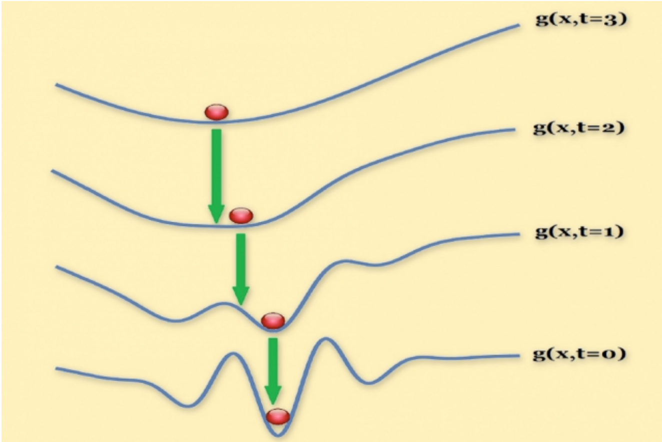


Figure 17.13: Smooth sailing, through a Gaussian smoothing.

- Here, we consider the so-called **Gaussian homotopy continuation** method [5], which may overcome the local minima problem for certain classes of optimization problems.
- The method begins by trying to find a convex approximation of an optimization problem, using a technique called **Gaussian smoothing**.
- Gaussian smoothing converts the cost function into a related function that gives not the value that the cost function would, but a weighted average of all the surrounding values.
- This has the effect of smoothing out any abrupt dips or ascents in the cost function's graph, as shown in Figure 17.13.
- The weights assigned the surrounding values are determined by a Gaussian function, or normal distribution.

Problem 17.16. Let $f(x) = x^4 - \frac{4}{3}x^3 - 4x^2 + 11$. Perform ***two GD iterations*** to estimate $\min_x f(x)$, starting from $x_0 = 1$ and setting $\gamma_n = 0.1/(n+1)$.

Solution. ***Clue***: $f'(x) = 4x^3 - 4x^2 - 8x = 4x(x^2 - x - 2)$ and $\min_x f(x) = f(2) = 1/3$.

Algorithm 17.17. *The above problem is implemented as follows.*

GD.m
```
%%-----------------------------------------------------
%% An Example for Gradient Descent Method
%%-----------------------------------------------------
f  = @(x) x.^4 -4/3*x.^3 -4*x.^2 +11;
df = @(x) 4*x.^3 -4*x.^2 -8*x;

x0 = 1.0;
xn = gd1D(df,x0,tol=1.0e-5,itmax=1000);
    fprintf("min f = %.7f @ xn= %.10f\n",feval(f,xn),xn);
```

gd1D.m
```
function xn = gd1D(df,x0,tol,itmax)
    % function xn = gd1D(df,x0,tol)
    % Input:   df: derivative of f;  x0: initial value
    % Default: gamma = 0.1/n
if nargin==2, tol=1.0e-5; itmax=1000; end

xn = x0;
for n=1:itmax
    gamma = 0.1/n;
    h = gamma*feval(df,xn); xn = xn - h;
    if (abs(h)<tol)
        fprintf('gd1D.m: converged @ n = %d (tol=%g)\n',n,tol); break;
    end
end
```

The Result
```
[Sat Oct.26] octave GD.m
gd1D.m: converged @ n = 8 (tol=1e-05)
min f = 0.3333333 @ xn= 2.0000182623
```

Exercises 17.2

By solving problems below, you will learn numerical characteristics of GD, particularly the importance of ***initial values*** (x_0) and ***step lengths*** (γ_n).

1. CAS Implement GD as in Algorithm 17.17 in **MATLAB**[3] and test it as follows.
   (a) Run `GD.m` with {`x0=3.0; itmax=10;`}.
   (b) Plot $y = f(x)$ with the iterates x_n, $n = 1, \cdots, 10$, being located on the x-axis.
   (c) First edit `gd1D.m` to replace `gamma=0.1/n` with `gamma=0.1/(n+3)` and then run `GD.m` again with {`x0=3.0; itmax=10;`}.
   (d) Plot $y = f(x)$ again with the iterates x_n, $n = 1, \cdots, 10$, being located on the x-axis.
   (e) Discuss your experiments, focusing on the importance of initial values (x_0) and step lengths (γ_n).

2. CAS Now, consider the last four digits of your student ID (say, $a\,b\,c\,d$). Let

$$g(x) = \widehat{a}\,x^4 - b\,x^3 - c\,x^2 + d,$$

   where $\widehat{a} = \max(a, 1)$.
   (a) Plot $y = g(x)$.
   (b) Examine the figure to find an accurate initial value x_0 for solving $\min_x g(x)$.
   (c) Edit `gd1D.m` to set `gamma` for GD to converge as fast as possible.
   (d) Report your experiments.

[3]***MATLAB*** (matrix laboratory) is a multi-paradigm numerical computing environment and proprietary programming language developed by MathWorks.

APPENDIX **A**

Review for 12 Selected Sections

In this appendix, we will review the following 12 sections by considering major concepts and representing problems.

Contents of Appendix A

A.1. (§14.4) Tangent Planes and Linear Approximations

Definition A.1. *Given $z = f(x, y)$, the **linear (tangent plane) approximation of f near (a, b)** is*

$$L(x, y) \equiv z_0 + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b), \tag{A.1}$$

where $z_0 = f(a, b)$.

Note: The equation of the **tangent plane** is

$$z - z_0 = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b),$$

or equivalently
$$-f_x(a, b)\,(x - a) - f_y(a, b)\,(y - b) + (z - z_0) = 0. \tag{A.2}$$

A level surface form of $z = f(x, y)$ can be rewritten as

$$F(x, y, z) = z - f(x, y) = 0;$$

its gradient beclomes
$$\nabla F = \langle -f_x,\, -f_y,\, 1\rangle. \tag{A.3}$$

Preveal A.2. (§ 16.6. Parametric Surfaces and Their Areas): Let a surface S be formed by the graph of $z = f(x, y)$ and parametrized by $\mathbf{r}(x, y) = \langle x,\, y,\, f(x, y)\rangle$. Then

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x,\, -f_y,\, 1\rangle; \tag{A.4}$$

see (16.62) on p. 210.

Theorem A.3. *If f_x and f_y **exist** near (a, b) and **continuous** at (a, b), then f is **differentiable** at (a, b).*

Definition A.4. *For a differentiable function $z = f(x, y)$, the **(total) differential** is*

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy, \tag{A.5}$$

where dx and dy represent the change in the x and y directions, respectively.

Problem A.5. Find an equation for the tangent plane to the elliptic paraboloid $z = x^2 + 4y^2$ at the point $(1, 1, 5)$.

Solution.

Ans: $z - 5 = 2 \cdot (x - 1) + 8 \cdot (y - 1) \Leftrightarrow z = 2x + 8y - 5.$

Problem A.6. Let $f(x, y) = \ln(x + 1) + \cos(x/y)$. Explain why the function is differentiable at $(0, 2)$.

Problem A.7. Use a linear approximation to estimate $f(2.2, 4.9)$, provided that $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$.

Solution.

Ans: 6.3

A.2. (§14.6) Directional Derivatives and Gradient Vector

Claim A.8. *For a unit vector* $\mathbf{u}$*, the* ***directional derivative*** *for a differential function* f *is*

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot \mathbf{u}.$$

Theorem A.9. *Let* f *be differentiable. Then,*

$$\max_{\mathbf{u}} D_{\mathbf{u}}f = |\nabla f| \tag{A.6}$$

Note: The gradient vector ∇f is directing the fastest increasing direction.

Tangent Plane and Normal Line to a Level Surface

Suppose S is a surface given as $F(x,y,z) = k$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ is on S. Then the **tangent plane** to S at $\mathbf{x}_0$ is

$$\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = F_x(\mathbf{x}_0)(x-x_0) + F_y(\mathbf{x}_0)(y-y_0) + F_z(\mathbf{x}_0)(z-z_0) = 0. \tag{A.7}$$

The **normal line** to S at $\mathbf{x}_0$ is

$$\frac{x-x_0}{F_x(\mathbf{x}_0)} = \frac{y-y_0}{F_y(\mathbf{x}_0)} = \frac{z-z_0}{F_z(\mathbf{x}_0)}. \tag{A.8}$$

Problem A.10. Let $f(x,y) = x + \sin(xy)$.

1. Find the directional derivative of f at the point $(1,0)$ in the direction given by the angle $\theta = \pi/3$.
2. In what direction does f have the maximum rate of change? What is the maximum rate of change?

Solution.

Ans: (a) $(1+\sqrt{3})/2$ (b) $\sqrt{2}$

Problem A.11. Find the equations of the tangent plane and the normal line at $P(0,0,1)$ to $x+y+z=e^{xyz}$.

Solution.

Ans: (a) $x+y+z=1$ (b) $x=y=z-1$

A.3. (§14.8) Lagrange Multipliers

Consider the optimization problem

$$\left[\begin{array}{l}\max / \min f(\mathbf{x}) \\ \quad \text{subject to } \underline{g(\mathbf{x}) = c}\end{array}\right.$$

Strategy A.12. (Method of Lagrange multipliers). For the max/min values of the optimization problem,

(a) Find all values of $x, y, z,$ and λ such that

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \text{ and } g(x,y,z) = c.$$

(b) Evaluate f at all these points, to find the maximum and minimum.

Example A.13. Use Lagrange Multipliers to prove that the rectangle with maximum area that has a give perimeter p is a square.

Problem A.14. Find the maximum and minimum values of $f(x, y) = 2x^2 + (y-1)^2$ on the **circle** $x^2 + y^2 = 4$.

Solution. $\nabla f = \lambda \nabla g \Longrightarrow \begin{bmatrix} 4x \\ 2(y-1) \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Therefore, $\begin{cases} 2x = \lambda\, x & \text{①} \\ y - 1 = \lambda\, y & \text{②} \\ x^2 + y^2 = 4 & \text{③} \end{cases}$

From ①, $x = 0$ or $\lambda = 2$.

Ans: min: $f(0, 2) = 1$; max: $f(\pm\sqrt{3}, -1) = 10$

Problem A.15. Find the maximum and minimum values of $f(x, y) = 2x^2 + (y-1)^2$ on the **disk** $x^2 + y^2 \le 4$.

Solution. ***Hint***: You should check values at **critical points** as well.

Ans: min: $f(0, 1) = 0$; $f(\pm\sqrt{3}, -1) = 10$

A.4. (§15.2) Double Integrals over General Regions

Multiple integrals can be computed with **iterated integral** where the given domain must be covered ***once-and-only-once***, without missing and without overlap. Furthermore, you should be able to change the order of integration properly.

Problem A.16. (Problem 15.16). Find the volume of the solid that lies under the plane $z = 1 + 2y$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. Try for both orders.

Ans: $28/5$

Problem A.17. Evaluate the integral by reversing the order of integration:

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$$

Solution.

Ans: $\frac{1}{3}(e^8 - 1)$

Self-study A.18. Sketch the region of integration and change the order of integration.

$$\int_{-2}^2 \int_0^{\sqrt{4-y^2}} f(x, y)\, dx\, dy \qquad \int_0^{\ln 2} \int_{e^y}^2 f(x, y)\, dx\, dy$$

Solution.

A.5. (§15.7) Triple Integrals in Cylindrical Coordinates

Definition A.19. (Definition 15.53). *The conversion between the **Cylindrical Coordinates** and the Rectangular Coordinate system gives*

$(x, y, z)_R \leftarrow (r, \theta, z)_C$	$(r, \theta, z)_C \leftarrow (x, y, z)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

(A.9)

Note: The triple integral with a Cylindrical Domain E can be carried out by first separating the domain like

$$E = D \times [u_1(x, y), u_2(x, y)], \quad \text{where } D \text{ is a polar region}.$$

Problem A.20. Evaluate $\iiint_E y\,dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$, above the xy-plane, and below the plane $z = y + 3$.
Solution.

Ans: 20π

Self-study **A.21.** Use the cylindrical coordinates to find the volume of the solid E that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 8$.
Solution.

Ans: $\frac{32}{3}\pi(\sqrt{2} - 1)$

A.6. (§15.9) Change of Variables in Multiple Integrals

Definition A.22. *A change of variables is a* ***transformation*** $T : Q \to R$ *(from the uv-plane to the xy-plane),* $T(u,v) = (x,y)$*, where x and y are related to u and v by the equations*

$$x = g(u,v), \quad y = h(u,v). \qquad [\textbf{or}, \ \mathbf{r}(u,v) = \langle g(u,v), h(u,v)\rangle]$$

We usually take these transformations to be C^1***-Transformation****, meaning g and h have continuous first-order partial derivatives, and* ***one-to-one****.*

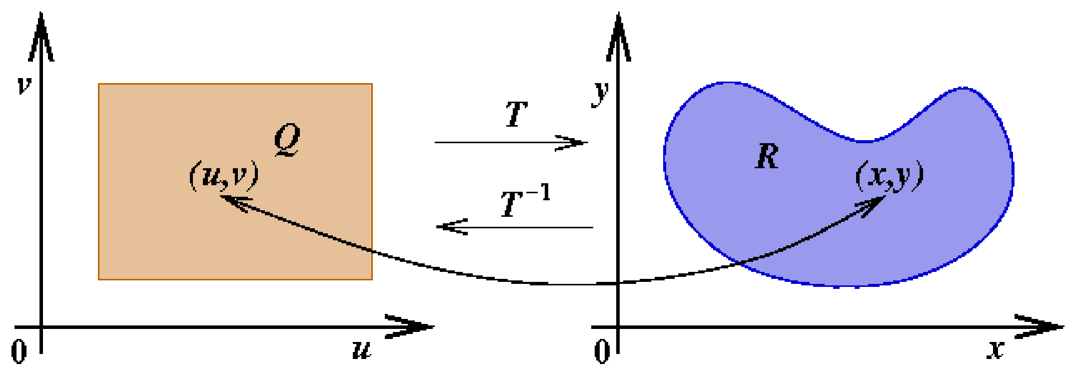


Figure A.1: Transformation: $R = T(Q)$, the **image** of T.

Definition A.23. *The* ***Jacobian of*** $T : x = g(u,v),\ y = h(u,v)$ *is*

$$\frac{\partial(x,y)}{\partial(u,v)} \stackrel{\text{def}}{=\!=} \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u\, y_v - x_v\, y_u. \tag{A.10}$$

Claim A.24. *Suppose $T : Q \to R$ is an one-to-one C^1 transformation whose Jacobian is nonzero. Then*

$$\iint_R f(x,y)\, dA = \iint_Q f(x(u,v), y(u,v)) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\, dv \tag{A.11}$$

Note: In **linear algebra**, an $n \times n$ matrix A is considered as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Furthermore its **determinant** can be viewed as a *volume scaling factor*. For details, see Section 3.1 of *Introduction to Linear Algebra*:
https://skim.math.msstate.edu/LectureNotes/Linear_Algebra_LectureNote.pdf.

Problem A.25. Make an appropriate change of variables to evaluate the integral

$$\iint_R \sin(9x^2 + 4y^2)\, dA,$$

where R is the region in the **first quadrant** bounded by the ellipse $9x^2 + 4y^2 = 1$.
Solution.

Ans: $\frac{\pi(1-\cos 1)}{24}$

A.7. (§16.2) Line Integrals

Definition A.26. *If f is defined on a smooth curve C given by*

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle, \quad a \le t \le b, \tag{A.12}$$

*then **line integral of f along C** is*

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i, \tag{A.13}$$

if this limit exists. Here $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$.

The line integral defined in (A.13) can be evaluated as

$$\begin{aligned} \textstyle\int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\,\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\ &= \int_a^b f(x(t),y(t))\,|\mathbf{r}'(t)|\,dt. \end{aligned} \tag{A.14}$$

Definition A.27. *Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t), a \le t \le b$. Then the **line integral of $\boldsymbol{F}$ along C** is*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds = \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt. \tag{A.15}$$

*We say that work is the line integral with respect to arc length of the **tangential component** of force.*

Problem A.28. Evaluate the line integral $\int_C x\, e^{y^2 - z^2}\, ds$, where C is the line segment from $(0, 0, 0)$ to $(2, -2, 1)$.
Solution.

Ans: $e^3 - 1$

Problem A.29. Find the work done by the vector field $\boldsymbol{F}(x, y) = \langle x, ye^x \rangle$ on the particle that moves along the parabola $x = y^2 + 1$ from $(1, 0)$ to $(2, 1)$.
Solution.

Ans: $\frac{3}{2} + \frac{e^2 - e}{2}$

A.8. (§16.3) The Fundamental Theorem for Line Integrals

Let C be a curve represented by

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle \quad \text{or} \quad \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$$

Theorem A.30.

1. *Suppose that $\boldsymbol{F}$ is continuous, and is a* ***conservative vector field****; that is, $\boldsymbol{F} = \nabla f$ for some f. Then*

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{A.16}$$

2. *$\int_C \boldsymbol{F} \cdot d\mathbf{r}$ is* ***independent of path*** *in D if and only if $\int_C \boldsymbol{F} \cdot d\mathbf{r} = 0$ for every closed path in D.*
3. *Suppose $\boldsymbol{F}$ is a vector field that is continuous on an* ***open connected*** *domain D. If $\int_C \boldsymbol{F} \cdot d\mathbf{r}$ is* ***independent of path*** *in D, then $\boldsymbol{F}$ is* ***conservative*** *(i.e., there is f such that $\boldsymbol{F} = \nabla f$).*
4. *If $\boldsymbol{F} = \langle P, Q\rangle$ is conservative, where P and Q have continuous partial derivatives, then*

$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{A.17}$$

5. *When D is a simply-connected domain, the equality (A.17) implies conservativeness of $\boldsymbol{F}$.*

Roughly speaking: When $\boldsymbol{F} = \langle P, Q\rangle$ is smooth enough,

conservativeness ⇔ independence of path ⇔ $Q_x = P_y$

Problem A.31. Find the work done by

$$\boldsymbol{F} = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$$

in moving an object from $A(1,1)$ to $B(2,4)$.
Solution. First, check if $\boldsymbol{F}$ is conservative: $Q_x = 3\sqrt{y},\ \ P_y = 2\cdot\frac{3}{2}y^{1/2} = 3\sqrt{y}$.

Ans: 30

Problem A.32. Given $\boldsymbol{F}(x,y) = \langle e^y + y\cos x,\ xe^y + \sin x\rangle$,

(a) Find a potential.
(b) Evaluate $\int_C \boldsymbol{F}\cdot d\mathbf{r}$, where C is parameterized as

$$\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t\rangle,\ \ 0\le t\le \pi.$$

Solution.

Ans: (a) $f(x,y) = xe^y + y\sin x + K$ (b) $-e^{\pi} - 1$

A.9. (§16.4) Green's Theorem

Theorem **A.33.** ***(Green's Theorem)***. *Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ has **continuous partial derivatives** on an open region including D, then*

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{A.18}$$

The theorem gives the following formulas for the area of D:

$$A(D) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx \tag{A.19}$$

Problem **A.34.** Evaluate $\oint_C \boldsymbol{F}\cdot d\mathbf{r}$, where $\boldsymbol{F} = \langle e^{-x} + y^2, e^{-y} + x^2 + 2xy\rangle$ and C is the circle $x^2 + (y-1)^2 = 1$ oriented clockwise.

Solution. ***Hint***: Check the orientation of the curve.

Ans: 0

Problem A.35. Use the identity (an example of Green's Theorem)

$$A(D) = \iint_D dA = \oint_{\partial D} x\, dy$$

to show that the area of D (the shaded region) is 6. You have to compute **the line integral for each of four line segments** of the boundary. For the slant line segment, in particular, you should introduce an appropriate parameterization for the line integral.

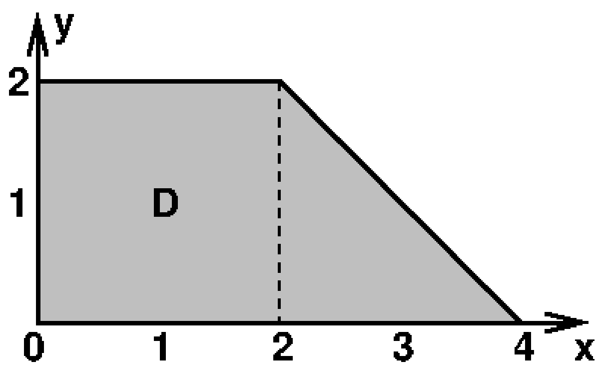


Figure A.2

A.10. (§16.7) Surface Integrals

Suppose that the surface S has a parametric representation

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v) \in D.$$

Then, surface integrals of **scalar functions** give

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA. \tag{A.20}$$

Remark A.36.

- $dS = |r_u \times r_v|\,dA$.
- *For line integrals,* $\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$.
- *When* $z = g(x,y)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$. *Thus the formula (16.64) reads*

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\,dA. \tag{A.21}$$

Surface Integrals of Vector Fields. Let r be a parametric representation of S, from $D \subset \mathbb{R}^2$. The flux **across the surface** S can be measured by

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} &\overset{\text{def}}{=\!=} \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS \\ &= \iint_D \boldsymbol{F}(\mathbf{r})\cdot\left(\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right)|\mathbf{r}_u\times\mathbf{r}_v|\,dA \\ &= \iint_D \boldsymbol{F}(\mathbf{r})\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.\end{aligned} \tag{A.22}$$

Note that $\boldsymbol{F}\cdot\mathbf{n}$ and $\boldsymbol{F}(\mathbf{r})\cdot(\mathbf{r}_u\times\mathbf{r}_v)$ are scalar functions.

Remark A.37. Line integrals of vector fields is defined to measure quantities **along the curve**. That is, for C parametrized by $\mathbf{r} : [a, b] \to C$,

$$\begin{aligned} \int_C \boldsymbol{F} \cdot d\mathbf{r} &\overset{\text{def}}{=\!=} \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds \\ &= \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt = \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \end{aligned} \tag{A.23}$$

Surfaces defined by $z = g(x, y)$:

- A vector representation: $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$.
- Normal vector: $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$.
- Thus, when $\boldsymbol{F} = \langle P,\, Q,\, R\rangle$,

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \boldsymbol{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA = \iint_D (-P\, g_x - Q\, g_y + R)\, dA. \tag{A.24}$$

Problem A.38. Evaluate $\iint_S (x^2 + y^2 + z)\, dS$, where S is the surface whose side S_1 is given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \le 1$ in the plane $z = 0$, and whose top S_3 is the disk $x^2 + y^2 \le 1$ in the plane $z = 1$.

Solution. ***Hint***: Use (A.19). ***Clue***: $S_1 : x = \cos\theta,\ y = \sin\theta,\ z = z;\ (\theta, z) \in D \equiv [0, 2\pi] \times [0, 1]$. Then $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$.

Ans: $3\pi + \frac{1}{2}\pi + \frac{3}{2}\pi = 5\pi$

Problem A.39. Find the flux of $\boldsymbol{F} = \langle x, y, 1\rangle$ across a upward helicoid: $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 2,\ \ 0 \le v \le \pi$.

Solution. ***Hint***: Use (A.22). ***Clue***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.

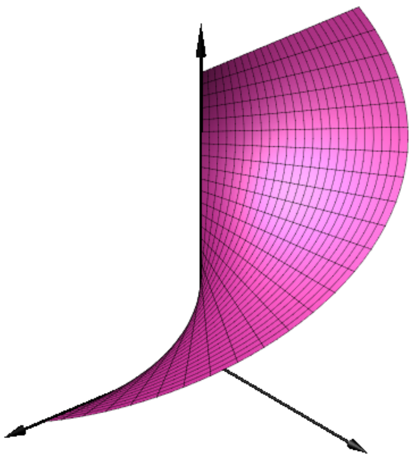


Figure A.3

Ans: 2π

Problem A.40. Evaluate $\iint\limits_S \boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F}=\langle x,y,z\rangle$ and S is the boundary of the solid region E enclosed by the paraboloid $z=1-x^2-y^2$ and the plane $z=0$.

Solution. ***Clue***: For S_1 (the upper part), use the formula in (A.24). For S_2 (the bottom: $z=0$), you may try to get $\boldsymbol{F}\cdot\mathbf{n}$, where $\mathbf{n}=-\mathbf{k}$.

Ans: $\frac{3\pi}{2}+0=\frac{3\pi}{2}$

A.11. (§16.8) Stokes's Theorem

Stokes's Theorem is a high-dimensional version of Green's Theorem studied in § 16.4.

Recall: **(Green's Theorem, p. 274).** Let C be positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ have **continuous partial derivatives** on an open region including D, then

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \oint_C P\,dx + Q\,dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot \mathbf{k}\,dA. \tag{A.25}$$

Theorem A.41. (Stokes's Theorem) *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a vector field whose components have* ***continuous partial derivatives*** *on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \tag{A.26}$$

Remark A.42.

- *See Figure 16.29(left) on p. 216, for an oriented surface of which the boundary has positive orientation.*
- ***Computation*** *of the surface integral: for $\mathbf{r} : D \to S$,*

$$\iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \overset{\text{def}}{=\!=} \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot \mathbf{n}\,dS = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot (\mathbf{r}_u \times \mathbf{r}_v)\,dA. \tag{A.27}$$

- *Green's Theorem is a special case in which S is flat and lies on the xy-plane ($\mathbf{n} = \mathbf{k}$). Compare the last terms in (A.25) and (A.27).*

Try to solve problems in Section 16.8, once more.

A.12. (§16.9) The Divergence Theorem

Theorem A.43. (Divergence Theorem) *Let E be a simple solid region and S be the boundary surface of E, given with positive (outward) orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ have* ***continuous partial derivatives*** *on an open region that contains E. Then*

$$\oiint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV. \tag{A.28}$$

Problem A.44. Use the Divergence Theorem to evaluate the (total) flux $\iint_S \boldsymbol{F}\cdot dS$, where

$$\boldsymbol{F}(x,y,z) = (x + y^2 + \cos z)\,\mathbf{i} + [\sin(\pi z) + xe^{-z}]\,\mathbf{j} + z\,\mathbf{k}$$

and S is a part of the cylinder $x^2+y^2=4$ that lies between $z=0$ and $z=1$.

Solution.

Ans: 8π

Problem A.45. Use the Divergence Theorem to evaluate

$$\iint_S \left(x^2 + y\,\sin x + \frac{z^2}{2}\right) dS,$$

where S is the unit sphere $x^2 + y^2 + z^2 = 4$.

Solution. ***Clue***: What is n?

Ans: 32π

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