Calculus Lectures

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Seongjai Kim

Department of Mathematics and Statistics
Mississippi State University
Mississippi State, MS 39762 USA
Email: skim@math.msstate.edu

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Seongjai Kim, Professor of Mathematics, Department of Mathematics and Statistics, Mississippi State University, Mississippi State, MS 39762 USA. Email: skim@math.msstate.edu.

Prologue

This lecture note is closely following two books:

- Stewart's Calculus, 8th Ed. [3]
- Thomas' Calculus, 15th Ed. [4]

In organizing the part of *Vector Calculus*, I am indebted by *Cedar Crest College Calculus IV Lecture Notes*, Dr. James Hammer [2].

Projects. Several projects are included for students to experience **computer algebra**. Computer algebra (also called **symbolic computation**) is a scientific area that refers to the study and development of algorithms and software for manipulating mathematical expressions and other mathematical objects; it emphasizes **exact computation** with expressions containing variables that have no given value and are manipulated as symbols. In practice, you can use a **computer algebra system** (**CAS**) to effectively handle complex math equations and problems that would be simply too complicated/time-consuming to do by hand. The projects are organized using **Matlab** or **Maple**.

Programming Scripts. Also added are some of programming scripts in Matlab/Maple. The end of each section includes exercise problems. For problems indicated by the CAS sign CAS, you are recommended to use a CAS to solve the problem.

Currently the lecture note is not fully grown up; other useful techniques and interesting examples would be soon incorporated. Any questions, suggestions, comments will be deeply appreciated.

Seongjai Kim
April 22, 2024

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CHAPTER 1

Functions

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Contents of Chapter 1

1.1. Functions and Their Graphs

Introduction. Functions are a tool for describing the real world in mathematical terms. A **function** can be represented by **an equation, a graph, a numerical table, or a verbal description**; we will use all four representations throughout the lecture note. This section reviews these ideas.

1.1.1. Functions; Domain and Range

Definition 1.1. A **function** f from a set D to a set Y is a rule that **assigns a unique value** $f(x) \in Y$ to each $x \in D$. That is,

$$\begin{aligned} f:\ & D \to Y \\ & x \mapsto f(x), \quad \text{where } f(x) \text{ is a unique value for } x \end{aligned} \tag{1.1}$$

- A function produces the same output for the same input.
- Let y be the output value of f at x. Then,

  $$y = f(x),$$

  where x is the **independent variable**, representing the input value to f, and y is the **dependent variable** or **output value of** f at x.

Definition 1.2. The set D of all possible input values is called the **domain** of the function. The set of all output values $f(x)$, as x varies throughout D, is called the **range** of the function. The range might not include every element in the set Y.

- For a function $y = f(x)$, if the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the function gives real y-values. This is called the **natural domain** of f.

Example 1.3. Find the natural domain and associated range of each functions. The domain in each case is the values of x for which **the formula makes sense.**

(a) $y = x^2$ (b) $y = 1/x$ (c) $y = \sqrt{1 - x^2}$

Solution.

1.1.2. Graphs of Functions

Definition 1.4. If f is a function with domain D, its **graph** consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is

$$\boldsymbol{G}(f) := \{(x, f(x)) \mid x \in D\}. \tag{1.2}$$

Example 1.5. Graph the functions

(a) $y = x^2$ over $[-2, 2]$ (b) $y = 1/x$ over $[-2, 2]$ (c) $y = \sqrt{1 - x^2}$

Solution.

Representing a Function Numerically

Note: We have seen how a function may be represented **algebraically by a formula** and **visually by a graph**.

- Another way to represent a function is **numerically**, **through a table of values**.
- From an appropriate table of values, a graph of the function can be obtained, possibly with the aid of a computer. The graph consisting of only the points in the table is called a **scatterplot**.

Example 1.6.

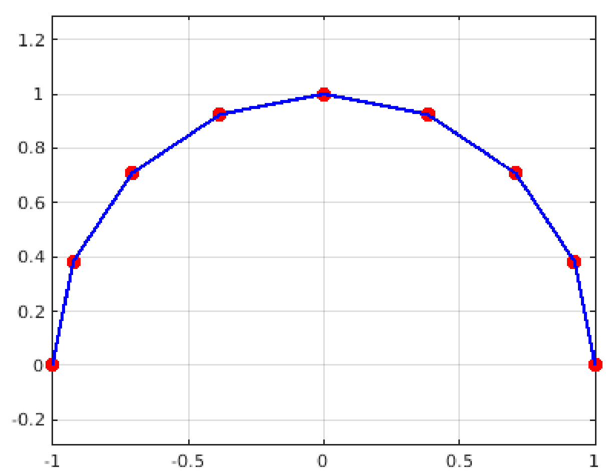
A Matlab code

```
a=0; b=pi;
n=8; h=(b-a)/n; angle = (a:h:b)';

% y = sqrt(1-x^2)
X = cos(angle); Y = sin(angle);
Data = [X Y]

plot(X,Y,'r.','MarkerSize',30);
hold on; axis equal; grid on
plot(X,Y,'b-','LineWidth',2);
hold off

print -dpng 'plot-Data.png';
```

Data set

```
        x           y
   -------------------
     1.0000           0
     0.9239      0.3827
     0.7071      0.7071
     0.3827      0.9239
     0.0000      1.0000
    -0.3827      0.9239
    -0.7071      0.7071
    -0.9239      0.3827
    -1.0000      0.0000
```

A **Matlab** code generated a data set in a table form. The data can be visualized using a computer program.



1.1.3. The Vertical Line Test for a Function

Claim 1.7. Vertical Line Test

- Not every curve in the coordinate plane can be the graph of a function.
- A function f can have only one value $f(x)$ for each x in its domain, so **no vertical line** can intersect the graph of a function more than once.
- If a is in the domain of the function f, then the vertical line $x = a$ will intersect the graph of f at the single point $(a, f(a))$.

Example 1.8. Are they functions?

(a) Circle $x^2 + y^2 = 1$ (b) Upper semicircle (c) Lower semicircle

Solution.

Ans: (a) no. (b) yes. (c) yes.

Example 1.9. Derive an equation of the circle centered at (h, k) and having radius r.

Solution.

1.1.4. Various Function Types

Piecewise-Defined Functions

Example 1.10. Graph the piecewise-defined functions.

(a) $f(x) = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$ (b) $g(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 0 \end{cases}$

Solution.

Example 1.11.

- The function whose value at any number x is *the greatest integer less than or equal to* x is called the **greatest integer function** or the **integer floor function**. e.g., $\lfloor \mathbf{2.3} \rfloor = \mathbf{2}, \ \lfloor \mathbf{1.0} \rfloor = \mathbf{1}, \ \lfloor \mathbf{-1.5} \rfloor = \mathbf{-2}.$
- The function whose value at any number x is *the smallest integer greater than or equal to* x is called the **least integer function** or the **integer ceiling function**. e.g., $\lceil \mathbf{2.3} \rceil = \mathbf{3}, \ \lceil \mathbf{1.0} \rceil = \mathbf{1}, \ \lceil \mathbf{-1.5} \rceil = \mathbf{-1}.$

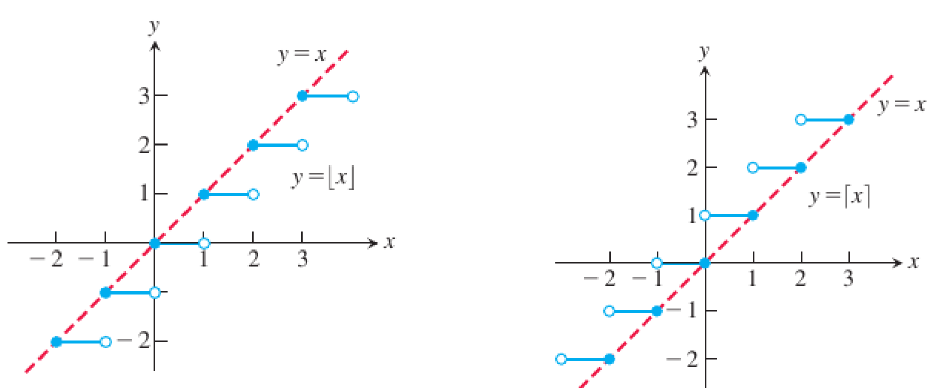


Figure 1.1: The greatest integer function (left) and the least integer function (right).

Increasing and Decreasing Functions

Definition 1.12. Let f be a function defined on an interval I and let x_1 and x_2 be two distinct points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_1) > f(x_2)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

Note: It is important to realize that the definitions of increasing and decreasing functions must be satisfied for every pair of points x_1 and x_2 in I with $x_1 < x_2$. Because we use the inequality $<$ to compare the function values, instead of $\leq$, it is sometimes said that f is **strictly increasing** or **strictly decreasing** on I.

Example 1.13. For each of given functions, find regions on which it is either increasing or decreasing.

(a) $y = x^2$

(b) $y = x^3$

Solution.

Ans: (a) $(-\infty, 0]$: decreasing; $[0, \infty)$: increasing

Even Functions and Odd Functions: Symmetry

Definition 1.14. A function $y = f(x)$ is an

$$\begin{array}{lll} \textbf{even} \text{ function of } x & \text{if} & f(-x) = f(x) \\ \textbf{odd} \text{ function of } x & \text{if} & f(-x) = -f(x) \end{array} \tag{1.3}$$

for every x in the function's domain.

Note: The names **'even'** and **'odd'** come from **powers of x**.

Example 1.15. Find if the given functions are even, odd, or neither.

(a) $y = 2x^2 - 2$ (b) $y = x^3 - 4x$ (c) $y = x^2 + x$

Solution.

Remark 1.16. Symmetry

- The graph of an even function is **symmetric about the y-axis**. A **reflection** across the y-axis leaves the graph unchanged.
- The graph of an odd function is **symmetric about the origin**. A **rotation** of 180° about the origin leaves the graph unchanged.

1.1.5. Common Functions

Linear Functions

- A function of the form $f(x) = mx + b$ is called a **linear function**. The function $f(x) = x$, where $m = 1$ and $b = 0$, is called the **identity function**.

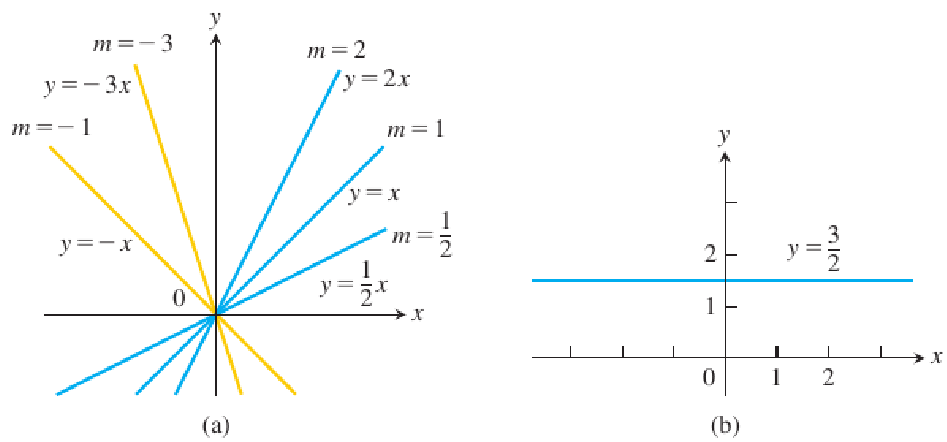


Figure 1.2: (a) Lines through the origin with slope m. (b) A constant function with slope $m = 0$.

- Two variables y and x are **proportional** (to one another), if one is always a constant multiple of the other – that is, if $y = kx$ for some nonzero constant k.
  - If the variable y is proportional to the reciprocal $1/x$, then sometimes it is said that y is **inversely proportional** to x.

Power Functions

A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider.

(a) $f(x) = x^a$ with $a = n$, a positive integer.

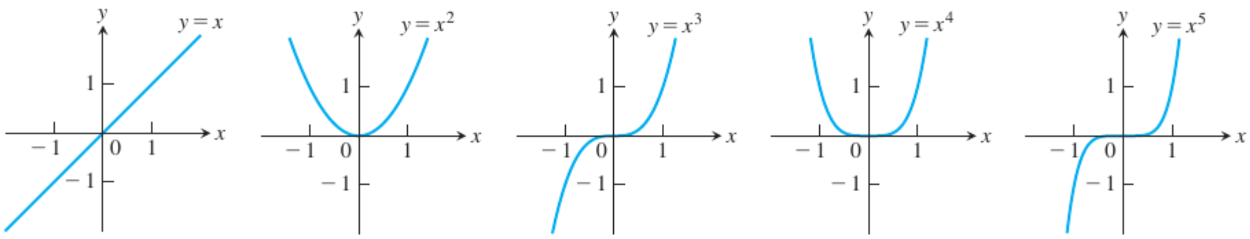


Figure 1.3: Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$.

Each curve passes through the origin and the point $(1, 1)$. *Are they symmetric? Where are they increasing? What happens when x approaches $-\infty$ or ∞?*

(b) $f(x) = x^a$ with $a = -1$ or $a = -2$.

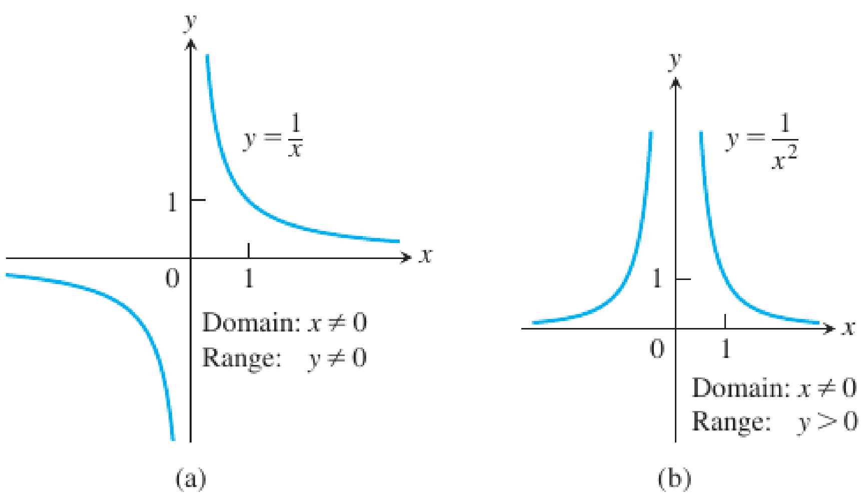


Figure 1.4: Graphs of the power functions $f(x) = x^a$. (a) $a = -1$. (b) $a = -2$.

(c) $a = 1/2,\ 1/3,\ 3/2$, and $2/3$.

- $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ are the **square root** and **cube root** functions, respectively.
- Note that $x^{3/2} = (x^{1/2})^3$ and $x^{2/3} = (x^{1/3})^2$.

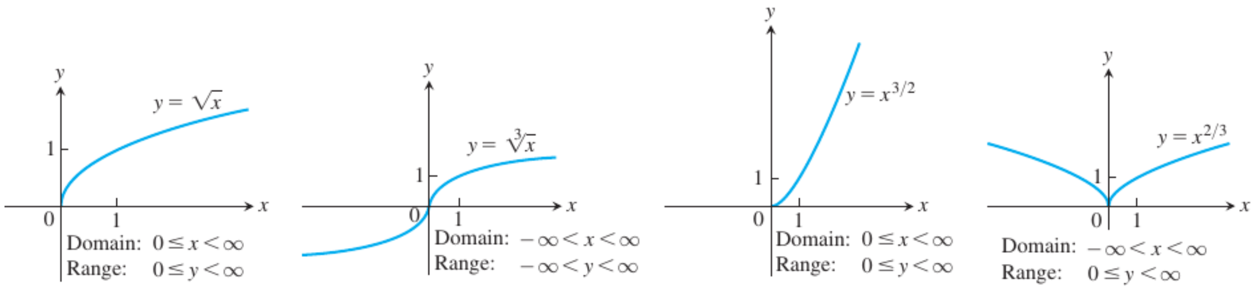


Figure 1.5: Graphs of the power functions $f(x) = x^a$ for $a = 1/2,\ 1/3,\ 3/2$, and $2/3$.

Polynomials

A function p is a **polynomial** if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0, \tag{1.4}$$

where n is a nonnegative integer, $a_0, a_1, \cdots, a_n \in \mathbb{R}$, and $a_n \neq 0$.

- n is called the **degree** of the polynomial
- $a_0, a_1, \cdots, a_n$ are called the **coefficients** of the polynomial
- p is called a **quadratic polynomial** when $n = 2$, while it is a **cubic polynomial** if $n = 3$.
- All polynomials have domain (∞, ∞).

Example 1.17. Sketch the graph of each polynomial function.

(a) $y = x^3 - 3x^2 + 2x$

(b) $y = (x+1)(x-1)^3(x-2)^2$

Rational Functions

A **rational function** is a quotient or ratio

$$f(x) = \frac{p(x)}{q(x)}, \tag{1.5}$$

where p and q are polynomials. The **domain** of a rational function is the set of all real x for which $q(x) \neq 0$.

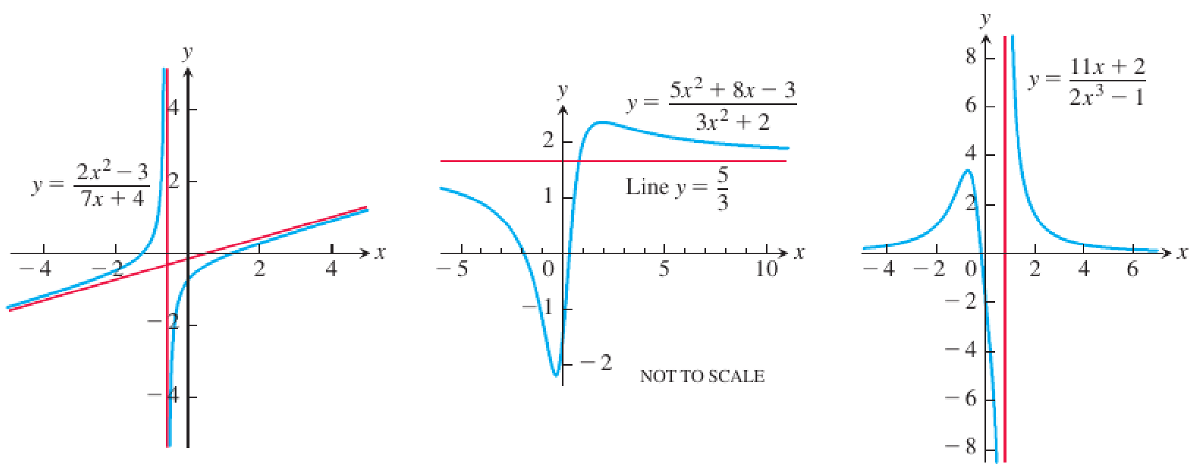


Figure 1.6: Graphs of three rational functions. The straight red lines approached by the graphs are called **asymptotes** and are not part of the graphs.

Self-study 1.18. Sketch the graph of $f(x) = \dfrac{x^2 - 2x}{x - 1}$

Note: Other common functions are trigonometric functions, exponential functions, and logarithmic functions, which we will deal with in detail later in this chapter.

Exercises 1.1

1. Sketch the graph and find the domain and range of each function.

   (a) $f(x) = 1 - 2\sqrt{x}$

   (b) $g(t) = \dfrac{3}{t-4}$

   (c) $h(x) = \sqrt{4x - x^2}$

   Ans: (b) D: $\{t \neq 4\}$. R: $\{y \neq 0\}$.

2. Consider the point (x, y) lying on the graph of the line $2x + y = 1$. Let L be the distance from the point (x, y) to the origin $(0, 0)$. Write L as a function of x.
   Clue: Start with the distance: $L = (x^2 + y^2)^{1/2}$, which comes from the **Pythagorean theorem**.

   Ans: $L = \sqrt{5x^2 - 4x + 1}$

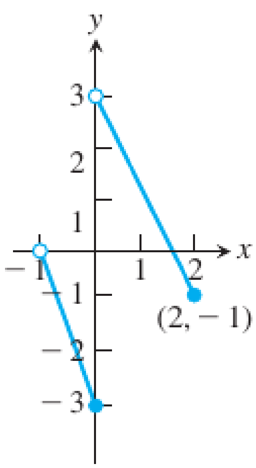
3. Graph the following equations and explain why they are not graphs of functions of x.

   (a) $|y| = 2x$

   (b) $y^2 = x^2$

   Hint: (a) Consider cases that $|y|$ is nonnegative or negative. (b) It can be written as $|y| = |x|$.

4. Find a formula for the piecewise-defined function.



5. Say whether the function is even, odd, or neither. Give reasons for your answer.

   (a) $f(x) = x^3 - 5x$

   (b) $f(x) = x^{-5}$

   (c) $g(t) = |t^3| + t$

   (d) $g(x) = \dfrac{x^2 - 1}{x^2 + 1}$

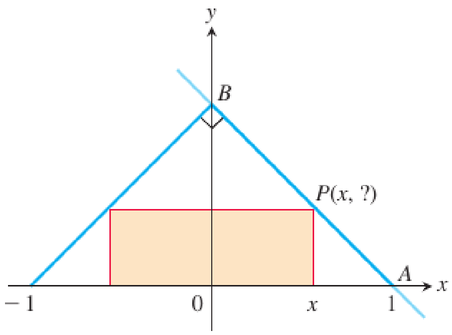
   Ans: (c) neither, $\because$ $g(-t) \neq g(t)$ and $g(-t) \neq -g(t)$. (d) even, $\because$ $g(-x) = g(x)$.

6. The accompanying figure shows a rectangle inscribed in an *isosceles right triangle* whose hypotenuse is 2 units long.

   (a) Express the y-coordinate of P in terms of x.

   (b) Express the area of the rectangle in terms of x.

   Clue: (a) You might start by writing an equation for the line AB.



Ans: (b) $A = x(1 - x)$.

1.2. Combining Functions; Shifting and Scaling Graphs

Sums, Differences, Products, and Quotients

- Like numbers, functions can be added, subtracted, multiplied, and divided to produce new functions.
- If f and g are functions, we define functions $f+g$, $f-g$, fg, f/g by the formulas

$$\begin{aligned} (f+g)(x) &= f(x)+g(x) \\ (f-g)(x) &= f(x)-g(x) \\ (fg)(x) &= f(x)\,g(x) \\ (f/g)(x) &= f(x)/g(x) \end{aligned} \tag{1.6}$$

  - **Domain of $f+g$, $f-g$, and fg**: every x that belongs to the domains of both f and g; that is, $x \in D(f) \cap D(g)$.
  - **Domain of f/g**: $x \in D(f) \cap D(g)$, except $g(x) = 0$.

Example 1.19. The functions defined by the formulas

$$f(x) = \sqrt{x} \text{ and } g(x) = \sqrt{1-x}$$

have domains $D(f) = [0, \infty)$ and $D(g) = (-\infty, 1]$. The points common to these domains are the points in

$$[0, \infty) \cap (-\infty, 1] = [0, 1].$$

Find the formulas and domains for the various algebraic combinations of the two functions, where $f \cdot g = fg$.

Solution. *Let's complete filling the following table.*

Function	Formula	Domain
$f+g$		
$f-g$		
$f \cdot g$	$(f \cdot g)(x) = f(x)g(x) = \sqrt{x(1-x)}$	
f/g		
g/f	$\left(\frac{g}{f}\right)(x) = \frac{g(x)}{f(x)} = \sqrt{\frac{1-x}{x}}$	$(0, 1]$ ($x = 0$ excluded)

Composite Functions

Definition 1.20. If f and g are functions, the **composite** function $f \circ g$ ("f composed with g") is defined by

$$(f \circ g)(x) \;=\; f(g(x)). \tag{1.7}$$

The domain of $f \circ g$ consists of the numbers x in the domain of g for which g(x) lies in the domain of f:

$$D(f \circ g) = \{x \in G(g) \mid g(x) \in D(f)\}. \tag{1.8}$$

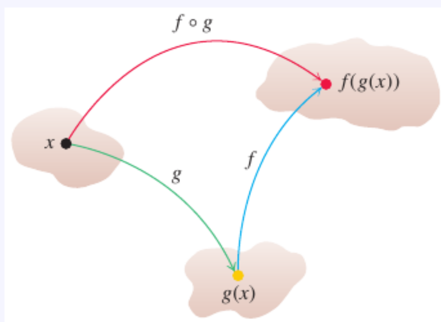


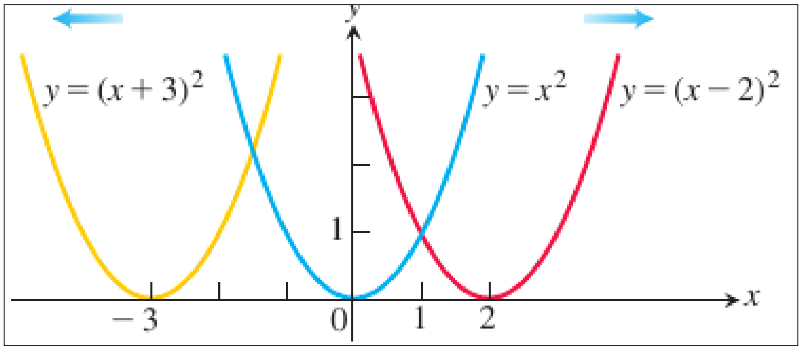
Figure 1.7: Arrow diagram for $f \circ g$.

Example 1.21. Let $f(x) = \sqrt{x}$ and $g(x) = x + 1$. Find $f \circ g$, $g \circ f$, $f \circ f$, and $g \circ g$, and their domains.

Solution.

Composition		Formula	Domain
$(f \circ g)(x)$	=	$f(g(x)) = \sqrt{g(x)} = \sqrt{x+1}$	$[-1, \infty)$
$(g \circ f)(x)$	=		
$(f \circ f)(x)$	=	$f(f(x)) = \sqrt{f(x)} = \sqrt{\sqrt{x}} = x^{1/4}$	$[0, \infty)$
$(g \circ g)(x)$	=		

Shifting a Graph of a Function



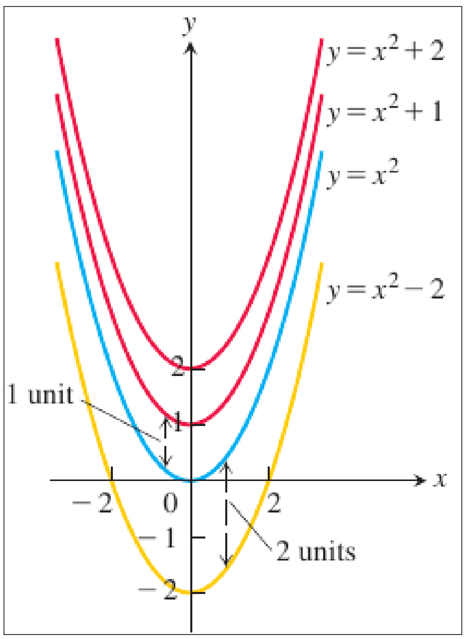


Figure 1.8: Shift from the graph of $y = x^2$.

Observation 1.22. Note that $\rightarrow$ and $\uparrow$ are positive directions of x and y, respectively.

$$\begin{array}{llrll} \text{shift the graph} & \rightarrow & \mathbf{2} & \Longrightarrow & \mathbf{x} \rightarrow (\mathbf{x}-\mathbf{2}) \\ \text{shift the graph} & \rightarrow & -\mathbf{3} & \Longrightarrow & \mathbf{x} \rightarrow (\mathbf{x}+\mathbf{3}) \\ \hline \text{shift the graph} & \uparrow & \mathbf{2} & \Longrightarrow & \mathbf{y} \rightarrow (\mathbf{y}-\mathbf{2}) \\ \text{shift the graph} & \uparrow & -\mathbf{2} & \Longrightarrow & \mathbf{y} \rightarrow (\mathbf{y}+\mathbf{2}) \end{array}$$

Shift Formula

If the graph of $y = f(x)$ **is shifted** (c, d), then the resulting graph is formulated as

$$\boldsymbol{y - d = f(x - c).} \tag{1.9}$$

Example 1.23. Sketch the graph of the functions.

(a) $y = -(x-3)^2$ (b) $y - 2 = -x^2$ (c) $y = -(x+1)^2 + 4$

Solution.

Ans: (a) vertex= $(3, 0)$. (c) vertex= $(-1, 4)$.

Summary 1.24. If the graph of $\boldsymbol{y = f(x)}$ **is shifted** (c, d), then the resulting graph is formulated with

$$\begin{array}{ll} \boldsymbol{x} & \text{being replaced by } \boldsymbol{x - c} \\ \boldsymbol{y} & \text{being replaced by } \boldsymbol{y - d} \end{array} \tag{1.10}$$

Why does this formula make sense algebraically?

- The equation
$$y = f(x) \tag{1.11}$$
expresses the relation between x and y of each point (x, y) on the graph of f.
- Let the graph be shifted (c, d). Let $\boldsymbol{(x, y)}$ **be shifted to** $\boldsymbol{(X, Y)}$.
- Then, we should find the **relation between** $\boldsymbol{X}$ **and** $\boldsymbol{Y}$, to formulate the shifted graph.
- Since $(X, Y) = (x + c, y + d)$, we may get
$$(x, y) = (X - c, Y - d). \tag{1.12}$$
- Using (1.11) and (1.12), we obtain
$$Y - d = f(X - c). \tag{1.13}$$

So (1.10) follows.

Scaling a Graph of a Function

Scaling Formula

- If the graph of $y = f(x)$ is scaled **horizontally** by a factor of $b > 0$, then the resulting graph is formulated as

$$y = f(x/b). \tag{1.14}$$

- If the graph of $y = f(x)$ is scaled **vertically** by a factor of $a > 0$, then the resulting graph is formulated as

$$y/a = f(x). \tag{1.15}$$

- If the graph of $\boldsymbol{y = f(x)}$ **is scaled by factors of** $(\boldsymbol{b}, \boldsymbol{a})$, then the resulting graph is formulated as

$$\boldsymbol{y/a = f(x/b)} \quad \text{or} \quad \boldsymbol{y = a\, f(x/b)} \tag{1.16}$$

Note: The reference book (Thomas' Calculus) explains scaling in terms of "stretch" and "compress".

$$\begin{array}{lcl} \text{stretched by a factor of } b & \Leftrightarrow & \text{scaling factor} = b \\ \text{compressed by a factor of } b & \Leftrightarrow & \text{scaling factor} = 1/b \end{array}$$

Reflecting a Graph of a Function

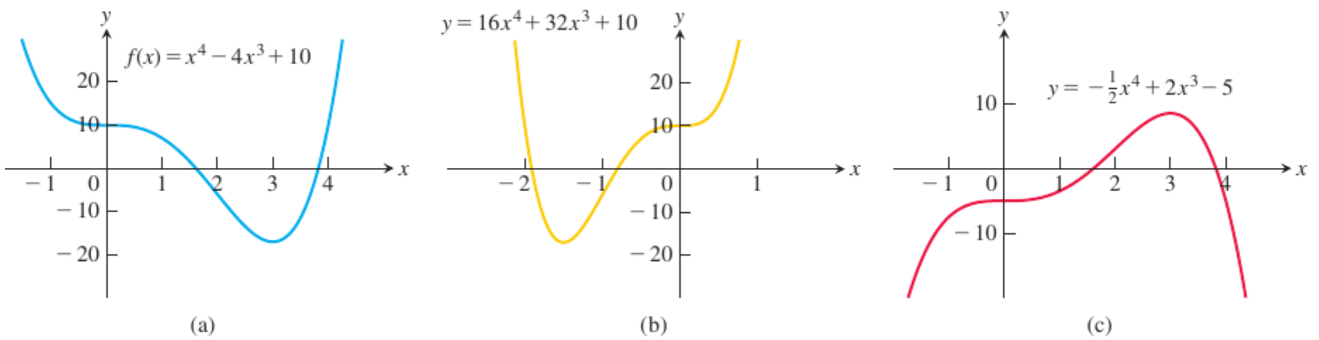
Reflecting Formula

Reflection is the same as scaling by a factor of -1. That is,

$$\begin{array}{lcl} \text{reflect } G(f) \text{ across the } x\text{-axis} & \Leftrightarrow & y \text{ is replaced by } -y \\ \text{reflect } G(f) \text{ across the } y\text{-axis} & \Leftrightarrow & x \text{ is replaced by } -x \end{array} \tag{1.17}$$

Example 1.25. The graph in (a) depicts $y = f(x)$. Find the scaling factor and the reflection for graphs in (b) and (c) to be expressed in the form

$$y = pf(qx).$$



Solution.

(b) $c = 1/2$ & y-axis reflection

(c) $d = 1/2$ & x-axis reflection

Ans: (b) $y = f(-2x)$. (c) $\dfrac{y}{-1/2} = f(x) \Rightarrow y = -\dfrac{1}{2}f(x)$.

Exercises 1.2

1. If $f(x) = x - 1$ and $g(x) = 1/(x+1)$, find the following.

   (a) $f(g(x))$
   (b) $f(g(1/2))$
   (c) $g(f(x))$
   (d) $g(f(1/2))$

Ans: (c) $g(f(x)) = 1/(f(x)+1) = 1/x$. (d) 2.

2. Write a formula for $f \circ g \circ h$.

   (a) $f(x) = 3x + 4,\ g(x) = 2x - 1,\ h(x) = x^2$
   (b) $f(x) = \sqrt{2x+1},\ g(x) = \dfrac{1}{x+4},\quad h(x) = \dfrac{1}{x}$

Ans: (b) $\sqrt{\dfrac{5x+1}{4x+1}}$.

3. Copy and complete the following table.

	$g(x)$	$f(x)$	$(f \circ g)(x)$
a.	$x - 7$	$\sqrt{x}$	?
b.	$x + 2$	$3x$	?
c.	?	$\sqrt{x-5}$	$\sqrt{x^2-5}$
d.	$\dfrac{x}{x-1}$	$\dfrac{x}{x-1}$	?
e.	?	$1 + \dfrac{1}{x}$	x
f.	$\dfrac{1}{x}$	?	x

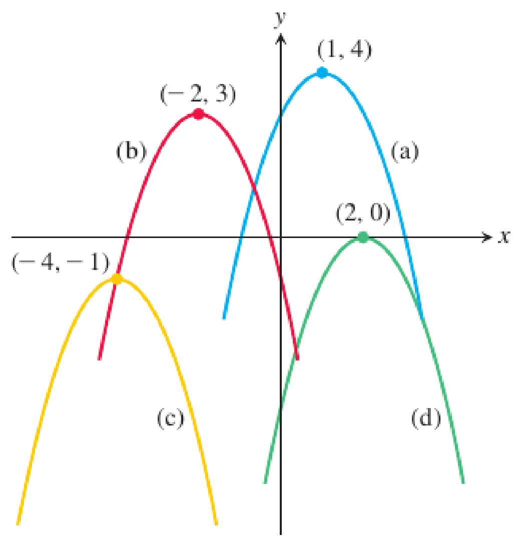
Ans: (c) x^2. (e) $1/(x-1)$.

4. Find a function $g(x)$ so that

   (a) $f(x) = \dfrac{x}{x-2}$ and $(f \circ g)(x) = x$.
   (b) $f(x) = 2x^3 - 4$ and $(f \circ g)(x) = x + 2$.

Ans: (a) $g(x) = \dfrac{2x}{x-1}$.

5. The accompanying figure shows the graph of $y = -x^2$ shifted to four new positions. Write an equation for each new graph.



6. **Challenge**[1] Assume that f is an **even** function, g is an **odd** function, and both f and g are defined on the entire real line $(-\infty, \infty)$. Which of the following (where defined) are even? odd?

(a) fg	(b) f/g	(c) g/f
(d) $f^2 = f\,f$	(e) $g^2 = g\,g$	(f) $f \circ g$
(g) $g \circ f$	(h) $f \circ f$	(i) $g \circ g$

Ans: (a) Odd. (c) Odd. (e) Even. (i) Odd.

7. **Challenge** Can a function be both even and odd? Give reasons for your answer.

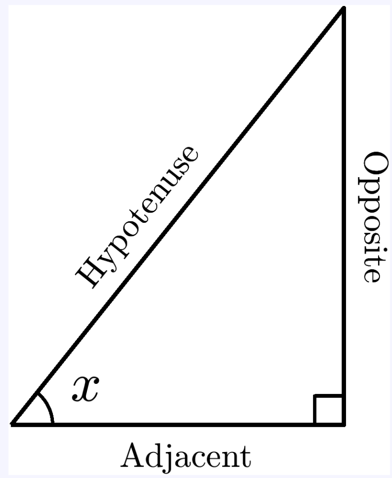
[1] **Challenge** indicates the problem may be harder than usual. Challenge it!

1.3. Trigonometric Functions

You might learn the following in your High school.

Sides of a right triangle:

- The **hypotenuse** of a right triangle is the side opposite the right angle. It must be the longest side of the right triangle.
- The **adjacent side** is the non-hypotenuse side that is adjacent to *the given angle*.
- The **opposite side** is the side across from a given angle.



SOH–CAH–TOA:

$$\begin{array}{lcl}
\hline
\sin x & = & \dfrac{\textit{opposite}}{\textit{hypotenuse}} & \Rightarrow & \text{SOH} \\
\hline
\cos x & = & \dfrac{\textit{adjacent}}{\textit{hypotenuse}} & \Rightarrow & \text{CAH} \\
\hline
\tan x & = & \dfrac{\textit{opposite}}{\textit{adjacent}} & \Rightarrow & \text{TOA} \\
\hline
\end{array}$$

Cosecant, Secant, Cotangent:

$$\csc x = \frac{1}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \cot x = \frac{1}{\tan x} \tag{1.18}$$

1.3.1. Angles and Sectors

Definition 1.26. An **angle** is the figure formed by two rays sharing a common endpoint, called the **vertex of the angle**.

The **angle** can be defined with the *unit circle*, the circle of radius 1.

"The angle is θ (radian), when the corresponding arc length is θ."

- The angle of the whole circle is 2π (radian).
- $2\pi = 360^\circ$
- $\pi = 180^\circ \quad \Rightarrow \quad \boxed{{}^\circ = \frac{\pi}{180}}$

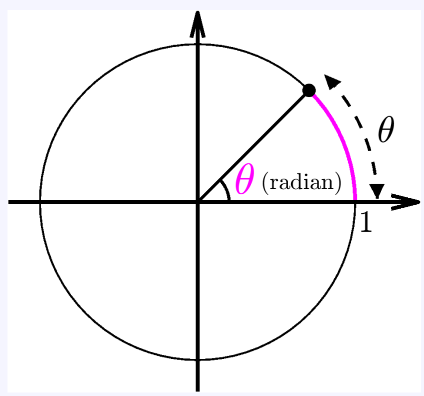


Figure 1.9: Geometric definition of the angle.

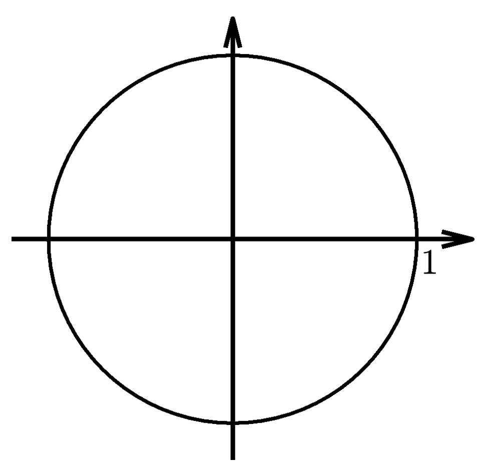
Example 1.27. Figure the following angles, using the unit circle.

(a) 30°

(b) -60°

(c) 420°

(d) $-\dfrac{3}{2}\pi$



Sectors

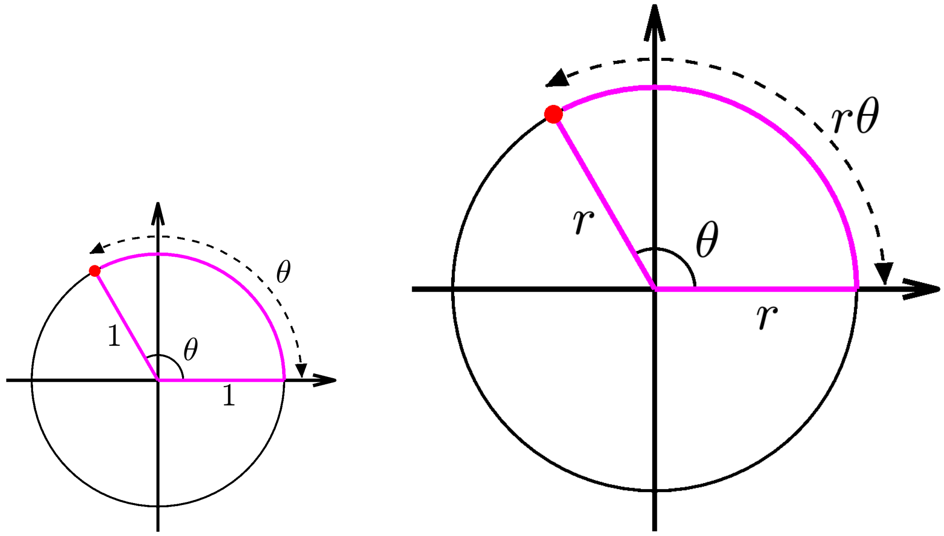


Figure 1.10: The angle and the **arc length** of the **sector**.

Definition 1.28. A **sector** (a.k.a. **circular sector**) is the the portion of a disc enclosed by two radii and a circular arc.

Formula 1.29. A sector is given as in the right of Figure 1.10. Then,

$$\text{Arc length } \ell = r\theta \qquad \text{Area } A = \frac{1}{2}r\ell = \frac{1}{2}r^2\theta$$

** A sector is like a triangle!*

Example 1.30. Find the arc length of the sectors.

(a) $r = 12$, angle = $\frac{5}{4}\pi$

(b) $r = 4$, angle = $240°$

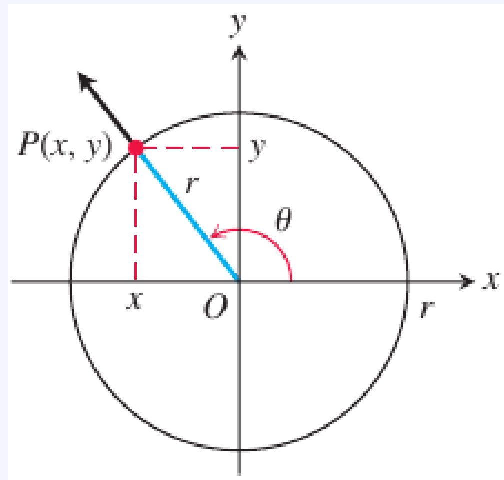
Solution.

Ans: (a) 15π. (b) $16\pi/3$.

1.3.2. The Six Basic Trigonometric Functions

Definition 1.31. The trigonometric functions of an **acute angle** are given in terms of the sides of a right triangle (SOH–CAH–TOA).

- We extend this definition to **obtuse and negative angles** by first placing the angle in standard position in a circle of radius r.
- We then define the trigonometric functions in terms of the coordinates of the point $P(x, y)$ where the angle's terminal ray intersects the circle.



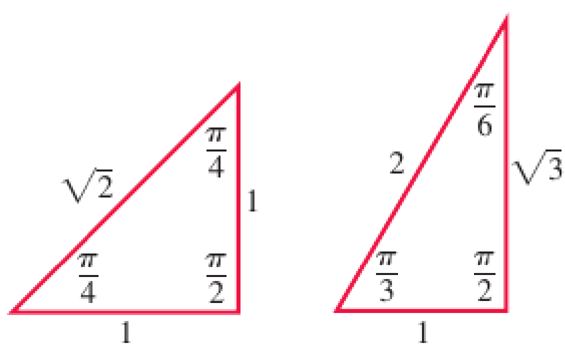
$$\begin{aligned}
&\textbf{sine}: \ \sin\theta = \frac{y}{r} && \textbf{cosecant}: \ \csc\theta = \frac{1}{\sin\theta} = \frac{r}{y} \\
&\textbf{cosine}: \ \cos\theta = \frac{x}{r} && \textbf{secant}: \ \sec\theta = \frac{1}{\cos\theta} = \frac{r}{x} \\
&\textbf{tangent}: \ \tan\theta = \frac{y}{x} && \textbf{cotangent}: \ \cot\theta = \frac{1}{\tan\theta} = \frac{x}{y}
\end{aligned} \tag{1.19}$$

Example 1.32. Evaluate the exact values of these trigonometric ratios for common angles.

(a) $\{\sin, \cos, \tan\}\frac{\pi}{4} = \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 1$

(b) $\{\sin, \cos, \tan\}\frac{\pi}{6} = \frac{1}{2}, \frac{\sqrt{3}}{2}, \frac{1}{\sqrt{3}}$

(c) $\{\sin, \cos, \tan\}\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \frac{1}{2}, \sqrt{3}$



Geometric interpretation of trigonometric functions

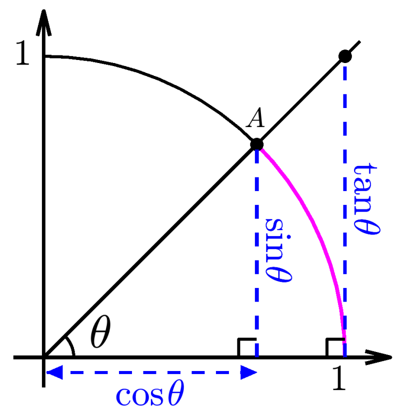


Figure 1.11: Geometric interpretation of trigonometric functions.

Definition 1.33. The **trigonometric functions** are defined with the **unit circle**, the circle of radius 1.

Point $A(x, y)$ is chosen on the unit circle as in the figure.

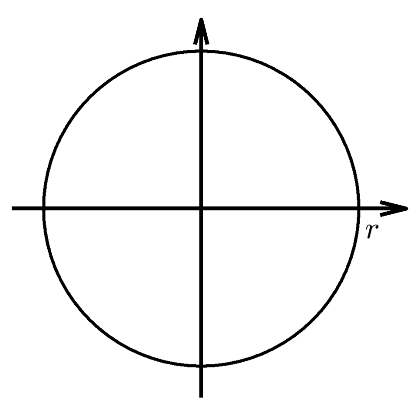
- $\boldsymbol{A(x, y) = (\cos\theta, \sin\theta)}$
- "**s**i**n**" is the vertical component of the right triangle.
- $\boldsymbol{\tan\theta = \frac{y}{x}}$ **= slope** $\quad \Rightarrow \quad \tan\theta = \dfrac{\sin\theta}{\cos\theta} = \dfrac{\tan\theta}{1}$
- These hold for all angles θ, i.e., all choices of A on the unit circle.

Example 1.34. Let θ be the angle made by the line segment from $O(0,0)$ to $P(-4,3)$. Evaluate the following.

(a) $\sin\theta$

(b) $\cos\theta$

(c) $\tan\theta$



Ans: (a) $3/5$. (b) $-4/5$. (c) $-3/4$.

Formula 1.35. Frequently Used Trigonometric Formulas:
For all angle x,

$$\sin^2 x + \cos^2 x = 1 \qquad \tan x = \frac{\sin x}{\cos x} = \text{slope}$$

Graphs of Trigonometric Functions

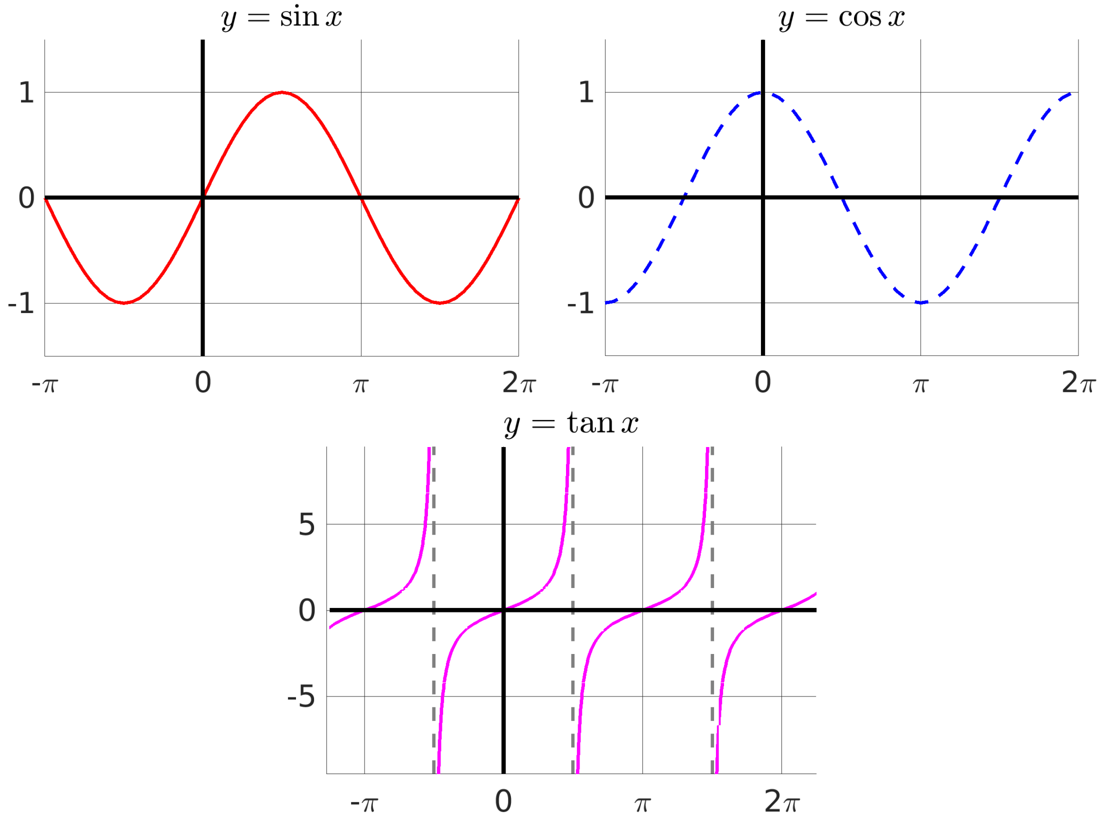


Figure 1.12: Graphs of $y = \sin x$, $y = \cos x$, and $y = \tan x$.

Periodicity of the Trigonometric Functions

Definition 1.36. A function $f(x)$ is **periodic** if there is a number $p > 0$ such that $f(x+p) = f(x)$ for every value of x. The smallest such value of p is the **period** of f.
For example,

$$\text{Period of “}\sin\text{” and “}\cos\text{”} = 2\pi$$
$$\text{Period of “}\tan\text{”} = \pi$$

Observation 1.37. $\sin(-x) = -\sin x$ **and** $\cos(-x) = \cos x$**.**

The following holds also when $\sin$ and $\cos$ are interchanged.

- When the graph of $y = \sin x$ is moved by $\pm\pi$, the result becomes $y = -\sin x$.
- When the graph of $y = \sin(\pm x)$ is moved by $\pm\frac{\pi}{2}$ (or, $\pm 3\pi/2$), the result should match with either $y = \cos x$ or $y = -\cos x$.

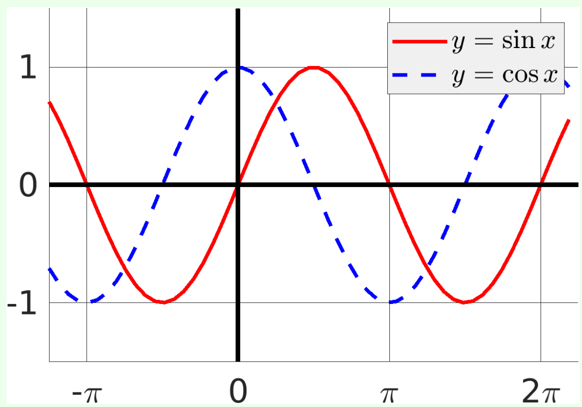


Figure 1.13: Graphs of $y = \sin x$ and $y = \cos x$, superposed.

Example 1.38. Which one is the same as $\cos\left(\dfrac{3\pi}{2} - x\right)$?

A. $\cos x$

B. $-\cos x$

C. $\sin x$

D. $-\sin x$

E. None of these

Transformations of Trigonometric Graphs

The rules for shifting, scaling, and reflecting the graph of a function, studied in §1.2 and summarized in the following, apply to the trigonometric functions:

$$\frac{y-d}{a} = f\left(\frac{x-c}{b}\right) \quad \text{or} \quad y = a\, f\left(\frac{x-c}{b}\right) + d, \tag{1.20}$$

where (b, a) are scaling factors and (c, d) are shifting units.

Definition 1.39. The transformation rules applied to the sine function give the **general sine function** or **sinusoid** formula

$$f(x) = A \sin\left(\frac{2\pi}{B}(x - C)\right) + D, \tag{1.21}$$

where where $|A|$ is the **amplitude**, $|B|$ is the **period**, C is the **horizontal shift**, and D is the **vertical shift**.

Example 1.40. Identify A, B, C, and D for the sine function and sketch its graph.

$$y = 2\sin\left(\frac{x}{2} + \pi\right) - 1$$

Solution.

Ans: $y = 2\sin\left(\frac{2\pi}{4\pi}(x - (-2\pi))\right) - 1$

1.3.3. Trigonometric Formulas

Trigonometric Identities

$$\begin{aligned}
\cos^2\theta + \sin^2\theta &= 1 \\
1 + \tan^2\theta &= \sec^2\theta \\
1 + \cot^2\theta &= \csc^2\theta
\end{aligned}$$

Angle-Sum Formulas

$$\begin{aligned}
\sin(A+B) &= \sin A\cos B + \cos A\sin B \\
\cos(A+B) &= \cos A\cos B - \sin A\sin B
\end{aligned}$$

Product Formulas

$$\begin{aligned}
\sin A\sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)] \\
\sin A\cos B &= \frac{1}{2}[\sin(A-B) + \sin(A+B)] \\
\cos A\cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)]
\end{aligned} \tag{1.22}$$

Double-Angle Formulas

$$\begin{aligned}
\sin 2\theta &= 2\sin\theta\cos\theta \\
\cos 2\theta &= \cos^2\theta - \sin^2\theta
\end{aligned}$$

Half-Angle Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

The Law of Cosines

If a, b, and c are sides of a triangle ABC, then

$$c^2 = a^2 + b^2 - 2ab\cos C. \tag{1.23}$$

Exercises 1.3

1. On a circle of radius 10 m, how long is an arc that subtends a central angle of
   (a) $7\pi/5$
   (b) $120°$

2. Graph the functions. What is the period of each function?
   (a) $-\cos(2\pi x)$
   (b) $2\sin\left(2x - \dfrac{\pi}{4}\right) + 1$

   Ans: (b) Period: π.

3. Use the Half-Angle Formulas to find the exact function values.
   (a) $\cos^2 \dfrac{\pi}{8}$
   (b) $\sin^2 \dfrac{3\pi}{8}$

   Ans: (b) $1/2 + \sqrt{2}/4$.

4. Solve for the angle θ, where $0 \le \theta \le 2\pi$
   (a) $\cos^2\theta = \dfrac{3}{4}$
   (b) $\cos^2\theta - \dfrac{3}{2}\sin\theta = 0.$

   Hint: (b). $\cos^2\theta = 1 - \sin^2\theta$. Thus the equation reads $\sin^2\theta + \dfrac{3}{2}\sin\theta - 1 = 0$, which is a quadratic equation of $x = \sin\theta$.

   Ans: (b). $\theta = \pi/6,\ 5\pi/6$.

5. A triangle has sides $a = 2$ and $b = 3$ and angle $C = 60°$. Find the length of side c.

   Ans: $\sqrt{7}$.

1.4. Graphing with Software

- Through the course, you will learn how to
  - visualize functions, solve equations, and analyze data.
- You may use
  - Maple, Mathematica, Matlab, or **Python**

  which are available on computers campus-wide.
- However, for a bit simpler work in Calculus, you may try online graphing software. For example,
  - **GeoGebra** https://www.geogebra.org/

  for which you may find **Tutorials** https://www.geogebra.org/a/14. Learn about “CAS Calculator”.

Note: In modern society, becoming a good programmer would be quite advantageous. Thus, first, try to be familiar with computational environments for computer software. I recommend you to start with one of Maple, Mathematica, and Matlab. The simplest one is **Matlab**.

Mathematica

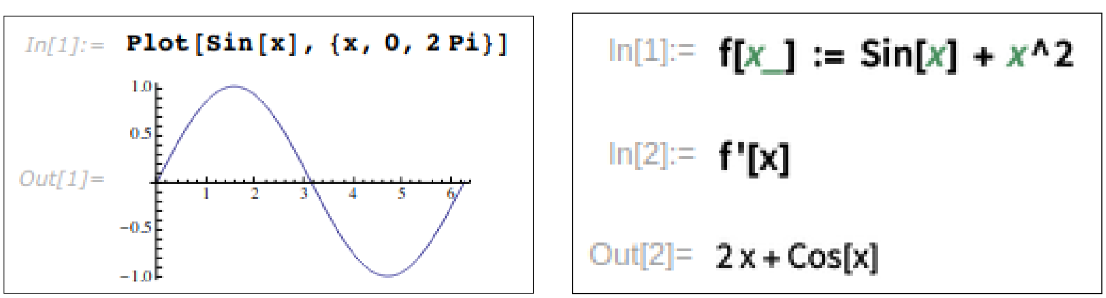


Figure 1.14: Examples in Mathematica.

Maple

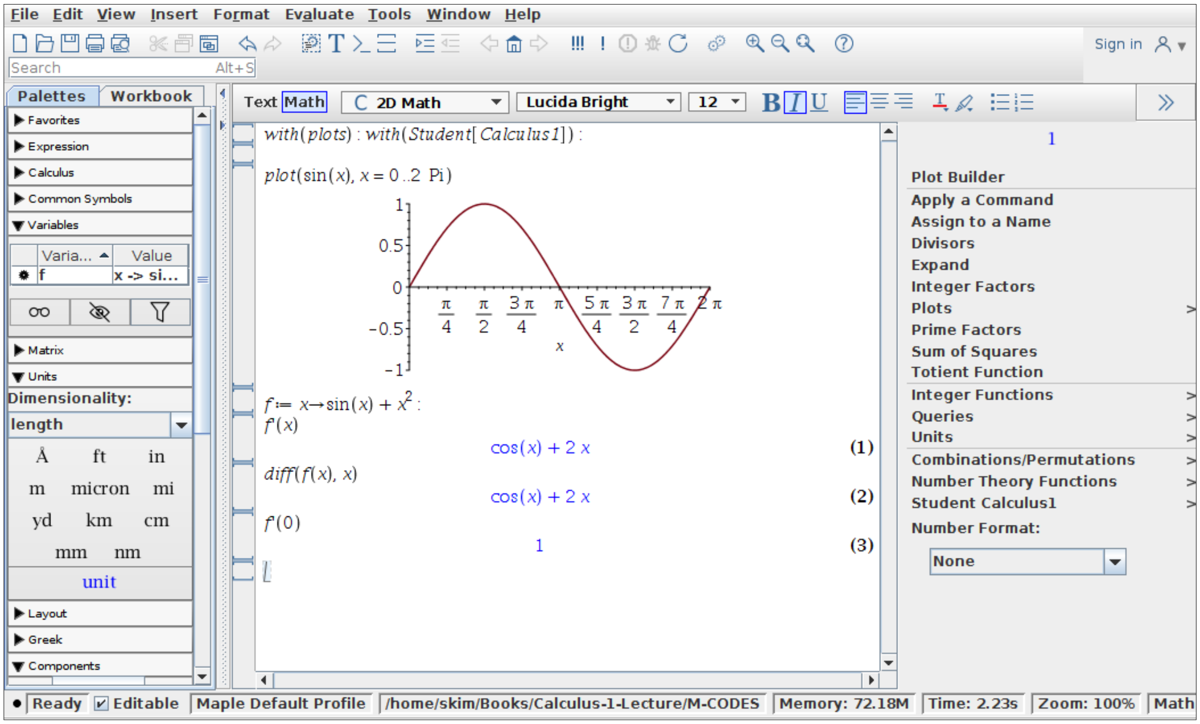


Figure 1.15: An example code in Maple.

Example 1.41. Let's play with Maple a little more.

maple-plot-derivative.mw

```
with(plots, implicitplot):
implicitplot(x^2+2 x = -y^2+4 y-1, x=-3..1, y=0..4)
implicitplot(r=1-cos(theta), r=0..2, theta=0..2 Pi, coords=polar,
     size=[250, 250], axis=[color=red], background=green)
g := x -> if x<3 then x+1 else -x^2+13 end if:
plot(g, 0..5, size=[300,200])
```

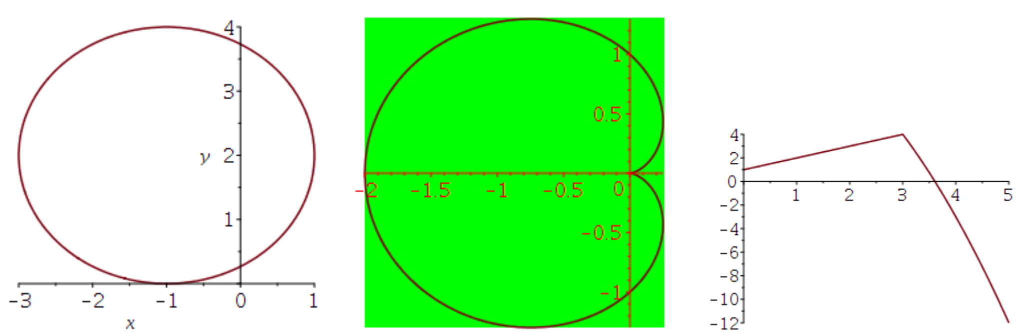


Figure 1.16: Maple plots: A circle, a **cardioid**, and a piecewise-defined function.

Matlab

y_sin_x.m

```
x = linspace(-pi, 2*pi, 100);
y = sin(x);

plot(x,y,'r',linewidth=2); grid on
xticks([-pi,0,pi,2*pi]); xticklabels({'-\pi','0','\pi','2\pi'})
yticks([-1,-0.5,0,0.5,1])
title('Matlab: $y=\sin x $','fontsize',18,'interpreter','latex')
axis tight
    if exist('__octave_config_info__')  %octave
        print('y=sin-x-matlab.png')
    else
        exportgraphics(gcf,'y=sin-x-matlab.png','Resolution',100)
    end
```

Python

y_sin_x.py

```
import numpy as np
from matplotlib import pyplot as plt

x = np.linspace(-np.pi, 2*np.pi, 100)
y = np.sin(x)

plt.plot(x,y,'r',linewidth=2)
plt.grid(color = 'green', linestyle = '--', linewidth = 0.5)
plt.xticks([-np.pi,0,np.pi,2*np.pi],['$-\pi$',0,'$\pi$','$2\pi$'])
plt.yticks([-1,-0.5,0,0.5,1])
plt.title('Python: $y=\sin x $',fontsize=18)
#plt.show()
plt.savefig('y=sin-x-python.png',bbox_inches='tight')
```

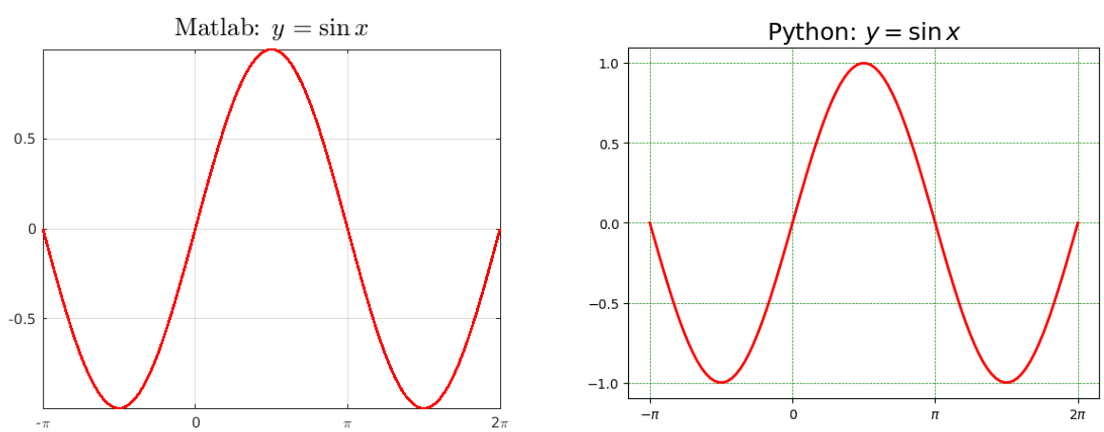


Figure 1.17: `y=sin-x-matlab.png` and `y=sin-x-python.png`

Exercises 1.4

1. CAS[2] Find an appropriate graphing software viewing window for the given function and use it to display its graph. The window should give a picture of the overall behavior of the function. There is more than one choice, but incorrect choices can miss important aspects of the function.

 (a) $f(x) = x^4 - 4x^3 + 15$

 (b) $f(x) = x\sqrt{9 - x^2}$

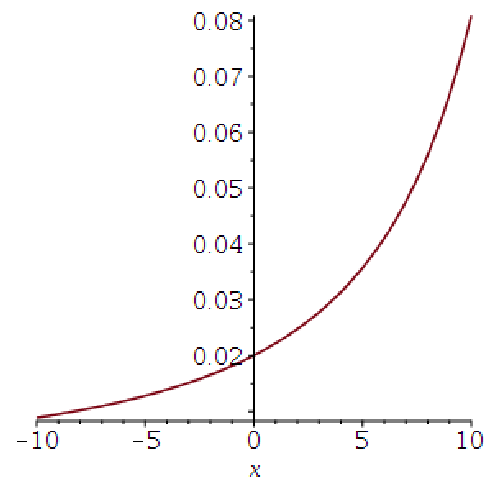
 (c) $y = x^{1/3}(x^2 - 8)$

 (d) $y = \dfrac{8}{(x-20)^2 - 1}$

Note: **Finding a Viewing Window**:

If you do not specify the viewing window, then the CAS will use a default window.

- For example, for (d), if you implement $plot\left(\dfrac{8}{(x-20)^2-1}\right)$ in Maple, then you will see the figure right.
- You should specify a viewing window, as in Example 1.41, not to miss important aspects of the function.
- A good strategy for the determination of a viewing window is to (1) ***start with a large window*** and then (2) ***specify a right one to see important features in detail***.



2. CAS Graph the function $y = \cos x + \dfrac{1}{100}\sin 100x$

3. CAS Graph four periods of the function $f(x) = -\tan 2x$.

[2] Problems with the sign CAS must be done with a **Computer Algebra System** such as Maple, Mathematica, or Matlab.

1.5. Exponential Functions

Definition 1.42. A function of the form

$$f(x) = a^x, \quad \text{where } a > 0 \text{ and } a \neq 1, \tag{1.24}$$

is called an **exponential function** (with base a).

- All exponential functions have domain $(-\infty, \infty)$ and range $(0, \infty)$, so an exponential function never assumes the value 0.
- All exponential functions are either increasing ($a > 1$) or decreasing ($0 < a < 1$) over the whole domain.

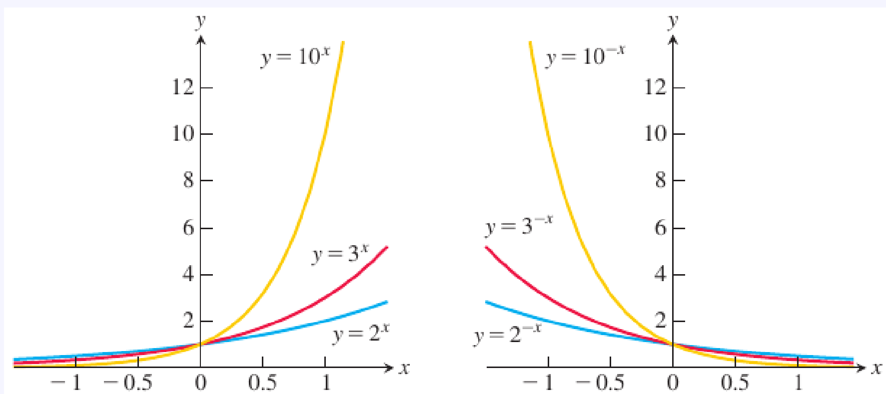


Figure 1.18: Exponential functions.

Example 1.43. Sketch the graph of the function $f(x) = 3 - 2^x$ and determine its domain and range.

Solution.

Example 1.44. Table 1.1 shows data for the population of the world in the 20th century. Figure 1.19 shows the corresponding scatter plot.

- The pattern of the data points in Figure 1.19 suggests an exponential growth.
- Use an **exponential regression** algorithm to find a model of the form

$$P(t) = a \cdot b^t, \tag{1.25}$$

where $t = 0$ corresponds to 1900.

Table 1.1

t (years since 1900)	Population P (millions)
0	1650
10	1750
20	1860
30	2070
40	2300
50	2560
60	3040
70	3710
80	4450
90	5280
100	6080
110	6870

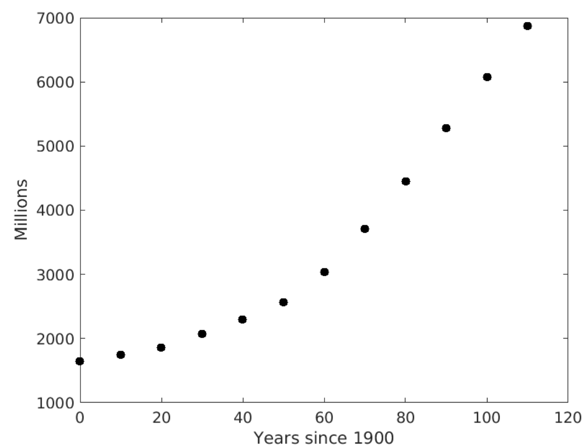


Figure 1.19: Scatter plot for world population growth.

Solution.

population.m
```
Data =[0 1650; 10 1750; 20 1860; 30 2070;
      40 2300; 50 2560; 60 3040; 70 3710;
      80 4450; 90 5280; 100 6080; 110 6870];
m = size(Data,1);

% exponential model, through linearization
A = ones(m,2);
A(:,2) = Data(:,1);
r = log(Data(:,2));
lm = (A'*A)\(A'*r);
a = exp(lm(1)), b = exp(lm(2)),
```

```
plot(Data(:,1),Data(:,2),'k.','MarkerSize',20)
   xlabel('Years since 1900');
   ylabel('Millions'); hold on
   print -dpng 'population-data.png'
t = Data(:,1);
plot(t,a*b.^t,'r-','LineWidth',2)
   print -dpng 'population-regression.png'
   hold off
```

The program results in

$$a = 1.4365 \times 10^3, \quad b = 1.0140.$$

Thus the exponential model reads

$$P(t) = (1.4365 \times 10^9) \cdot (1.0140)^t. \quad \textbf{(1.26)}$$

Figure 1.20 shows the graph of this exponential function together with the original data points. We see that the exponential curve fits the data reasonably well. □

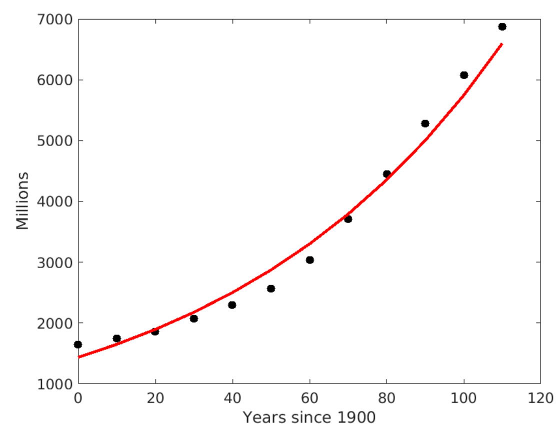


Figure 1.20: Exponential model for world population growth.

Integer and Rational Exponents

- When $x = n$ is a positive integer,

$$a^n = \underbrace{a \cdot a \cdot \ldots \cdot a}_{n \text{ times}}.$$

- When $x = -n$ for some positive integer n,

$$a^{-n} = \frac{1}{a^n} = \left(\frac{1}{a}\right)^n.$$

- When $x = p/q$ is a rational number,

$$a^{p/q} = \sqrt[q]{a^p} = (\sqrt[q]{a})^p$$

Laws of Exponents

If $a > 0$ and $b > 0$, the following rules hold for all real numbers x and y.

1. $a^x \cdot a^y = a^{x+y}$
2. $\dfrac{a^x}{a^y} = a^{x-y}$
3. $(a^x)^y = (a^y)^x = a^{xy}$
4. $a^x \cdot b^x = (ab)^x$
5. $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

The Number e

Of all possible bases for an exponential function, there is one that is most convenient for the purposes of calculus. The choice of a base a is influenced by the way the graph of $y = a^x$ crosses the y-axis.

- As we will see later, that some of the formulas of calculus will be greatly simplified, if we choose the base a so that **the slope of the tangent line to $y = a^x$ at $x = 0$ is exactly** 1. See Exercise 5, p. 44.
- In fact, there is such a number and it is denoted by the letter e, called the **Euler's number**. It will be further studied as a project when you learn integration; see Section P.3, p. 707.
  (The notation e was chosen by the Swiss mathematician **Leonhard Euler** in 1727, probably because it is the first letter of the word exponential.)
- It turns out that the number e lies between 2 and 3. Later, we will see that the value of e, correct to six decimal places, is

$$e \approx 2.718282 \tag{1.27}$$

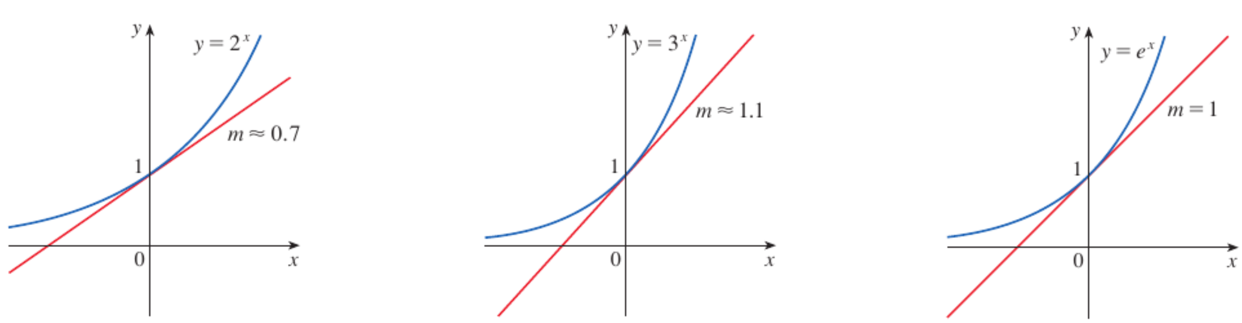


Figure 1.21: The number e.

Remark 1.45. Properties of the Natural Exponential Function
The exponential function $f(x) = e^x$ is an increasing continuous function with domain $\mathbb{R}$ and range $(0, \infty)$. Thus $e^x > 0$ for all x and the x-axis is a **horizontal asymptote** of $f(x) = e^x$.

Example 1.46. Find the domain of the following functions.

(a) $f(x) = \dfrac{1+x}{e^{\cos x}}$

(b) $f(x) = \dfrac{1-e^{x^2}}{1-e^{1-x^2}}$

Solution.

Example 1.47. Graph the function $y = \frac{1}{2}e^{-x} + 1$ and state the domain and range.

Solution.

Exercises 1.5

1. Use the Laws of Exponents to rewrite and simplify each expression.

   (a) $16^2 \cdot 16^{-1.75}$ (b) $\dfrac{3^{5/3}}{3^{2/3}}$ (c) $(\sqrt{3})^{1/2} \cdot (\sqrt{12})^{1/2}$ (d) $(36^{\sqrt{2}})^{\sqrt{2}/4}$

   Ans: (a) 2. (d) 6.

2. Make a rough sketch by hand of the graph of the function.

   (a) $y = e^{|x|}$ (b) $h(x) = 2\left(\dfrac{1}{2}\right)^x - 3$

3. The population of **Starkville**, Mississippi, was 2,689 in the year 1900 and 25,495 in 2020. Assume that the population in Starkville grows exponentially with the model

   $$P_n = P_0 \cdot (1+r)^n,$$

   where n is the elapsed year and r denotes the growth rate per year. Then, we can find $r = 0.018921 (= 1.8921\%)$.

   (a) Estimate the population in 1950 and 2000.
   (b) Approximately when is the population going to reach 50,000?

   Ans: (b) 2056.

4. Let $f(x) = 5^x$. Show that

   $$\frac{f(x+h) - f(x)}{h} = 5^x \left(\frac{5^h - 1}{h}\right)$$

5. The number e is determined so that the slope of the graph of $y = e^x$ at $x = 0$ is exactly 1. Let h be a point near 0. Then

   $$S(h) := \frac{e^h - e^0}{h - 0} = \frac{e^h - 1}{h}$$

   represents the **average slope** of the graph between the two points $(0,1)$ and (h, e^h). Use your calculator to evaluate $S(h)$, for $h = 0.1,\ 0.01,\ 0.001,\ 0.0001$. What can you say about the results?

   Ans: For example, $S(0.01) = 1.0050$.

1.6. Inverse Functions and Logarithms

1.6.1. Inverse Functions

Key Idea 1.48. Let $f : X \to Y$ be a function. For simplicity, consider

$$y = f(x) = 2x + 1. \tag{1.28}$$

- Then, f is a rule that performs two actions: $\times 2$ and followed by $+1$.
- The **reverse** of f must be: -1 followed by $\div 2$.
  - Let $y \in Y$. Then the reverse of f can be written as

    $$x = (y - 1)/2 =: g(y) \tag{1.29}$$

    The function g is the **inverse function** of f.
  - However, it is conventional to choose x for the independent variable. Thus it can be formulated as

    $$y = g(x) = (x - 1)/2. \tag{1.30}$$

- Let's summarize the above:

  (a) **Solve $y = f(x)$ for x**: $x = (y - 1)/2 =: g(y)$.
  (b) **Exchange x and y**: $y = g(x) = (x - 1)/2$.

Note: The first step for finding the inverse function of f is to solve $y = f(x)$ for x, to get $x = g(y)$. Here the required is **for g to be a function**.

Definition 1.49. A function f is called a **one-to-one function** if it never takes on the same value twice; that is,

$$f(x_1) \neq f(x_2) \quad \text{whenever } x_1 \neq x_2. \tag{1.31}$$

Claim 1.50. Horizontal Line Test.
A function is **one-to-one** if and only if no horizontal line intersects its graph more than once.

- $f(x) = x^3$ is one-to-one.
- The function in Figure 1.23 is not one-to-one because $f(x_1) = f(x_2)$ for $x_1 \neq x_2$.

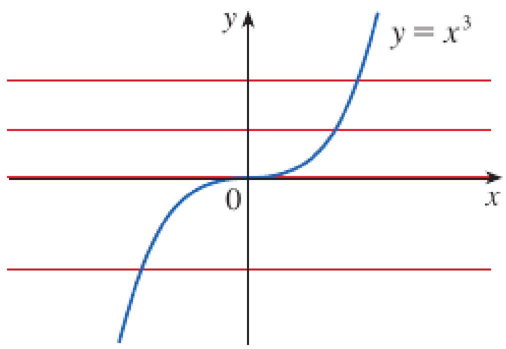


Figure 1.22: $f(x) = x^3$ is one-to-one.

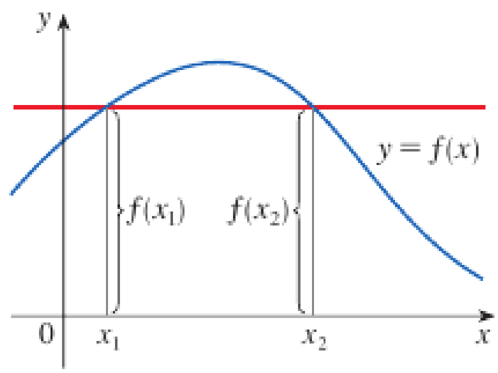


Figure 1.23: This function is not one-to-one because $f(x_1) = f(x_2)$ for $x_1 \neq x_2$.

Example 1.51. Check if the function is one-to-one.

1. $f(x) = x^2$
2. $g(x) = x^2$, $x \geq 0$
3. $h(x) = x^3$

Solution.

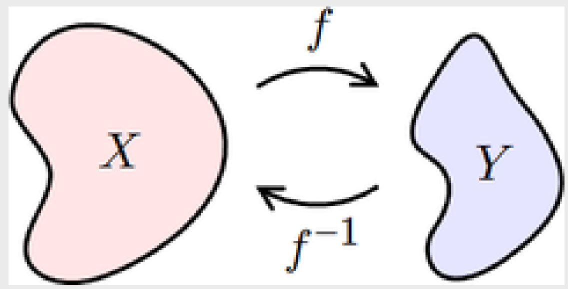
Definition 1.52. Let f be a one-to-one function with domain X and range Y. Then its **inverse function** f^{-1} has domain Y and range X and is defined by

$$f^{-1}(y) = x \iff f(x) = y, \tag{1.32}$$

for any $y \in Y$.

Remark 1.53.
The definition says that if f maps x into y, then f^{-1} maps y back into x. From (1.32), we can obtain the **cancellation equations**

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for all } x \in X \\ f(f^{-1}(y)) &= y \quad \text{for all } y \in Y \end{aligned} \tag{1.33}$$



Example 1.54. For example, if $f(x) = x^3$, then $f^{-1}(x) = x^{1/3}$ and so that the cancellation equations read

$$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(x^3) = (x^3)^{1/3} = x \\ f(f^{-1}(y)) &= f(y^{1/3}) = (y^{1/3})^3 = y \end{aligned}$$

Example 1.55. Assume f is a one-to-one function.

(a) If $f(1) = 5$, what is $f^{-1}(5)$?

(b) If $f^{-1}(8) = -10$, what is $f(-10)$?

Solution.

Caution 1.56.

- Do not mistake the -1 in f^{-1} for an exponent.

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}. \tag{1.34}$$

- If f were not one-to-one, then its inverse would not be uniquely defined and cannot be a function. $\Rightarrow$ An inverse function does not exist.

Strategy 1.57. How to Find the Inverse Function of a One-to-One Function f: Write $y = f(x)$.

Step 1: Solve this equation for x in terms of y (if possible).

Step 2: Interchange x and y; the resulting equation is $y = f^{-1}(x)$.

Example 1.58. Find the inverse of the function $h(x) = \dfrac{6-3x}{5x+7}$.

Solution.

Example 1.59. Find the inverse of the function $f(x) = x^3 + 2$, expressed as a function of x.

Solution. Write $y = x^3 + 2$.

Step 1: Solve it for x:

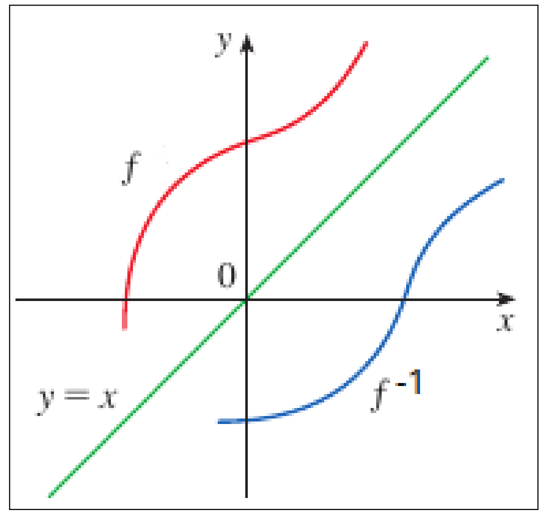
$$x^3 = y - 2 \quad \Rightarrow \quad \boldsymbol{x = \sqrt[3]{y-2}}.$$

Step 2: Exchange x and y:

$$\boldsymbol{y = \sqrt[3]{x-2}}.$$

Therefore the inverse function is

$$f^{-1}(x) = \sqrt[3]{x-2}.$$



Observation 1.60. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Example 1.61. Let $f(x) = x^2 - 2x + 1$, $x \geq 1$. Then the range of f is $[0, \infty)$.

(a) Find the inverse of f.

(b) Find its domain and range.

(c) Plot the graphs of f and f^{-1} in the same coordinates.

Solution.

1.6.2. Logarithmic Functions

Recall: If $a > 0$ and $a \neq 1$, the exponential function $f(x) = a^x$ is either increasing or decreasing and so it is one-to-one by the Horizontal Line Test. It therefore has an inverse function.

Definition 1.62. The **logarithmic function with base a**, written $y = \log_a x$, is the inverse of $y = a^x$ ($a > 0$, $a \neq 1$). That is,

$$\log_a x = y \iff a^y = x. \tag{1.35}$$

Example 1.63. Find the inverse of $y = 2^x$.

Solution.

1. **Solve $y = 2^x$ for x:**

$$x = \log_2 y$$

2. **Exchange x and y:**

$$y = \log_2 x$$

Thus the graph of $y = \log_2 x$ must the reflection of the graph of $y = 2^x$ about $y = x$.

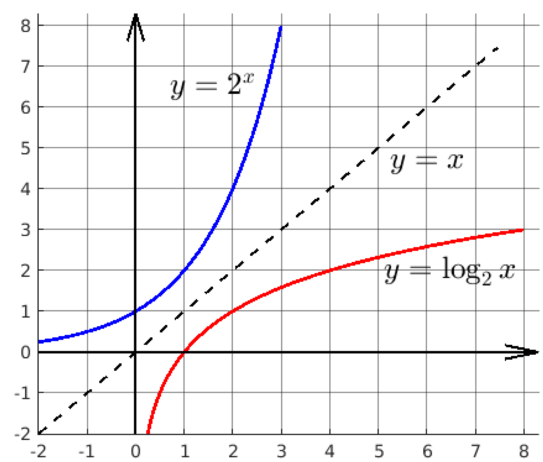


Figure 1.24: Graphs of $y = 2^x$ and $y = \log_2 x$.

Note:

- Equation (1.35) represents the action of "**solving for x**"
- The domain of $y = \log_a x$ must be the range of $y = a^x$, which is $(0, \infty)$.

The Natural Logarithm and the Common Logarithm

Of all possible bases a for logarithms, we will see later that the most convenient choice of a base is the number e.

Definition 1.64.

- The logarithm with base e is called the **natural logarithm** and has a special notation:

$$\log_e x = \ln x \tag{1.36}$$

- The logarithm with base 10 is called the **common logarithm** and has a special notation:

$$\log_{10} x = \log x \tag{1.37}$$

Remark 1.65.

- From your calculator, you can see buttons of LN and LOG, which represent $\ln = \log_e$ and $\log = \log_{10}$, respectively.
- When you implement **a code on computers**, the functions $\ln$ and $\log$ can be called by "`log`" and "`log10`", respectively.

Properties of Logarithms

- **Algebraic Properties**: for $(a > 0,\ a \neq 1)$

$$\begin{array}{lrcl} \text{Product Rule:} & \log_a xy & = & \log_a x + \log_a y \\ \text{Quotient Rule:} & \log_a \dfrac{x}{y} & = & \log_a x - \log_a y \\ \text{Power Rule:} & \log_a x^\alpha & = & \alpha \log_a x \\ \text{Reciprocal Rule:} & \log_a \dfrac{1}{x} & = & -\log_a x \end{array} \tag{1.38}$$

- **Inverse Properties**

$$\begin{array}{ll} a^{\log_a x} = x,\ \ x > 0; & \log_a a^x = x,\ \ x \in \mathbb{R} \\ e^{\ln x} = x,\ \ x > 0; & \ln e^x = x,\ \ x \in \mathbb{R} \end{array} \tag{1.39}$$

Example 1.66. Use the laws of logs to expand $\ln\left(\dfrac{x^2\sqrt{x^2+3}}{3x+1}\right)$.

Solution.

Example 1.67. Simplify the following.

(a) $\log_3 75 - 2\log_3 5$

(b) $2\log_5 100 - 4\log_5 50$

Solution.

Example 1.68. Solve for x.

(a) $e^{5-3x} = 3$.

(b) $\log_3 x + \log_3(x-2) = 1$

(c) $\ln(\ln x) = 0$

Solution.

Ans: (a) $x = \frac{1}{3}(5 - \ln 3)$. (b) $x = 3$. **(Caution: $x = -1$ cannot be a solution.)**

Claim 1.69.

(a) Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}. \tag{1.40}$$

(b) Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a}, \quad (a > 0,\ a \neq 1) \tag{1.41}$$

which is called the **Change-of-Base Formula**.

Proof. (a). $a^x = e^{\ln(a^x)} = e^{x \ln a}$.
(b). $\ln x = \ln(a^{\log_a x}) = (\log_a x)(\ln a)$, from which one can get (1.41). □

1.6.3. Inverse Trigonometric Functions

Note: Trigonometric functions are periodic.

- When we try to find the inverse trigonometric functions, we have a difficulty: because the trigonometric functions are not one-to-one.
- The difficulty is overcome by restricting the domains of these functions so that they become one-to-one.

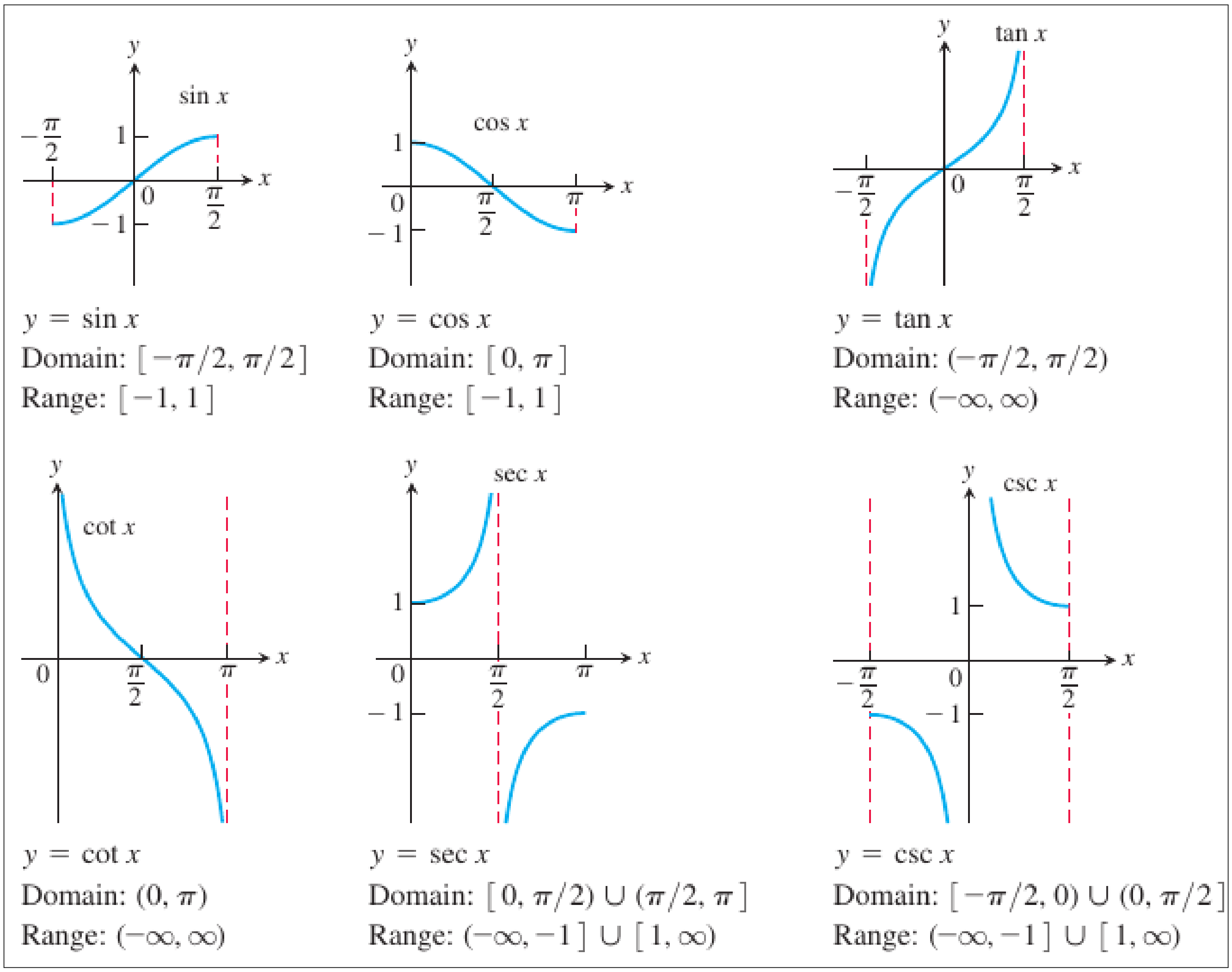


Figure 1.25: Domain restrictions that make the trigonometric functions one-to-one.

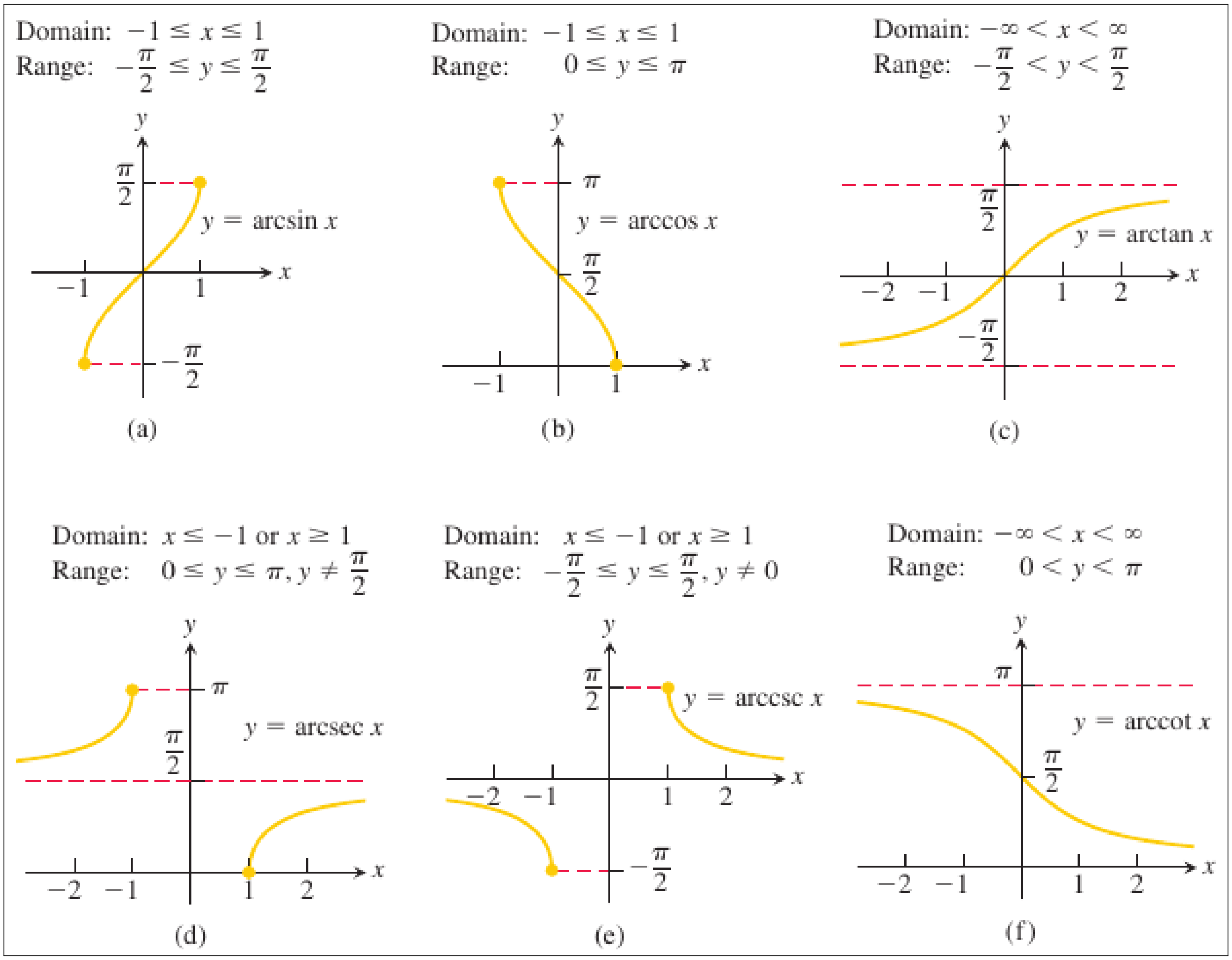


Figure 1.26: Graphs of the six basic inverse trigonometric functions.

Definition 1.70. (The Arcsine and Arccosine Functions)

$$y = \textbf{arcsin}\, x \text{ is the number in } [-\pi/2, \pi/2] \text{ for which } \sin y = x$$
$$y = \textbf{arccos}\, x \text{ is the number in } [0, \pi] \text{ for which } \cos y = x$$

Example 1.71. Evaluate (a) $\arcsin\left(\frac{\sqrt{3}}{2}\right)$ and (b) $\arccos\left(-\frac{1}{2}\right)$.

Solution.

Ans: (b). $2\pi/3$.

Example 1.72. Is it correct? If not, why?

(a) $\arcsin\left(\sin\frac{9\pi}{4}\right) = \frac{9\pi}{4}$

(b) $\cos(\arccos 2) = 2$

Solution.

Definition 1.73. (Inverses of $\tan x$, $\cot x$, $\sec x$, and $\csc x$)

$y = \textbf{arctan}\,x$ is the number in $(-\pi/2, \pi/2)$ for which $\tan y = x$
$y = \textbf{arccot}\,x$ is the number in $(0, \pi)$ for which $\cot y = x$
$y = \textbf{arcsec}\,x$ is the number in $[0, \pi]\backslash\{\pi/2\}$ for which $\sec y = x$
$y = \textbf{arccsc}\,x$ is the number in $[-\pi/2, \pi/2]\backslash\{0\}$ for which $\csc y = x$

Example 1.74. Find the exact value for the expression.

(a) $\cos^{-1}(-1)$

(b) $\arctan(-1)$

(c) $\sin\left(\arcsin\left(-\frac{1}{\sqrt{2}}\right)\right)$

Solution.

Example 1.75. Simplify the expression $\cos(\tan^{-1} x)$.
Solution. Let $y = \tan^{-1} x$. Then $y \in (-\pi/2, \pi/2)$ and $\tan y = x$. Here

$$\begin{cases} \text{What we must find :} & \cos(y) \\ \text{What we know :} & \tan(y) = x \end{cases} \tag{1.42}$$

Now, $\sec^2 y = 1 + \tan^2 y = 1 + x^2$.
$\Rightarrow \sec y = \sqrt{1+x^2}$ (because $\sec y > 0$ for $y \in (-\pi/2, \pi/2)$)
$\Rightarrow \cos(y) = \dfrac{1}{\sqrt{x^2+1}}$.

Ans: $\cos(\tan^{-1} x) = \dfrac{1}{\sqrt{x^2+1}}$.

Note: You can reach the final outcome from (1.42), with a **geometric manipulation**.

Self-study 1.76. Simplify the expression $\sin(\tan^{-1} x)$.
Solution.

Ans: $\sin(\tan^{-1} x) = \dfrac{x}{\sqrt{x^2+1}}$.

Exercises 1.6

1. Let $f(x) = \sqrt{1-x^2}, \;\; 0 \le x \le 1$.

   (a) What symmetry does the graph have?

   (b) Show that f is its own inverse.

   Hint: (b) Use Strategy 1.57.

   Ans: (a). It is symmetric about $y = x$.

2. Find a formula for the inverse of the function; identify its domain and range.

   (a) $f(x) = x^2 - 2x$, $x \le 1$ (b) $f(x) = \dfrac{x+3}{x-2}$

   Ans: (b). $f^{-1}(x) = (2x+3)/(x-1)$; D: $\{x \neq 1\}$; R: $\{y \neq 2\}$

3. Simplify to find the exact value of each expression.

   (a) $\log_5(1/125)$ (b) $\ln(\ln e^{e^{30}})$

   (c) $e^{-2\ln 5}$ (d) $e^{\ln(\ln e^2)}$

   Ans: (b) 30. (d) 2.

4. Solve each equation for x.

   (a) $\ln(x+3) = 2$ (b) $\ln(\ln x) = 1$

   Ans: (b) $x = e^e$.

5. For $\log_2 x + \log_2(x-3) = 2$.

   (a) Solve it for x.

   (b) How many solutions does it have? If there is only one solution, why?

6. Find the exact value of each expression.

   (a) $\tan^{-1}\sqrt{3}$ (b) $\arcsin(\sin(5\pi/4))$

   (c) $\sec^{-1} 2$ (d) $\cos(\sin^{-1}(\frac{5}{13}))$

   Ans: (d) $12/13$.

CHAPTER 2

Limits and Continuity

In This Chapter:

Topics	Applications/Properties
Rates of Change	
Average Rates of Change	Secant Lines
Tangent Lines to Curves	
Limit of a Function	
Limit Laws	
Sandwich Theorem	Basics of Computer Programming
The Precise Definition of a Limit	
One-Sided Limits	
Continuity	
Limits of Continuous Functions	
Intermediate Value Theorem	
Limits Involving Infinity	
Horizontal Asymptotes	
Oblique Asymptotes	
Vertical Asymptotes	

Contents of Chapter 2

2.1. Rates of Change and Tangent Lines to Curves

Note: In the late 16th century, **Galileo** discovered that a solid object dropped from rest (initially not moving) near the surface of the earth and allowed to fall freely will fall a distance proportional to the square of the time it has been falling.

- This type of motion is called **free fall**.
- It assumes negligible air resistance to slow the object down, and that gravity is the only force acting on the falling object.
- If y denotes the distance fallen in feet after t seconds, then the **Galileo's law of free-fall** is

$$y = 16t^2 \text{ ft.} \tag{2.1}$$

The Galileo's law of free-fall states that, in the absence of air resistance, all bodies fall with the same acceleration, independent of their mass.

Average and Instantaneous Speed

Average Speed. When $f(t)$ measures the distance traveled at time t,

$$\text{Average speed over } [t_0, t_1] = \frac{\text{distance traveled}}{\text{elapsed time}} = \frac{f(t_1) - f(t_0)}{t_1 - t_0} \tag{2.2}$$

Example 2.1. A rock breaks loose from the top of a tall cliff. What is its average speed

(a) during the first 2 sec of fall?

(b) during the 1-sec interval between second 1 and second 2?

Solution.

Ans: (a) 32 ft/sec. (b) 48 ft/sec.

Example 2.2. Find **the speed** of the falling rock in Example 2.1 at $t = 1$ and $t = 2$.

Solution. We can calculate the average speed of the rock over a time interval $[t_0, t_0 + h]$, having length $\Delta t = (t_0 + h) - (t_0) = h$, as

$$\frac{\Delta y}{\Delta t} = \frac{16(t_0 + h)^2 - 16t_0^2}{h} \frac{\text{ft}}{\text{sec}}. \tag{2.3}$$

- We cannot use this formula to calculate the **instantaneous speed** at the exact moment t_0 by simply substituting $h = 0$, because we cannot divide by zero.
- But we can use it to calculate average speeds over shorter and shorter time intervals.
- When we do so, by taking smaller and smaller values of h, we see a pattern.

Table 2.1: Average speeds over short time intervals $[t_0, t_0 + h]$.

Length of time interval h	Average speed over interval of length h starting at $t_0 = 1$	Average speed over interval of length h starting at $t_0 = 2$
1	48	80
0.1	33.6	65.6
0.01	32.16	64.16
0.001	32.016	64.016
0.0001	32.0016	64.0016

- The ***average speed*** on intervals starting at $t_0 = 1$ seems to approach a limiting value of 32, as the length of the interval decreases.
- This suggests that the rock is falling at a ***speed*** of 32 ft/sec at $t_0 = 1$.

Let's confirm this algebraically.

- When $t_0 = 1$, Equation (2.3) reads

$$\begin{aligned}\frac{\Delta y}{\Delta t}(t_0 = 1) &= \frac{16(1+h)^2 - 16(1)^2}{h} = \frac{16(1+2h+h^2) - 16(1)^2}{h} \\ &= \frac{32h + 16h^2}{h} = 32 + 16h \;\to\; 32, \;\; \text{as } \; h \to 0.\end{aligned} \tag{2.4}$$

- Similarly, when we set $t_0 = 2$, Equation (2.3) reads

$$\begin{aligned}\frac{\Delta y}{\Delta t}(t_0 = 2) &= \frac{16(2+h)^2 - 16(2)^2}{h} = \frac{16(4+4h+h^2) - 16(2)^2}{h} \\ &= \frac{64h + 16h^2}{h} = 64 + 16h \;\to\; 64, \;\; \text{as } \; h \to 0.\end{aligned} \tag{2.5}$$

As h gets closer and closer to 0, the average speed has the limiting value 64 ft/sec when $t_0 = 2$ sec, as suggested by Table 2.1. □

Average Rates of Change and Secant Lines

Definition 2.3. The **average rate of change** of $y = f(x)$ with respect to x over $[x_1, x_2]$, $x_2 = x_1 + h$, is

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1} = \frac{f(x_1 + h) - f(x_1)}{h}, \quad h \neq 0. \tag{2.6}$$

Remark 2.4. Geometrically, the rate of change of f over $[x_1, x_2]$ is the slope of the line through the points $P(x_1, f(x_1))$ and $Q(x_2, f(x_2))$ (Figure 2.1). In geometry, a line joining two points of a curve is called a **secant line**. Thus, the **average rate of change** of f over $[x_1, x_2]$ is identical with the slope of secant line PQ.

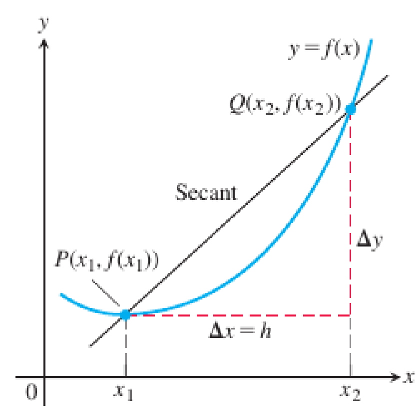


Figure 2.1: A secant to the graph of $y = f(x)$.

Defining the Tangent Line

We use an approach that analyzes the behavior of the secant lines that pass through P and nearby points Q as Q moves toward P along the curve (Figure 2.2).

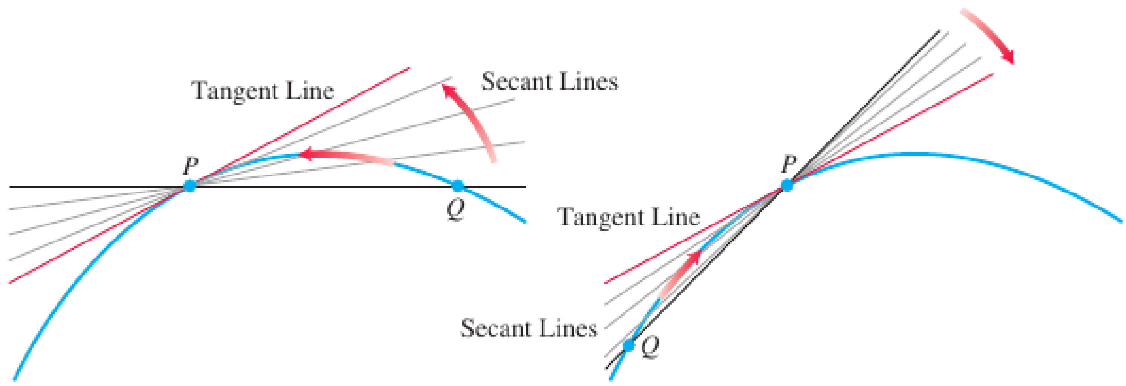


Figure 2.2: Secant lines and the tangent line to the curve at P.

Definition 2.5. (Rough Definition of the Tangent Line)
The **tangent line** to the curve at P is the **limit** of the secant lines, as $Q \to P$ from either side. (We clarify the limit idea in the next section.)

Example 2.6. Find the slope of the tangent line to the parabola $y = x^2$ at the point $(2, 4)$ by analyzing the slopes of secant lines through $(2, 4)$. Write an equation for the tangent line to the parabola at this point.
Solution. The slope of the secant line through $(2, 2^2)$ and $(2 + h, (2 + h)^2)$:

Ans: $y - 4 = 4(x - 2)$.

Remark 2.7. Examples 2.2 and 2.6 show that if the **average rate of change** of $y = f(x)$ converges to a certain number, as $h \to 0$, then the number can be viewed as the **instantaneous rate of change**, i.e., the **slope of the tangent line**.

Exercises 2.1

1. Use the method in Example 2.6 to find (1) the slope of the curve at the given point P, and (2) an equation of the tangent line at P.

   (a) $y = 7 - x^2$, $P(2, 3)$ (b) $y = x^3 - 12x$, $P(1, -11)$

   Ans: (b) slope: -9

2. The deck of a bridge is suspended 275 feet above a river. If a pebble falls off the side of the bridge, the height, in feet, of the pebble above the water surface after t seconds is given by $y = 275 - 16t^2$.

   (a) Find the average velocity of the pebble for the time period beginning when $t = 4$ and lasting
   (i) 0.1 seconds (ii) 0.05 seconds (iii) 0.01 seconds

   (b) Estimate the instantaneous velocity of the pebble when $t = 4$.

   Ans: (a)(ii). -128.8 ft/sec

3. The table shows the position of a motorcyclist after accelerating from rest. Find the average velocity for each time period:

   (a) $[2, 4]$ (b) $[3, 4]$ (c) $[4, 6]$

t (sec)	0	1	2	3	4	5	6
s (feet)	0	4.9	20.6	46.5	79.2	124.8	176.7

4. Let $f(x) = x^2 - x$.

   (a) Use the method in (2.4)-(2.5), p.62, to estimate the instantaneous slopes of f at four different points: $(x, f(x))$ for $x = 0, 1, 2, 3$.

   (b) Plot the results of (a) in the x-and-slope coordinates to find the best-fitting curve.

   Ans: (b) slope$(x) = 2x - 1$.

2.2. Limit of a Function and Limit Laws

Example 2.8. How does the function

$$f(x) = \frac{x^2 - 1}{x - 1}$$

behave near $x = 1$?

Solution. The given formula defines f for all real numbers x except $x = 1$ (since we cannot divide by zero). When $x \neq 1$, it can be simplified as

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x + 1, \text{ for } x \neq 1.$$

- The graph of f is the line $y = x + 1$ with the point $(1, 2)$ removed.
- This removed point is shown as a **hole** in Figure 2.3.
- Even though $f(1)$ is not defined, it is clear that we can make the value of $f(x)$ as close as we want to 2 by choosing x close enough to 1.

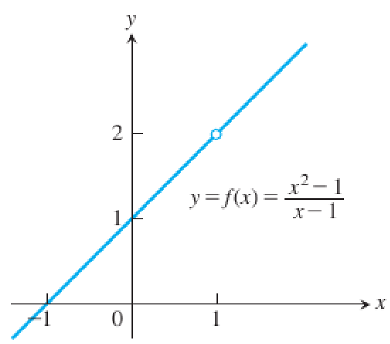


Figure 2.3: The graph of $y = (x^2 - 1)/(x - 1)$.

Intuitive Definition of a Limit

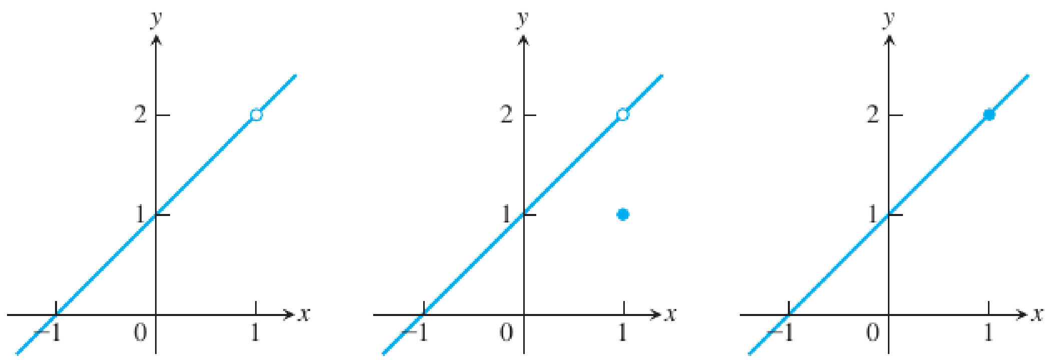
Definition 2.9. Suppose that $f(x)$ is defined on an open interval about a, **except possibly at** a. Then we write

$$\lim_{x \to a} f(x) = L, \tag{2.7}$$

and say **"the limit $f(x)$, as x approaches a, is L"**,
if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by restricting x to be sufficiently close to a (on either side of a) but not equal to a.

Example 2.10.

- The limit of a function does not depend on how the function is defined at the point being approached.
- It does not even matter whether the function is defined at the point.

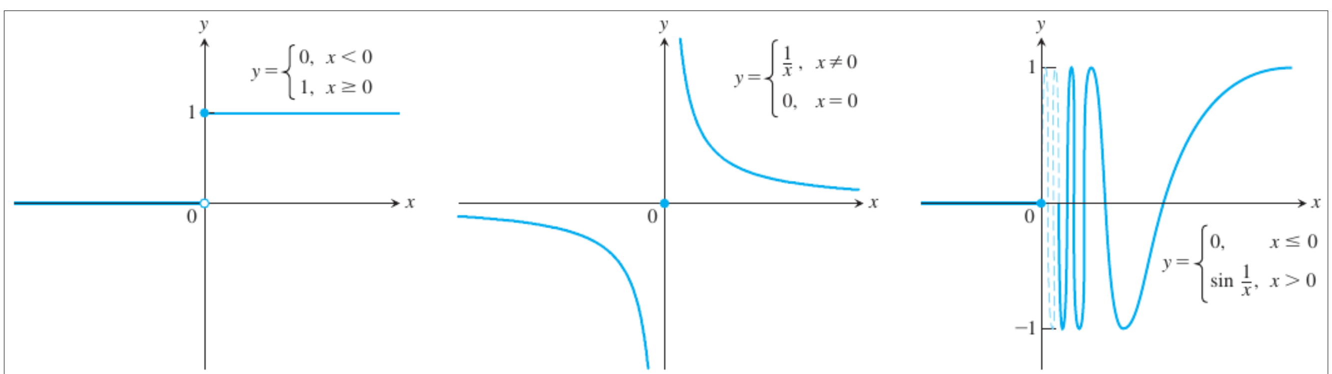


(a) $f(x) = \dfrac{x^2 - 1}{x - 1}$ (b) $g(x) = \begin{cases} \dfrac{x^2-1}{x-1}, & x \neq 1 \\ 1, & x = 1 \end{cases}$ (c) $h(x) = x + 1$

Example 2.11. Discuss the behavior of the following functions, explaining why they have no limit as $x \to 0$.

(a) $U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$ (b) $g(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ (c) $f(x) = \begin{cases} 0, & x \leq 0 \\ \sin\dfrac{1}{x}, & x > 0 \end{cases}$

Solution. (a) The **unit step function** $U(x)$ jumps at $x = 0$. (b) g *grows too large to have a limit*. (b) f *oscillates too much to have a limit*.

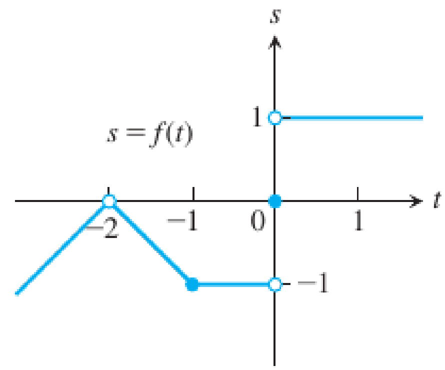


(a) Unit step function $U(x)$ (b) $g(x)$ (c) $f(x)$

Example 2.12. For the function $f(t)$ graphed below, find the following limits or explain why they do not exist.

(a) $\lim_{t\to -2} f(t)$ (b) $\lim_{t\to -1} f(t)$ (c) $\lim_{t\to 0} f(t)$ (d) $\lim_{t\to 1} f(t)$

Solution.



The Limit Laws

Theorem 2.13. *Limit Laws*
Let L, M, c, and k be real numbers, n a positive integer, and

$$\lim_{x\to a} f(x) = L \quad \textit{and} \quad \lim_{x\to a} g(x) = M. \tag{2.8}$$

Then,

1. *Sum Rule:* $\lim_{x\to a}(f(x) + g(x)) = L + M$
2. *Difference Rule:* $\lim_{x\to a}(f(x) - g(x)) = L - M$
3. *Constant Multiple Rule:* $\lim_{x\to a}(k \cdot f(x)) = k \cdot L$
4. *Product Rule:* $\lim_{x\to a}(f(x) \cdot g(x)) = L \cdot M$
5. *Quotient Rule:* $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$
6. *Power Rule:* $\lim_{x\to a}[f(x)]^n = L^n,$
7. *Root Rule:* $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L} = L^{1/n}$

(If n is even, we assume that $f(x) \geq 0$ for x in an interval containing a.)

The Key: Apply $\texttt{lim}_{x\to a}$ to each term, if the limit exists.

Example 2.14. Find the limits.

(a) $\displaystyle\lim_{x\to 0}\left(x^4+\frac{\cos 2x}{100}\right)$ (b) $\displaystyle\lim_{x\to 1}\frac{x^2+x-2}{x^2-x}$

Solution.

Ans: (a) $1/100$. (b) 3.

Example 2.15. Evaluate the limits, by **rationalization**.

(a) $\displaystyle\lim_{x\to 0}\frac{\sqrt{x^2+100}-10}{x^2}$ (b) $\displaystyle\lim_{h\to 1}\frac{h-1}{\sqrt{3h+1}-2}$

Solution.

Ans: (a) $1/20$. (b) $4/3$.

The Sandwich Theorem

Theorem 2.16. ***Sandwich Theorem***
Suppose that $g(x) \le f(x) \le h(x)$ for all x in some open interval containing a, except possibly at $x = a$ itself. Suppose also that

$$\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L.$$

Then $\lim_{x\to a} f(x) = L$.

Example 2.17. Prove that for all θ,

$$\begin{array}{ll} \text{(a)} & -|\theta| \le \sin\theta \le |\theta| \\ \text{(b)} & 0 \le 1-\cos\theta \le |\theta| \end{array} \qquad (2.9)$$

Solution. ***Hint***: Use the geometric definition of the angle in Figure 1.9, p. 26, and the geometric interpretation of trigonometric functions in Figure 1.11, p. 29.

Example 2.18. Use the Sandwich Theorem to verify the following

(a) $\displaystyle\lim_{\theta\to 0} \sin\theta = 0$

(b) $\displaystyle\lim_{\theta\to 0} \cos\theta = 1$

(c) For any function f, $\displaystyle\lim_{x\to a} |f(x)| = 0$ implies $\displaystyle\lim_{x\to a} f(x) = 0$.

Solution. ***Clue***: (b) Use (2.9) with $\cos\theta = 1-(1-\cos\theta)$

Basics of Computer Programming

The **very basics of computer programming** is to understand how to deal with

- recursive iterations (loops)
- conditional statements
- calling functions
- data input/output

Example 2.19. Estimate

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \tag{2.10}$$

by choosing $n = 10^i,\ i = 1, 2, \cdots, 7$.

Solution.

estimate_e.m

```
for i=1:7
    n = 10^i;
    f = (1+1/n)^n;
    % print out the quantities
    fprintf('n= %6g; f(n)= %.8g\n',n,f)
end
```

Result

```
n=     10; f(n)= 2.5937425
n=    100; f(n)= 2.7048138
n=   1000; f(n)= 2.7169239
n=  10000; f(n)= 2.7181459
n= 100000; f(n)= 2.7182682
n=  1e+06; f(n)= 2.7182805
n=  1e+07; f(n)= 2.7182817
```

Using Maple:

maple-estimate-e.mw

```
for i to 7 do
    n := 10^i;
    f := (1 + 1/n)^n;
    # print out the quantities
    printf("n= %6g; f= %.8g\n",n,f);
end do:
```

The result must be the same as that of the Matlab program.

As you can see from the example, **computer programming** is not about **computational languages** but about **mathematical logic**.

Note: The number e can be defined as the limit in (2.10).

Exercises 2.2

1. Explain why the limits do not exist.

   (a) $\lim\limits_{x\to 0} \dfrac{x}{|x|}$ (b) $\lim\limits_{x\to 1} \dfrac{1}{x-1}$

   Hint: (a) Consider two cases: $x < 0$ and $x > 0$.

2. Find the limits.

   (a) $\lim\limits_{x\to 6} 8(x-5)(x-7)$ (b) $\lim\limits_{x\to 2} \dfrac{2x+5}{11-x^3}$

   Ans: (b) 3.

3. Find the limits.

   (a) $\lim\limits_{t\to 1} \dfrac{t^2+t-2}{t^2-1}$ (b) $\lim\limits_{x\to 4} \dfrac{4-x}{5-\sqrt{x^2+9}}$

   Ans: (b) $5/4$.

4. Evaluate the limit
$$\lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$$
   for the given function f and the value of x.

   (a) $f(x) = x^2,\ x = -2$
   (b) $f(x) = 1/x,\ x = 2$

   Ans: (b) $-1/4$.

5. Solve the following.

   (a) If $\lim\limits_{x\to 4} \dfrac{f(x)-5}{x-2} = 1$, find $\lim\limits_{x\to 4} f(x)$.
   (b) If $\lim\limits_{x\to 2} \dfrac{f(x)-5}{x-2} = 1$, find $\lim\limits_{x\to 2} f(x)$.
   (c) If $\lim\limits_{x\to 2} \dfrac{f(x)-5}{x-2} = 2$, find $\lim\limits_{x\to 2} f(x)$.

   Ans: (b) 5.

6. **Challenge** If $\lim\limits_{x\to 2} \dfrac{x^2+ax+b}{x-2} = 1$, find the constants a and b.

2.3. The Precise Definition of a Limit

Example 2.20. Consider the function $f(x) = 2x-1$ near $x = 4$. **Intuitively it seems clear** that $f(x)$ is close to 7 when x is close to 4, so we may write

$$\lim_{x\to 4} f(x) = 7. \tag{2.11}$$

However, **how close to $x = 4$ does x have to be** so that **$f(x)$ differs from 7 by, say, less than 1 unit**?

Solution. **Asked**: For what values of x is $|f(x) - 7| < 1$?

- To find the answer, express $f(x)$ explicitly in terms of x:

$$|f(x) - 7| = |(2x - 1) - 7| = |2x - 8| < 1.$$

- We solve the inequality:

$$\begin{aligned} &-1 < 2x - 8 < 1 \\ &\Rightarrow 7 < 2x < 9 \\ &\Rightarrow 3.5 < x < 4.5 \\ &\Rightarrow -0.5 < x - 4 < 0.5 \end{aligned}$$

- Thus, keeping x within 0.5 units of $x = 4$ will keep $f(x)$ within 1 unit of $y = 7$. That is,

$$|f(x) - 7| < 1, \quad \text{whenever } |x - 4| < 0.5. \quad \square \tag{2.12}$$

Remark 2.21. In general: For the function $f(x) = 2x - 1$, for arbitrary $\varepsilon > 0$, there exists a number $\delta = \delta(\varepsilon) > 0$ such that

$$|f(x) - 7| < \varepsilon, \quad \text{whenever } |x - 4| < \delta. \tag{2.13}$$

- In the above example, if $\varepsilon = 1$, then we have found $\delta = 0.5$.
- Another example: if $\varepsilon = 0.01$, then we can set $\delta = 0.005$.

The Precise Definition of a Limit

Definition 2.22. Let $f(x)$ be defined on an open interval about a, except possibly at a itself. We say that

the limit of $f(x)$ is L as x approaches a,

and write

$$\lim_{x\to a} f(x) = L, \tag{2.14}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta. \tag{2.15}$$

Note: The value of f at $x = a$ does not influence the existence of a limit; $x = a$ is excluded. The above definition is sometimes called the **ε-δ definition of the limit**.

Example 2.23. Use Definition 2.22 to show

$$\lim_{x\to 1}(5x - 3) = 2.$$

Solution. Let $f(x) = 5x - 3$ and $\varepsilon > 0$ be set arbitrarily. Then, we should find $\delta > 0$ such that

$$|f(x) - 2| < \varepsilon \text{ whenever } 0 < |x - 1| < \delta. \tag{2.16}$$

- As in Example 2.20, start with $|f(x) - 2| < \varepsilon$:

$$\begin{aligned} |f(x) - 2| = |(5x - 3) - 2| = |5x - 5| &< \varepsilon \\ 5|x - 1| &< \varepsilon \\ |x - 1| &< \varepsilon/5 \end{aligned}$$

- Thus, we can take $\delta = \varepsilon/5$. Such a δ would satisfy (2.16). □

Remark 2.24. In the last example, the value of $\delta = \varepsilon/5$ is not the only value that satisfies (2.16). Any smaller positive δ will do as well. The definition does not ask for the "best" positive δ, just one that will work.

Example 2.25. For the limit $\lim_{x\to 5}\sqrt{x-1} = 2$, find a δ that works for $\varepsilon = 1$. That is, find a δ such that

$$|\sqrt{x-1} - 2| < 1 \text{ whenever } 0 < |x-5| < \delta.$$

Solution.

Ans: $\delta = 3$.

Strategy 2.26. The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x-a| < \delta$$

can be accomplished in two steps.

1. Solve the inequality $\boldsymbol{|f(x) - L| < \varepsilon}$ to find an open interval $\boldsymbol{(c, d)}$ containing a.
2. Find a value of a $\delta > 0$ that places the open interval $\boldsymbol{(a-\delta, a+\delta)}$ **inside the interval** $\boldsymbol{(c, d)}$.

Exercises 2.3

1. Each of the following gives a function $f(x)$ and numbers L, a, and $\varepsilon > 0$. In each case, an open interval about a on which the inequality $|f(x) - L| < \varepsilon$ holds. Then give a value for $\delta > 0$ such that
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$
   (a) $f(x) = 1/x, \ L = 1/4, \ a = 4, \ \varepsilon = 0.05$
   (b) $f(x) = \sqrt{x - 7}, \ L = 4, \ a = 23, \ \varepsilon = 1$

   Ans: (a) $(10/3, 5)$, $\delta = 2/3$.

2. Each of the following gives a function $f(x)$ and numbers a and $\varepsilon > 0$. Find $L = \lim_{x \to a} f(x)$. Then find a number $\delta > 0$ such that
$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta.$$
   (a) $f(x) = \dfrac{x^2 - 4}{x - 2}, \ a = 2, \ \varepsilon = 0.05$
   (b) $f(x) = \sqrt{1 - 5x}, \ a = -3, \ \varepsilon = 0.5$

   Ans: (b) $L = 4$, $\delta = 0.75$.

3. Use the ε-δ definition of the limit, Definition 2.22, to prove the limit statements.
   (a) $\displaystyle\lim_{x \to 9} \sqrt{x - 5} = 2.$
   (b) $\displaystyle\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = -6.$

   Hint: (b) Start with $\left|\dfrac{x^2 - 9}{x + 3} + 6\right| < \varepsilon$. When $x \neq -3$, it can be written as $|(x - 3) + 6| < \varepsilon$. Now, solve the inequality.

4. **Challenge** Use Definition 2.22 to prove that $\lim_{x \to a} \sqrt{x} = \sqrt{a}$.
   Hint: Use $|\sqrt{x} - \sqrt{a}| = \dfrac{|x - a|}{\sqrt{x} + \sqrt{a}}$.

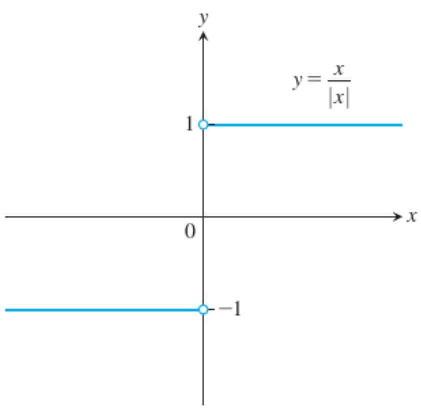
2.4. One-Sided Limits

Definition 2.27. A **one-sided limit** is the value the function $f(x)$ approaches, as x approaches a specified point from **one side only**, either from the left or from the right.

- If x approaches a from the right, the limit L^+ is a **right-hand limit** or *limit from the right*; we write
$$\lim_{x\to a^+} f(x) = L^+. \tag{2.17}$$
- If x approaches from the left, the limit L^- is a **left-hand limit** or *limit from the left*; we write
$$\lim_{x\to a^-} f(x) = L^-. \tag{2.18}$$

Example 2.28. Let $f(x) = x/|x|$. Then $\lim_{x\to 0} f(x)$ does not exist; see Exercise 1 of Section 2.2. Find one-sided limits for $f(x)$ at $x = 0$.

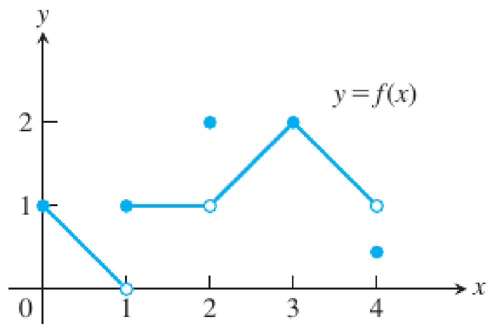
Solution.



Theorem 2.29. *Suppose that a function f is defined on an* ***open interval*** *containing a, except perhaps at a itself. Then $f(x)$ has a limit as x approaches a if and only if it has left-hand and right-hand limits there and these one-sided limits are equal:*

$$\lim_{x\to a} f(x) = L \iff \lim_{x\to a^-} f(x) = L \quad \text{and} \quad \lim_{x\to a^+} f(x) = L \tag{2.19}$$

Example 2.30. For the function graphed, complete the table.



a	$\lim_{x\to a^-} f(x)$	$\lim_{x\to a^+} f(x)$	$\lim_{x\to a} f(x)$
0	does not exist	1	1
1			
2			
3			
4			1

Note: Equation (2.19) is exceptionally applied at domain endpoints.

Definition 2.31. Precise Definitions of One-Sided Limits

(a) Assume the domain of f contains an interval (a, c) to the right of a. We say that $f(x)$ has **right-hand limit** L at a, and write

$$\lim_{x\to a^+} f(x) = L, \tag{2.20}$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } a < x < a + \delta. \tag{2.21}$$

(b) Assume the domain of f contains an interval (b, a) to the left of a. We say that $f(x)$ has **left-hand limit** L at a, and write

$$\lim_{x\to a^-} f(x) = L, \tag{2.22}$$

if for every number $\varepsilon > 0$ there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } a - \delta < x < a. \tag{2.23}$$

Example 2.32. Prove that $\lim_{x\to 0^+}\sqrt{x}=0$.

Solution. Let $\varepsilon>0$ be given. Then we should find $\delta>0$ such that

$$|\sqrt{x}-0|<\varepsilon \ \text{ whenever } \ 0<x<0+\delta, \tag{2.24}$$

or

$$\sqrt{x}<\varepsilon \ \text{ whenever } \ 0<x<\delta.$$

Squaring both sides of this last inequality gives

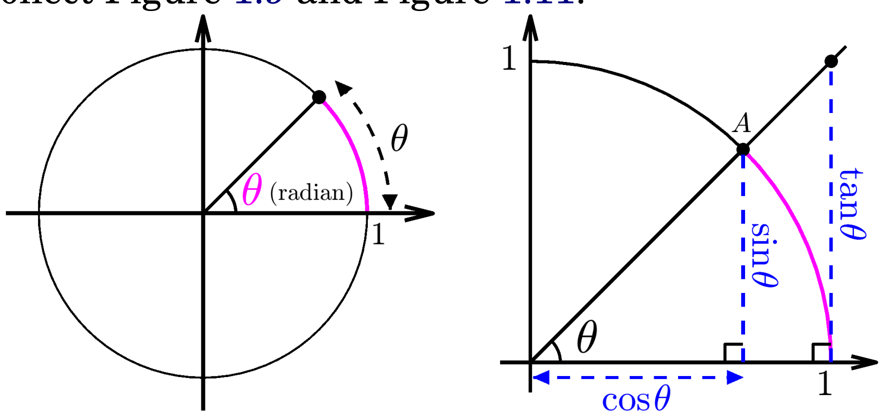
$$x<\varepsilon^2 \ \text{ whenever } \ 0<x<\delta.$$

Thus we may choose $\delta=\varepsilon^2$, with which (2.24) follows. □

Theorem 2.33. Limits involving $\sin\theta/\theta$

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta}=1. \qquad (\theta \textit{ in radians}) \tag{2.25}$$

Solution. Collect Figure 1.9 and Figure 1.11.



It follows from the geometric interpretations that for $|\theta|$ small,

$$\begin{array}{ll} \sin\theta<\theta<\tan\theta, & \text{when } \ \theta>0 \\ \sin\theta>\theta>\tan\theta, & \text{when } \ \theta<0 \end{array}$$

Dividing the inequalities by $\sin\theta$ reads

$$1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta} \quad\Longrightarrow\quad 1>\frac{\sin\theta}{\theta}>\cos\theta.$$

Since $\lim\limits_{\theta\to 0}\cos\theta=1$, (2.25) follows from the Sandwich Theorem. □

Example 2.34. Show that (a) $\lim_{x\to 0} \frac{\sin 2x}{5x} = \frac{2}{5}$ and (b) $\lim_{x\to 0} \frac{\cos x - 1}{x} = 0.$

Solution. ***Hint***: (b) Use the half-angle formula $\cos x = 1 - 2\sin^2(x/2)$.

sinc function

Definition 2.35. The **sinc function**, denoted by $\mathrm{sinc}\,(x)$, has two forms, normalized and unnormalized.

(a) The **unnormalized sinc function** is defined by

$$\mathrm{sinc}\, x = \begin{cases} (\sin x)/x, & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{2.26}$$

which is also called the **sampling function**.

(b) The **normalized sinc function** is defined by

$$\mathrm{sinc}\, x = \begin{cases} (\sin \pi x)/(\pi x), & x \neq 0 \\ 1, & x = 0 \end{cases} \tag{2.27}$$

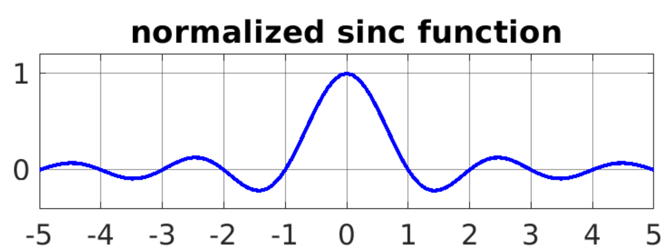
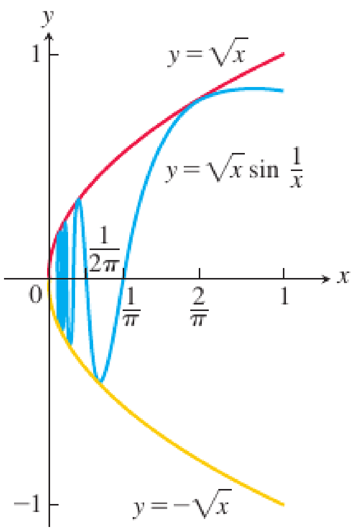


Figure 2.4: The normalized sinc function.

Exercises 2.4

1. Let $g(x) = \sqrt{x}\sin(1/x)$, as graphed in the figure.



   (a) Does $\lim_{x\to 0^+} g(x)$ exist? If so, what is it? If not, why not?

   (b) Does $\lim_{x\to 0^-} g(x)$ exist? If so, what is it? If not, why not?

   (c) Does $\lim_{x\to 0} g(x)$ exist? If so, what is it? If not, why not?

Ans: (c) Yes. 0.

2. Let $f(x) = \begin{cases} \sqrt{1-x^2}, & 0 \le x < 1 \\ 1, & 1 \le x < 2 \\ 2, & x = 2 \end{cases}$ Graph the function and answer these questions.

   (a) What are the domain and range of f?

   (b) At what points a, if any, does $\lim_{x\to a} f(x)$ exist?

   (c) At what points does the left-hand limit exist but not the right-hand limit?

   (d) At what points does the right-hand limit exist but not the left-hand limit?

Ans: (b) $[0,1)\cup(1,2]$. (c) $x = 2$.

3. Find the limits.

   (a) $\displaystyle\lim_{x\to 1^+} \frac{\sqrt{x}\,(x-1)}{|x-1|}$ (b) $\displaystyle\lim_{x\to 1^-} \frac{\sqrt{x}\,(x-1)}{|x-1|}$

   (c) $\displaystyle\lim_{x\to 0^+} \frac{|\sin x|}{x}$ (d) $\displaystyle\lim_{x\to 0^-} \frac{|\sin x|}{x}$

Ans: (b) -1.

4. Use (2.25) to find limits.

   (a) $\displaystyle\lim_{h\to 0^+} \frac{h}{\sin 3h}$ (b) $\displaystyle\lim_{x\to 0} \frac{\tan 2x}{x}$

   (c) $\displaystyle\lim_{t\to 0} \frac{\tan(1-\cos t)}{1-\cos t}$ (d) $\displaystyle\lim_{\theta\to 0} \sin\theta\cot 2\theta$

Ans: (c) 1. (d) $1/2$.

5. Let $f(x) = \begin{cases} x^2 \sin(1/x), & x < 0 \\ \sqrt{x}, & x > 0 \end{cases}$

   (a) Find $\lim\limits_{x \to 0^-} f(x)$.

   (b) Find $\lim\limits_{x \to 0^+} f(x)$.

   (c) Based on your conclusions in parts (a) and (b), can you say anything about $\lim\limits_{x \to 0} f(x)$? Give reasons for your answer.

Ans: (c) The limit exists and is 0. Why?

2.5. Continuity

2.5.1. Definition of Continuity

Note: When the graph of f has **no breaks** at a, we would say that f **is continuous at** a.

Definition 2.36. A function f is **continuous** at a if

$$\lim_{x\to a} f(x) = f(a). \tag{2.28}$$

Remark 2.37. Continuity Test
A function $f(x)$ is continuous at a point $x = a$ if and only if it meets the following three conditions.

$$\begin{array}{lll} 1. & f(a) \text{ is defined} & (\Leftrightarrow \quad a \text{ lies in the domain of } f) \\ 2. & \lim_{x\to a} f(x) \text{ exists} & (f \text{ has a limit as } x \to a) \\ 3. & \lim_{x\to a} f(x) = f(a) & (\text{the limit equals the function value}) \end{array} \tag{2.29}$$

Example 2.38. If any of three conditions in (2.29) fail, then f is **discontinuous** at a. There are four different types of discontinuities:

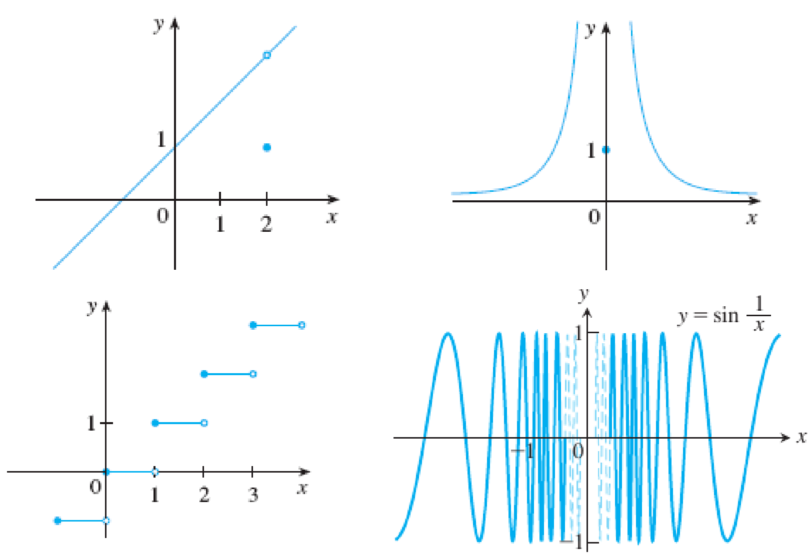


Figure 2.5: **Removable discontinuity**, **infinite discontinuity**, **jump discontinuity**, and **oscillating discontinuity**.

Example 2.39. Where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $g(x) = \begin{cases} \dfrac{1}{x^2}, & \text{if } x \neq 0 \\ 1, & \text{if } x = 0 \end{cases}$

Solution.

Definition 2.40. One-Sided Continuity

- A function f is **right-continuous** at a (or **continuous from the right**) if
$$\lim_{x \to a^+} f(x) = f(a). \tag{2.30}$$
- A function f is **left-continuous** at a (or **continuous from the left**) if
$$\lim_{x \to a^-} f(x) = f(a). \tag{2.31}$$

Example 2.41. State the numbers at which f is discontinuous. For each number, state whether f is continuous from the right, from the left, or neither.

Solution.

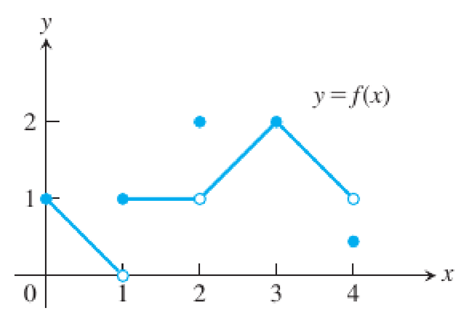


Figure 2.6: The figure used in Example 2.30.

Theorem 2.42. *Properties of Continuous Functions*
If f and g are continuous at a and k is a constant, then the following functions are also continuous at a:

(a) $f + g$
(b) $f - g$
(c) kf
(d) fg
(e) f/g, if $g(a) \neq 0$
(f) f^n, n a positive integer

Note: The following types of functions are continuous at every number in their **natural domains**:

- Polynomials
- Rational functions
- Root functions
- Trigonometric functions
- Exponential functions
- Logarithmic functions
- **Inverse functions**: When a continuous function defined on an interval has an inverse, the inverse function is itself a continuous function over its own domain.

Example 2.43. Where is the function $f(x) = \dfrac{\ln x + \tan^{-1} x}{x^2 - 1}$ continuous?

Solution.

Ans: $x > 0$, $x \neq 1$; which is its natural domain.

Example 2.44. Use the definition of continuity and the properties of limits to show that the following function is continuous at the given number a.

$$g(t) = \frac{t^2 + 5t}{2t + 1}, \quad a = 2.$$

Solution.

Example 2.45. Show that f is continuous on $(-\infty, \infty)$.

$$f(x) = \begin{cases} 1 - x^2, & \textbf{if } x \leq 1 \\ \ln x, & \textbf{if } x > 1 \end{cases}$$

Solution.

Example 2.46. Explain why the function is discontinuous at $x = 3$. Is it continuous from one side?

$$f(x) = \begin{cases} \dfrac{2x^2 - 5x - 2}{x - 3}, & \text{if } x \neq 3 \\ 6, & \text{if } x = 3 \end{cases}$$

Solution.

Example 2.47. For what value of the constant c is the function f continuous everywhere?

$$f(x) = \begin{cases} cx^2 + 2x, & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Solution.

2.5.2. More Properties of Continuous Functions

Continuity of Compositions of Functions

Theorem 2.48. *If f is continuous at a and g is continuous at $f(a)$, then the composition $g \circ f$ is continuous at a.*

Example 2.49. Show that the following functions are continuous on their natural domains.

(a) $y = \left|\dfrac{x-2}{x^2-2}\right|$

(b) $y = \left(\dfrac{\sin^2 x}{x^2+2}\right)^{1/2}$

Solution.

Limits of Continuous Functions

Theorem 2.50. *If $\lim_{x\to a} f(x) = b$ and g is continuous at b, then*

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(b) \tag{2.32}$$

Note: **Continuity and limit are commutative**, when the limit exists.

Example 2.51. Find $\displaystyle\lim_{x\to\pi/2} \cos\left(2x + \sin\left(\frac{3\pi}{2} + x\right)\right)$.

Solution.

Ans: -1

Theorem 2.52. *Intermediate Value Theorem (IVT)*
*Suppose that f is **continuous** on a **closed interval** $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.*

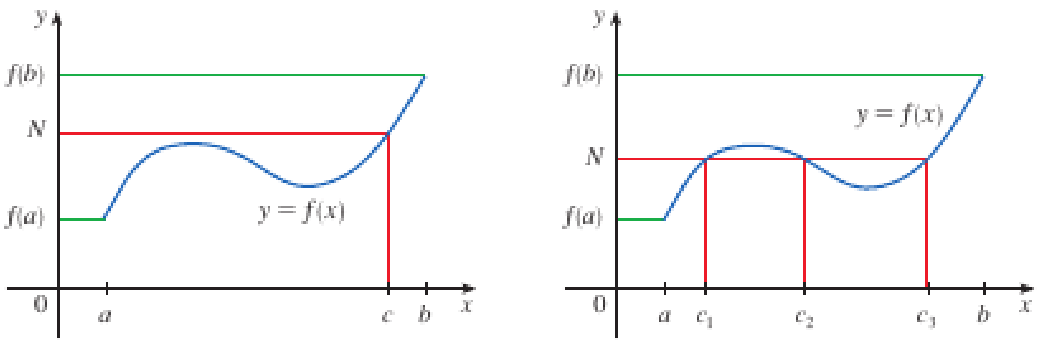


Figure 2.7: There is *at least* one such c that $f(c) = N$.

Remark 2.53. Consequences of the IVT

- **Connectedness of the Graph**: The IVT implies that the graph of a continuous function cannot have any breaks over the interval. It will be **connected** – a single, unbroken curve.
- **Root Finding**: We call a solution of the equation $f(x) = 0$ a **root of the equation** or a **zero of the function** f. The IVT tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

Example 2.54. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

(a) $e^x = 3 - 2x, \quad [0, 1]$

(b) $f(x) = x^3 + x^2 - 4x - 4, \quad [1, 3]$

Solution.

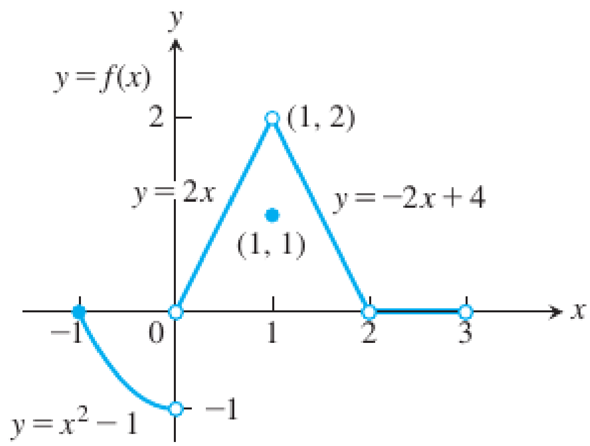
Continuous Extension to a Point

Example 2.55. Define $f(4)$ in a way that extends $f(x) = \dfrac{x^2 - 16}{x^2 - 3x - 4}$ to be continuous at $x = 4$.

Solution.

Exercises 2.5

Exercises 1-4 refer to the function f graphed.



1. (a) Does $f(-1)$ exist?
   (b) Does $\lim\limits_{x\to -1^+} f(x)$ exist?
   (c) Does $\lim\limits_{x\to -1^+} f(x) = f(-1)$?
   (d) Is f continuous at $x = -1$?

   Ans: All: yes

2. (a) Does $f(1)$ exist?
   (b) Does $\lim\limits_{x\to 1} f(x)$ exist?
   (c) Does $\lim\limits_{x\to 1} f(x) = f(1)$?
   (d) Is f continuous at $x = 1$?

   Ans: (c) no.

3. (a) Is f defined at $x = 2$?
   (b) Is f continuous at $x = 2$?

   Ans: no.

4. What value should be assigned to $f(2)$ to make the extended function continuous at $x = 2$?

5. At what points are the functions continuous?

   (a) $y = \dfrac{x+3}{x^2-3x-10}$

   (b) $f(x) = \begin{cases} \dfrac{x^3-8}{x^2-4}, & x \neq 1,\ \ x \neq -2 \\ 3, & x = 2 \\ 4, & x = -2 \end{cases}$

   Ans: (b) $x \neq -2$

6. Find the limits. Are the functions continuous at the point being approached?

   (a) $\lim\limits_{t\to 0} \sin\left(\dfrac{\pi}{2}\cos(\tan t)\right)$

   (b) $\lim\limits_{x\to 0} \tan\left(1 - \dfrac{\sin x}{x}\right)$

   Ans: (b) 0, no.

7. For what value of b is

$$g(x) = \begin{cases} \dfrac{x-b}{b+1}, & x \leq 0 \\ x^2 + b, & x > 0 \end{cases}$$

   continuous at every x?

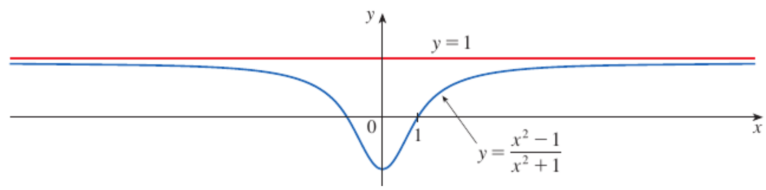
   Ans: $b = 0, -2$

8. Use the Intermediate Value Theorem to show that the equation $x^3 - 15x + 1 = 0$ has three solutions in the interval $[-4, 4]$.

9. (a) Show that the absolute value function $f(x) = |x|$ is continuous everywhere.
   (b) Prove that if f is a continuous function on an interval, then so is $|f|$.
   (c) Is the converse of the statement in part (b) also true?
   In other words, if $|f|$ is continuous, does it follow that f is continuous?
   If so, prove it. If not, find a counterexample.

2.6. Limits Involving Infinity; Asymptotes

Let's begin by investigating the behavior of the function f graphed



You can see

$$\lim_{x\to-\infty} \frac{x^2-1}{x^2+1} = 1 \quad \text{and} \quad \lim_{x\to\infty} \frac{x^2-1}{x^2+1} = 1$$

Definition 2.56. Intuitive Definition of a Limit at Infinity
Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x\to\infty} f(x) = L \tag{2.33}$$

means that the values of $f(x)$ can be made arbitrarily close to L by requiring x **to be sufficiently large**.

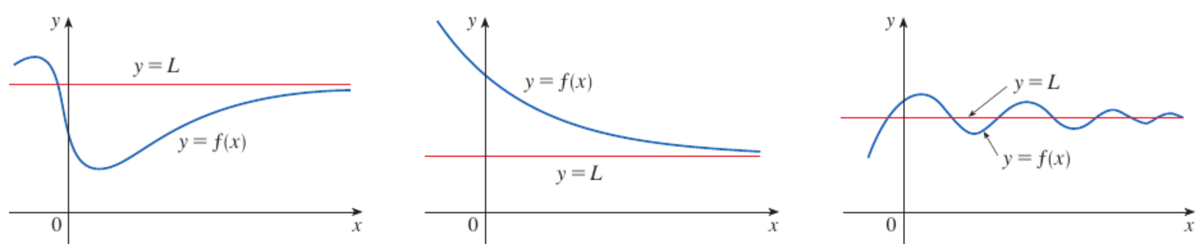


Figure 2.8: Examples illustrating $\lim_{x\to\infty} f(x) = L$.

Remark 2.57. Definition 2.56 can be stated formally:
$f(x)$ has the **limit** L **as** x **approaches infinity** and write

$$\lim_{x\to\infty} f(x) = L, \tag{2.34}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number M such that for all x in the domain of f,

$$|f(x) - L| < \varepsilon \;\text{ whenever }\; x > M. \tag{2.35}$$

Definition 2.58. Let f be a function defined on some interval $(-\infty, a)$. Then

$$\lim_{x\to-\infty} f(x) = L \tag{2.36}$$

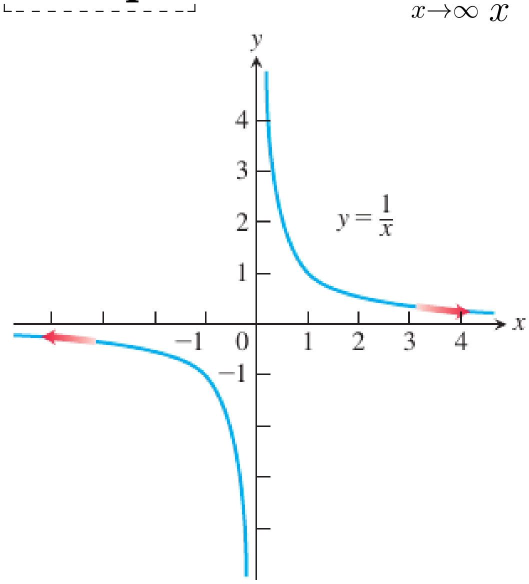
means that the values of $f(x)$ can be made arbitrarily close to L by requiring x **to be sufficiently large negative**.

2.6.1. Horizontal Asymptotes

Definition 2.59. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L. \tag{2.37}$$

Example 2.60. Find $\displaystyle\lim_{x\to\infty} \frac{1}{x}$ and $\displaystyle\lim_{x\to-\infty} \frac{1}{x}$.



Example 2.61. Evaluate the limit or show that it does not exist.
$$\lim_{x\to\infty} \frac{6x^2+8x-3}{3x^2+2}$$

Solution. ***Hint***: **Divide the numerator and the denominator by the highest power of x in the denominator.**

Example 2.62. Find the limits.

(a) $\displaystyle\lim_{x\to\infty} \frac{x^{-1}+x^{-4}}{x^{-2}+x^{-3}}$

(b) $\displaystyle\lim_{x\to-\infty} \left(\frac{x^2+x-2}{8x^2-3}\right)^{1/3}$

Solution.

Example 2.63. Evaluate the limits or show that they do not exist.

(a) $\lim_{x\to\infty} (\sqrt{x^2+1} - x)$ (b) $\lim_{x\to 2^+} \arctan\left(\dfrac{1}{x-2}\right)$ (c) $\lim_{x\to 2^-} \arctan\left(\dfrac{1}{x-2}\right)$

Solution.

Example 2.64. Find

(a) $\lim_{x\to\infty} \sin(1/x)$ (b) $\lim_{x\to\pm\infty} x\sin(1/x)$

Clue: (b) Let $t = 1/x$. Then it can be written as $\lim_{t\to 0^\pm} \dfrac{1}{t}\sin t$.

Solution.

Example 2.65. Find the horizontal asymptotes of the graph of
$f(x) = \dfrac{\sqrt{4x^6+1}}{2-x^3}$.

Solution. ***Clue***: When $x < 0$, $\sqrt{4x^6+1} = |x^3|\sqrt{4+1/x^6} = -x^3\sqrt{4+1/x^6}$.

Ans: $y = \pm 2$.

Self-study 2.66. Find the horizontal asymptotes of the graph of
$f(x) = \dfrac{x^3-2}{|x|^3+1}$.

Solution.

Ans: $y = \pm 1$.

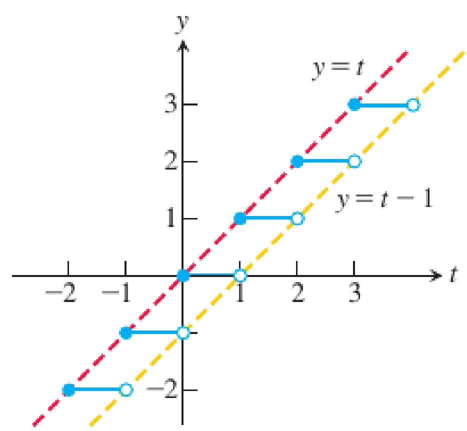
Example 2.67. Evaluate the limits or show that they do not exist.

(a) $\lim_{x\to 0^-} e^{1/x}$ (b) $\lim_{x\to 0^+} e^{1/x}$

Solution.

Example 2.68. Find $\lim_{x\to 0^+} x\left\lfloor \frac{1}{x} \right\rfloor$, where $\lfloor\cdot\rfloor$ denotes the *greatest integer function*.

Solution. ***Clue***: Let $t = 1/x$. For $t > 0$,
$t - 1 \le \lfloor t \rfloor \le t. \Rightarrow 1 - \frac{1}{t} \le \frac{1}{t}\lfloor t \rfloor \le 1.$



Ans: 1.

2.6.2. Oblique Asymptotes

Example 2.69. Find the **oblique asymptote** (or **slant line asymptote**) of the graph of

$$f(x) = \frac{x^2 - 3}{2x - 4}.$$

Then sketch the graph. **(Note that the numerator has one order higher than the denominator.)**

Solution. ***Hint***: Try to divide $2x - 4$ into $x^2 - 3$, to find a linear part and a remainder for f.

Ans: $y = x/2 + 1$.

2.6.3. Infinite Limits

Example 2.70. Discuss the behavior of

$$f(x) = \frac{1}{1 - x} \quad \text{as } x \to 1^{\pm}, \ -\infty, \ \infty.$$

Solution.

Definition 2.71. Infinite Limits

1. We say that $f(x)$ **approaches infinity as x approaches a**, and write
$$\lim_{x\to a} f(x) = \infty, \tag{2.38}$$
if for every positive real number K there exists a corresponding $\delta > 0$ such that
$$f(x) > K \quad \text{whenever } 0 < |x-a| < \delta. \tag{2.39}$$

2. We say that $f(x)$ **approaches negative infinity as x approaches a**, and write
$$\lim_{x\to a} f(x) = -\infty, \tag{2.40}$$
if for every negative real number K there exists a corresponding $\delta > 0$ such that
$$f(x) < K \quad \text{whenever } 0 < |x-a| < \delta. \tag{2.41}$$

Example 2.72. Prove that $\lim_{x\to 0} \dfrac{1}{x^2} = \infty$.

Solution. ***Clue***: Start with $f(x) = \dfrac{1}{x^2} > K$ for an arbitrary $K > 0$. What to do is to find a corresponding $\delta > 0$.

2.6.4. Vertical Asymptotes

Definition 2.73. A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x \to a^+} f(x) = \pm\infty \quad \text{or} \quad \lim_{x \to a^-} f(x) = \pm\infty. \tag{2.42}$$

Example 2.74. Find the vertical, horizontal, and oblique asymptotes of each curve.

(a) $f(x) = \dfrac{2}{x^2 - 3x}$

(b) $g(x) = \dfrac{x^2 + 2x}{x + 1}$

Solution.

Exercises 2.6

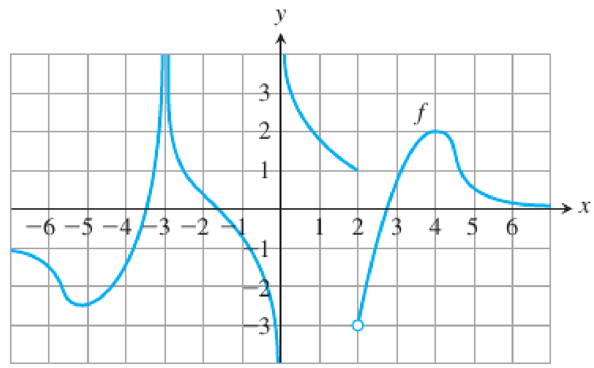
1. For the function f whose graph is given, determine the following limits.

   (a) $\lim_{x\to -3^-} f(x)$
   (b) $\lim_{x\to -3^+} f(x)$
   (c) $\lim_{x\to -3} f(x)$

   (d) $\lim_{x\to 0^-} f(x)$
   (e) $\lim_{x\to 0^+} f(x)$
   (f) $\lim_{x\to 0} f(x)$

   (g) $\lim_{x\to 2^-} f(x)$
   (h) $\lim_{x\to 2^+} f(x)$
   (i) $\lim_{x\to 2} f(x)$

   (j) $\lim_{x\to -\infty} f(x)$
   (k) $\lim_{x\to \infty} f(x)$



2. Find the limits.

   (a) $\displaystyle\lim_{x\to -\infty} \frac{\sqrt{x^2+1}}{x+1}$ (b) $\displaystyle\lim_{x\to \infty} \frac{x-3}{\sqrt{4x^2+25}}$

   Ans: (a) -1

3. Find the limits.

   (a) $\displaystyle\lim_{x\to 0^+} \frac{2}{3\,x^{1/3}}$ (b) $\displaystyle\lim_{x\to 0^-} \frac{2}{3\,x^{1/3}}$

   Ans: (b) $-\infty$.

4. Find the limits.

   (a) $\displaystyle\lim_{x\to \infty} \left(\sqrt{x^2+25}-\sqrt{x^2-1}\right)$ (b) $\displaystyle\lim_{x\to -\infty} \left(2x+\sqrt{4x^2+3x-2}\right)$

   Hint: Try multiplying or dividing by the conjugate.

   Ans: (b) $-3/4$.

5. For each, determine the domain and then use various limits to find the asymptotes and the range.

   (a) $y = 4 + \dfrac{3x^2}{x^2+1}$. (b) $y = \dfrac{2x}{x^2-1}$.

   Ans: (b) D: $x \neq \pm 1$. R: $(-\infty, \infty)$.
   Specify yourself two vertical asymptotes and a horizontal asymptote.

6. Graph the rational equations, including the graphs and equations of asymptotes.

   (a) $y = \dfrac{x^2}{x-1}$ (b) $y = \dfrac{x^2-1}{x}$

   Ans: (a) Asymptotes: $y = x+1$ and $x = 1$

CHAPTER **3**

Derivatives

In This Chapter:

Topics	**Applications/Properties**
Derivative & Differentiation Rules	
Powers, multiples, sums, & differences	Tangent Lines
Derivative of exponential functions	
Products and quotients	
Second- and higher-order derivatives	
Derivatives	
Trigonometric Function	
Chain Rule	Implicit Differentiation
Inverse Functions and Logarithms	
Related Rates	
Linearization and Differentials	

Contents of Chapter 3

3.1. Tangent Lines and the Derivative at a Point

Recall: In Definition 2.5, p. 63, the **tangent line** to the curve at P is *roughly* defined as the **limit** of the secant lines, as $Q \to P$ from either side.

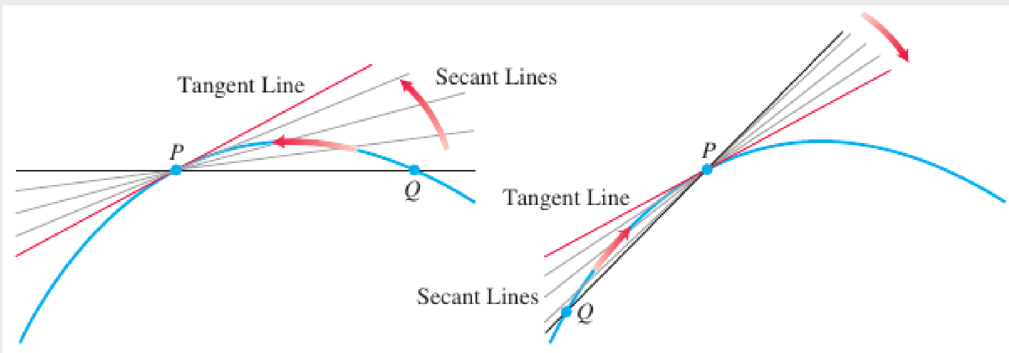


Figure 3.1: Secant lines and the tangent line to the curve at P.

Definition 3.1.
The **slope of the curve** $y = f(x)$ at the point $P(x_0, f(x_0))$ is the number

$$\lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \quad \text{(provided the limit exists).} \tag{3.1}$$

The **tangent line** to the curve at P is the line through P with this slope.

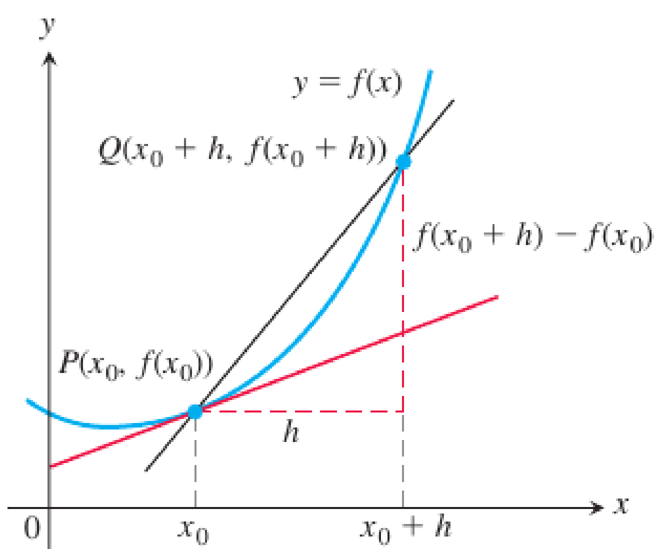


Figure 3.2: The slope at P, as the limit of the average rates of change.

Example 3.2. Find the slope of the parabola $y = x^2$ at the point $P(1, 1)$, and state an equation of the tangent line to the curve at P.

Solution.

Ans: slope $= 2$; $y - 1 = 2(x - 1)$

Example 3.3. In Examples 2.1 and 2.2 in Section 2.1, we studied the speed of a rock falling freely from rest near the surface of the earth. The rock fell

$$y = 16t^2 \text{ (feet)}$$

during the first t sec, and we used a sequence of average rates over increasingly short intervals to estimate the rock's speed at the instant $t = 1$. What was the rock's **exact speed** at this time?

Solution. Let $y = f(t) = 16t^2$. Then

$$f'(1) = \lim_{h \to 0} \frac{f(1 + h) - f(1)}{h} =$$

Definition 3.4. The **derivative** of a function $f(x)$ at a point x_0, denoted $f'(x_0)$, is

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}, \tag{3.2}$$

provided that the limit exists.

Remark 3.5. If we write $x = x_0 + h$, then (3.2) can be written as

$$f'(x_0) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{3.3}$$

Claim 3.6. The tangent line to $y = f(x)$ at $(x_0, f(x_0))$ is the line through point $(x_0, f(x_0))$ whose slope is equal to $f'(x_0)$, the derivative of f at x_0.

Example 3.7. If the tangent line to $y = f(x)$ at $(4, 3)$ passes through the point $(0, 2)$, find $f'(4)$.

Solution.

Summary 3.8. The following are all interpretations for **the limit of the difference quotient**

$$\lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}. \tag{3.4}$$

1. The **slope of the graph** of $y = f(x)$ at $x = x_0$
2. The **slope of the tangent line** to the curve $y = f(x)$ at $x = x_0$
3. The **rate of change** of $f(x)$ with respect to x at $x = x_0$
4. The **derivative** $f'(x_0)$

Example 3.9. Find and simplify the difference quotients $\dfrac{f(x+h)-f(x)}{h}$ for the functions, and then apply $\lim\limits_{h\to 0}$.

(a) $f(x) = x^2$ (b) $f(x) = x^3$

Solution.

Exercises 3.1

1. Find an equation for the tangent line to the curve at the given point. Then sketch the curve and tangent line together.

   (a) $y = x^3, \;\; (2, 8)$ (b) $y = (x-1)^2 - 1, \;\; (1, -1)$

   Ans: (b) $y = -1$

2. Find the slope of the curve at the point indicated.

   (a) $y = \sqrt{x}, \;\; x = 4$ (b) $y = \dfrac{1}{x-1}, \;\; x = 3$

   Ans: (b) $f'(3) = -1/4$

3. At what points does the graph of the function have a horizontal tangent line?

   $f(x) = x^3 - 3x$

   Hint: Find $f'(a) = \lim_{h \to 0} \dfrac{f(a+h) - f(a)}{h}$, and solve $f'(a) = 0$ for a.

   Ans: $(1, -2)$ and $(-1, 2)$

4. Find an equation of the straight line having slope $1/4$ that is tangent to the curve $y = \sqrt{x}$.

5. Let $f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

   (a) Is f continuous at $x = 0$?
   (b) Does $f'(0)$ exist?
   (c) Does f have a tangent line at the origin? If yes, what is it?

   Ans: (c) $y = 0$.

(Continued on the next page)

6. (a) $f(x) = x^3 + 2x, \quad a = 1$ (b) $g(x) = \cos x + 4\sin(2x), \quad a = \pi$

CAS Use a CAS to perform the following steps for the functions.

(i) Plot $y = f(x)$ over the interval $[a-1, a+4]$.

(ii) Define the difference quotient

$$Q(h) = \frac{f(a+h) - f(a)}{h}, \tag{3.5}$$

which is a function of the step size h, when holding a fixed.

(iii) Find the limit of Q as $h \to 0$; define the tangent line

$$y = T(x) := \Big(\lim_{h\to 0} Q(h)\Big) \cdot (x - a) + f(a) \tag{3.6}$$

(iv) Define the secant lines

$$y = S(x, h) := Q(h) \cdot (x - a) + f(a) \tag{3.7}$$

for $h = 3, 2, 1$.

(v) Graph the secant lines together with f and the tangent line over the interval in part (i).

Note: You may use Maple, Mathematica, or Matlab. The following shows a part of Matlab implementation.

- **Line 6**: In Matlab, to put graphs together, you should say "hold on" $\Rightarrow$ You can command "hold off" after adding all graphs.
- **Line 11**: The line is added earlier than Step (v), for your convenience.
- **After Line 11**: You should add `S(x,h)` defined as in (3.7).
- In order to graph e.g. `S(x,3)`, you may add a line
  `fplot(S(x,3),Interval, 'b--','LineWidth',1.5)`
- You result for (a) must look like Figure 3.3 below.

A part of tangent_secant.m

```
1  syms f(x) Q(h)   %also, views x and h as symbols
2  
3  f(x) = x^3+2*x; a=1;
4  Interval = [a-1,a+4];
5  fplot(f(x),Interval, 'k-','LineWidth',3) %Step 1
6  hold on
7  
8  Q(h) = (f(a+h)-f(a))/h;     % Step 2
9  slope = limit(Q(h),h,0);    % Step 3
10 T(x) = slope*(x-a)+f(a);
11 fplot(T(x),Interval, 'r-','LineWidth',2)
```

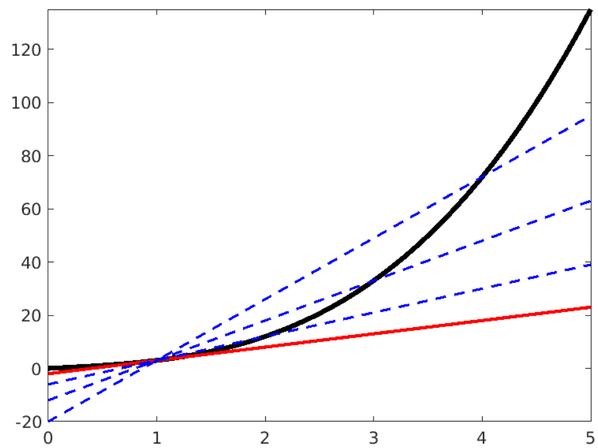


Figure 3.3: The result for Exercise 6 (a).

> **Note**: When you decide to use Maple, you may start with the following. Lines 7, 10, and 13–16 are Maple's outputs respectively for Lines 6, 9, and 12.

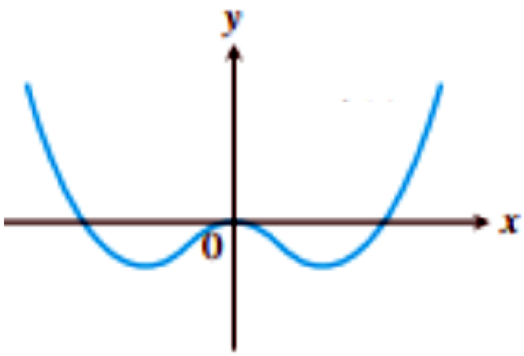
maple-tangent-secant.mw

```
1  with(plots): with(Student[Calculus1]):
2
3  f := x -> x^3 + 2*x:
4  a := 1:
5  Q := h -> (f(a + h) - f(a))/h:
6  slope := limit(Q(h), h = 0);
7                             slope := 5
8  T := x -> slope*(x - 1) + f(a):
9  T(x)
10                             5 x - 2
11 S := (x, h) -> Q(h)*(x - a) + f(a):
12 S(x, h)
13                /       3           \
14                \(1 + h)  - 1 + 2 h/ (x - 1)
15                ---------------------------- + 3
16                             h
```

3.2. The Derivative as a Function

Note: Since derivatives can also be thought of as **the slope of the tangent line** and also as **the rate of change**, we may sketch the graph of $f'(x)$ using the graph of $f(x)$.

Example 3.10. The graph of a function f is given below. Use it to sketch the graph of the derivative $f'(x)$.

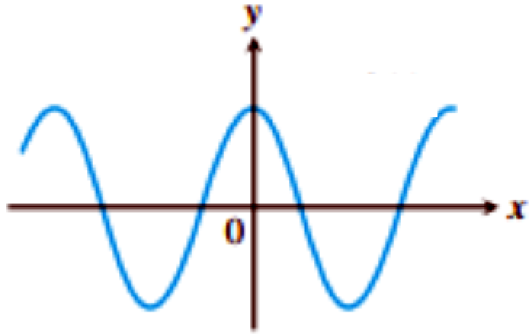


Solution.

Strategy 3.11. Graphing the Derivative

1. See where the rate of change of f is positive, negative, or zero.
2. Estimate the rough size of the growth rate at any x and its size in relation to the size of $f(x)$.
3. See where the rate of change itself is increasing or decreasing.

Example 3.12. The graph of a function f is given below. Use it to sketch the graph of the derivative $f'(x)$.



Solution.

Note: In the previous section, we have focused on finding the derivative of a function f at a particular point $(x_0, f(x_0))$, which is equal to the slope of the tangent line at that point.

- In this section, we investigate the derivative as a function derived from f by considering the limit at each point x in the domain of f.

Definition 3.13. The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{3.8}$$

provided the limit exists.

Remark 3.14. The domain of f' may be either **the same as** or **smaller than** the domain of f.

Example 3.15. Using the definition of derivative, find the derivative of $f(x) = 4 - 8x - 5x^2$.

Solution.

Example 3.16. Let $f(x) = \sqrt{x}$.

(a) Use the definition of derivative to find f'.

(b) State the domain of f'.

(c) Find the tangent line to the curve $y = f(x)$ at $x = 4$.

Solution.

Ans: (a) $f'(x) = 1/(2\sqrt{x})$

Notation 3.17. If we use the traditional notation for $y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, then some common alternative notations for the derivative are:

$$f'(x) = y' = \frac{df}{dx} = \frac{dy}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x). \tag{3.9}$$

The symbols D and $\dfrac{d}{dx}$ are called **differentiation operators**, because they indicate the operation of **differentiation** which is the process of calculating a derivative.

3.2.1. Differentiability

Differentiable on an Interval; One-Sided Derivatives

Definition 3.18. Differentiability

1. A function $y = f(x)$ is **differentiable on an open interval** (finite or infinite) if it has a derivative at each point of the interval.
2. It is **differentiable on a closed interval** $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\begin{cases} \lim_{h\to 0^+} \dfrac{f(a+h) - f(a)}{h} & \textbf{Right-hand derivative at } a \\ \lim_{h\to 0^-} \dfrac{f(b+h) - f(b)}{h} & \textbf{Left-hand derivative at } b \end{cases} \tag{3.10}$$

exist at the endpoints.

Example 3.19. In Example 3.16, we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Apply the definition to examine if the derivative exists at $x = 0$.

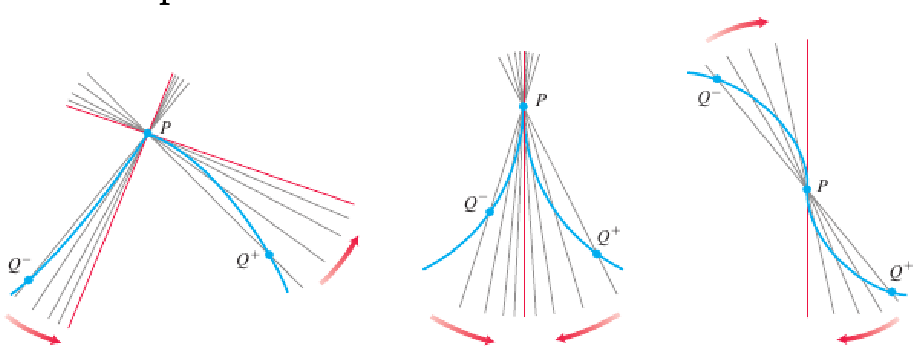
Solution.

Example 3.20. Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and on $(0, \infty)$ but has no derivative at $x = 0$.

Solution.

When Does a Function *Not* Have a Derivative at a Point?

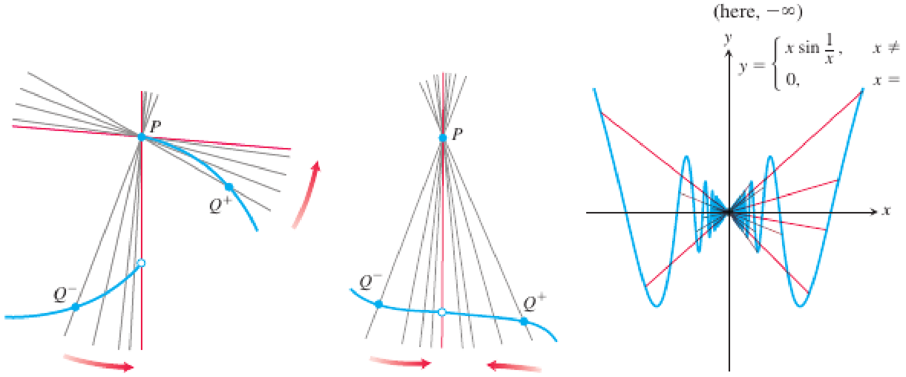
A function fails to have a derivative at a point for many reasons, including the examples below:



1. a *corner*, where the one-sided derivatives differ

2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other

3. a *vertical tangent line*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$)



4. a *discontinuity* (two examples shown)

5. wild oscillation

Theorem 3.21. Differentiability Implies Continuity: *If f has a derivative at $x = c$, then f is continuous at $x = c$.*

Proof. Given that $f'(c)$ exists, we must show that $\lim_{h\to 0} f(c+h) = f(c)$. For $h \neq 0$,

$$f(c+h) = f(c) + f(c+h) - f(c) = f(c) + \frac{f(c+h) - f(c)}{h} \cdot h.$$

Taking limits as $h \to 0$ reads

$$\begin{aligned}
\lim_{h\to 0} f(c+h) &= \lim_{h\to 0} \left[f(c) + \frac{f(c+h) - f(c)}{h} \cdot h \right] \\
&= f(c) + \lim_{h\to 0} \frac{f(c+h) - f(c)}{h} \cdot \lim_{h\to 0} h \\
&= f(c) + f'(c) \cdot 0 \\
&= f(c),
\end{aligned}$$

which completes the proof. □

Note: The converse of Theorem 3.21 is false in general.

Example 3.22. Where are the following functions continuous? Differentiable?

(a) $f(x) = |x|$

(b) $g(x) = x^{1/3}$

Solution.

Exercises 3.2

1. Find the indicated derivatives.

(a) $\dfrac{dy}{dx}$, if $y = x^3 + x^2$ (b) $v'(t)$, if $v(t) = t + \dfrac{1}{t}$

Ans: (b) $v'(t) = 1 - 1/t^2$

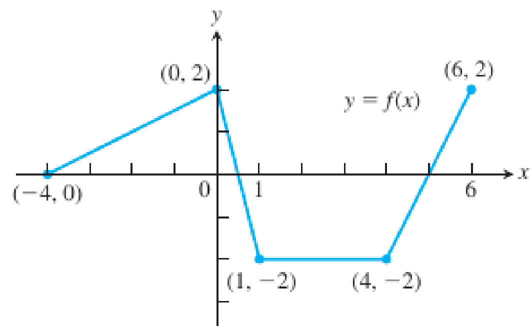
2. Differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

(a) $y = f(x) = \dfrac{1}{x}$, $(x, y) = (1, 1)$. (b) $w = g(z) = 1 + \sqrt{4 - z}$, $(z, w) = (3, 2)$.

Ans: (b) $w - 2 = -\dfrac{1}{2}(z - 3)$

3. (a) The graph in the accompanying figure is made of line segments joined end to end in the interval $[-4, 6]$. At which points of the interval is f' not defined? Give reasons for your answer.

(b) Graph the derivative of f. (The graph should show a step function.)

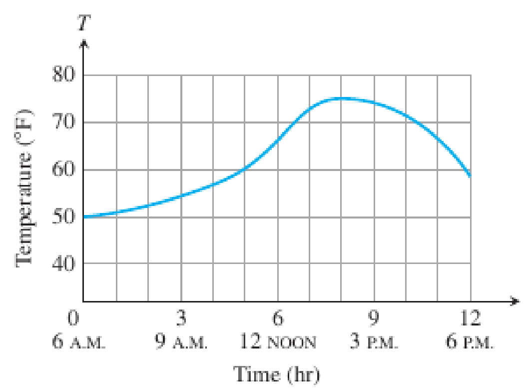


4. The given graph shows the outside temperature T in °F, between 6:00 AM and 6:00 PM.

(a) Estimate the rate of temperature change at the times 9:00 AM and 2:00 PM.

(b) At what time does the temperature increase most rapidly? Decrease most rapidly? What is the rate for each of those times?

(c) Use the graphical technique of Strategy 3.11 to graph the derivative of temperature T versus time t.



Ans: (a) 2.5°F/hr, 0°F/hr

5. Determine if the piecewise-defined function is differentiable at the origin. If it is differentiable, what is $f'(0)$?

(a) $f(x) = \begin{cases} 2x + 1 & x \le 0 \\ x^2 + 2x - 1 & x > 0 \end{cases}$ (b) $f(x) = \begin{cases} x - 1 & x \le 0 \\ \dfrac{1}{2}x + \sqrt{1 + x} - 2 & x > 0 \end{cases}$

Ans: (a) Not continuous $\Rightarrow$ not differentiable. (b) $f'(0) = 1$.

6. CAS Graph $y = 3x^2$ in a window that has $-2 \leq x \leq 2$, $0 \leq y \leq 4$. Then, on the same screen, graph
$$y = \frac{(x+h)^3 - x^3}{h},$$
for $h = -1,\ -0.2,\ 0.2,\ 1$. Explain what is going on.

3.3. Differentiation Rules

3.3.1. Powers, Multiples, Sums, and Differences

- **Derivative of a Constant Function**
  If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0. \tag{3.11}$$

- **Power Rule**: If n is ***any real number***, then

$$\frac{dx^n}{dx} = nx^{n-1}, \tag{3.12}$$

  for all x where the powers x^n and x^{n-1} are defined.

- **Constant Multiple Rule**
  If f is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}[cf(x)] = c\frac{d}{dx}f(x). \tag{3.13}$$

- **Sum and Difference Rules**: If f and g are both differentiable, then

$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x). \tag{3.14}$$

Example 3.23. Find the derivative of the following functions.

(a) $f(x) = 20^5$

(b) $g(t) = 5t + 4t^2$

Example 3.24. Find the derivative of the following functions.

(a) $f(x) = x^{3/2} + x^{-3}$

(b) $r(t) = \dfrac{2}{t^2} + \dfrac{8}{t^4}$

(c) $y = \dfrac{\sqrt{x} + x}{x^2}$

(d) $g(x) = \dfrac{1}{\sqrt{x}} + \sqrt[4]{x}$

3.3.2. Derivative of Exponential Functions

Let $f(x) = a^x$, $a > 0$, $a \neq 1$.

- We first evaluate $f'(0)$:

$$f'(0) = \lim_{h\to 0} \frac{a^h - a^0}{h} = \lim_{h\to 0} \frac{a^h - 1}{h}. \tag{3.15}$$

- When we apply the definition of the derivative to $f(x)$, we get

$$\begin{aligned} \frac{d}{dx}(a^x) &= \lim_{h\to 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h\to 0} \frac{a^x \cdot a^h - a^x}{h} \\ &= \lim_{h\to 0} a^x \cdot \frac{a^h - 1}{h} \\ &= a^x \cdot \lim_{h\to 0} \frac{a^h - 1}{h} \\ &= \boldsymbol{a^x \cdot f'(0)} \end{aligned} \tag{3.16}$$

Remark 3.25. In Section 1.5, the number e is introduced such that the slope of the tangent line to $y = e^x$ at $x = 0$ is exactly 1. That is,

$$\frac{d}{dx}(e^x)\Big|_{x=0} = \lim_{h\to 0} \frac{e^h - e^0}{h} = \lim_{h\to 0} \frac{e^h - 1}{h} = 1 \tag{3.17}$$

- **Derivative of the Natural Exponential Function**:

$$\frac{d}{dx}(e^x) = e^x \tag{3.18}$$

Example 3.26. Find the derivative of $f(x) = 3e^x + 4x^2 - \dfrac{1}{x} + 3$

Solution.

3.3.3. Products and Quotients

Note: A common misconception is that the derivative of a product is analogous to the sum and difference rules we saw previously:

$$f \text{ and } g \text{ are differentiable} \Rightarrow (fg)' = f'g'$$

However, this is **not true**.

Let both f and g be differentiable. Then

$$\begin{aligned}
\frac{d}{dx}[f(x)g(x)] &= \lim_{h\to 0} \frac{f(x+h)g(x+h) - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \frac{f(x+h)g(x+h) \boldsymbol{-f(x)g(x+h)} \ \boldsymbol{+f(x)g(x+h)} - f(x)g(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{f(x+h)-f(x)}{h} \cdot g(x+h) + f(x) \cdot \frac{g(x+h)-g(x)}{h}\right] \\
&= f'(x)g(x) + f(x)g'(x).
\end{aligned} \tag{3.19}$$

- **Product Rule**
  If both f and g are differentiable, then so is their product fg, and

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x) \tag{3.20}$$

Alternative notation:

$$(fg)' = f'g + fg' \quad \text{or} \quad \frac{d}{dx}[uv] = \frac{du}{dx}v + u\frac{dv}{dx}$$

Example 3.27. Find the derivative of the following:

(a) $y = (4x^2+3)(2x+5)$

(b) $y = 4e^x\left(x^3 + \dfrac{1}{x}\right)$

Solution.

Similarly to Product Rule, we can prove the following.

- **Quotient Rule**
  If both f and g are differentiable, then so is their product f/g, and

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2} \tag{3.21}$$

  Alternative notation:

$$(f/g)' = \frac{f'g - fg'}{g^2} \quad \text{or} \quad \frac{d}{dx}[u/v] = \frac{\dfrac{du}{dx}v - u\dfrac{dv}{dx}}{v^2}$$

Example 3.28. Find the derivative of the following:

(a) $G(x) = \dfrac{6x^4 - 5}{x + 1}$

(b) $y = \dfrac{6e^x}{1 + e^x}$

Solution.

Self-study 3.29. Find the derivative of $F(x) = \dfrac{x^2 e^x}{x^3 + 5e^x}$

Solution.

3.3.4. Second- and Higher-order Derivatives

Definition 3.30. If $y = f(x)$ is a differentiable function, then

- its derivative $f'(x)$ is also a function,
- so f' may have a derivative of its own, denoted $(f')' = f''$.

This new function is called the **second derivative** of f, because it is the derivative of the first derivative.

$$\text{Other notations: } f''(x) = \frac{d^2 f}{dx^2} = \frac{d^2 y}{dx^2} = \frac{d}{dx}\Big(\frac{dy}{dx}\Big) = y''.$$

Higher-Order Derivatives: If y'' is differentiable, its derivative $y''' = \dfrac{d}{dx}y'' = \dfrac{d^3 y}{dx^3}$ is the **third derivative** of y with respect to x.
The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^n y}{dx^n} = D^n y \tag{3.22}$$

denoting the **n-th derivative** of y w.r.t. x for positive integers n.

Example 3.31. When $f(x) = 6xe^x - 2x^2 + 3$, find f', f'', and f'''.

Example 3.32. Find the equation of the **tangent line** to the curve at the given point.

(a) $y = 2x^3 - x^2 + 2, \quad (1, 3)$

(b) $y = \dfrac{6e^x}{1 + e^x}, \quad (0, 3)$

Solution.

Example 3.33. Find the point(s) on the curve where the tangent line is horizontal, for $y = x^3 - 3x - 2$.

Solution.

Exercises 3.3

1. Find the first and second derivatives.

   (a) $y = \dfrac{4x^3}{3} - x + 2e^x$

   (b) $y = \sqrt[3]{x^{9.6}} - x^e$

   (c) $y = \dfrac{x^3}{3} + \dfrac{x^2}{2} + e^{-x}$

   (d) $w = re^{-r} + e^{-3/2}$

   Hint: (b) It can be rewritten as $y = x^{3.2} - x^e$. (c) Note that $e^{-x} = \dfrac{1}{e^x}$. Now you may apply Quotient Rule to get $[e^{-x}]' = -e^{-x}$. Later you will learn how to get derivative of e^{-x} more conveniently.

   Ans: (d) $w' = (1-r)e^{-r}$, $w'' = -(2-r)e^{-r}$.

2. Find the derivatives of all orders.

   (a) $y = \dfrac{x^4}{2} - \dfrac{3}{2}x^2 - x$

   (b) $y = (4x^2 + 3)(2 - x)$

Ans: (b) $y' = 8x(2-x) - (4x^2+3)$, $y'' = 8[(2-x) - x] - 8x = 16 - 24x$, $y''' = -24$, and $y^{(n)} = 0$, $n \geq 4$

3. Find the second derivatives of the functions.

   (a) $y = \dfrac{x^3 + 7}{x}$

   (b) $w = \left(\dfrac{1+z}{z}\right)(1 - z)$

   Ans: (b) $w'' = 2/z^3$

4. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

   Find the values of the following derivatives at $x = 0$.

   (a) $\dfrac{d}{dx}(uv)$

   (b) $\dfrac{d}{dx}(u/v)$

   (c) $\dfrac{d}{dx}(v/u)$

5. Let $f(x) = x^3 - 4x + 1$.

   (a) **Normal line to a curve**. Find an equation for the line perpendicular to the tangent line to the curve $y = f(x)$ at the point $(2, 1)$.

   (b) **Smallest slope**. What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

   (c) **Tangent lines having specified slope**. Find equations for the tangent lines to the curve at the points where the slope of the curve is 8.

6. **Challenge** Find the values of a and b that make the following function differentiable for all x-values.

(a) $f(x) = \begin{cases} ax, & \text{if } x < 0 \\ x^2 - 3x & \text{if } x \geq 0 \end{cases}$ (b) $g(x) = \begin{cases} ax + b, & \text{if } x > -1 \\ bx^2 - 4, & \text{if } x \leq -1 \end{cases}$

Hint: The limits of values and derivatives, measured from both sides, must be the same. For example, for (b), the limits of values are $-a+b$ and $b-4$. Thus you can conclude $a = 4$.

7. CAS **Newton's serpentine** is defined as

$$f(x) = \frac{4x}{x^2+1}.$$

(a) Plot the graph $y = f(x)$ over the interval $[-4, 4]$.
(b) On the same screen, plot the tangent line to the curve at $(1, 2)$
(c) Again, on the same screen, plot the curve $y = f'(x)$.

You may start with

newtons_serpentine.m
```
syms f(x)

f(x) = 4*x/(x.^2+1);
X = linspace(-4,4,101);
plot(X,f(X),'r-','linewidth',2); hold on
grid on, axis tight

diff(f(x))
```

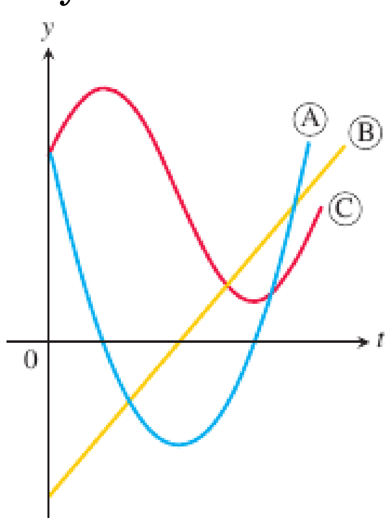
3.4. The Derivative as a Rate of Change

Motion Along a Line: Position, Velocity, Speed, and Acceleration

Definition 3.34. Let $s = f(t)$ be the **position** of a particle moving along a coordinate line. Then,

- Its average rate of change $\dfrac{\Delta s}{\Delta t}$ is the **average velocity** over a time interval Δt.
- Its derivative $\dfrac{ds}{dt} = v(t)$ represents the **instantaneous velocity**.
- **Speed** is the absolute value of velocity: $|v(t)| = \left|\dfrac{ds}{dt}\right|$.
- The instantaneous rate of change of velocity with respect to time is **acceleration**: $a(t) = v'(t) = s''(t)$.

Example 3.35. The graphs in the accompanying figure show the position s, the velocity $v = s'$, and the acceleration $a = s''$ of a moving body along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.



Example 3.36. A dynamite blast blows a heavy rock straight up with a launch velocity of 160 *ft*/*sec* (about 109 *mph*). It reaches a height of $s(t) = 160t - 16t^2$ *ft* after t seconds.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 *ft* above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?

Solution.

Example 3.37. At time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t\,m$.

(a) Find the body's acceleration each time the velocity is zero.

(b) Find the body's speed each time the acceleration is zero.

(c) Find the total **distance traveled** by the body from $t = 0$ to $t = 2$.

Solution.

Derivatives in Economics and Biology

Example 3.38. The **marginal cost of production** is the rate of change of cost with respect to the level of production, so it is $\frac{dc}{dx}$. Suppose that the dollar cost of producing x washing machines is

$$c(x) = 2000 + 100x - 0.1x^2.$$

(a) Find the average cost per machine of producing the first 100 washing machines.

(b) Find the marginal cost when 100 washing machines are produced.

(c) The marginal cost also represents the approximate cost of the next item to be produced. Show that the marginal cost when 100 washing machines are produced is approximately equal to the cost of producing one more washing machine after the first 100 have been made by calculating this cost directly.

Solution.

Example 3.39. When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t hours was

$$b = 10^6 + 10^4 t - 10^3 t^2.$$

Find the growth rates at a) $t = 0$ hours, b) $t = 5$ hours, and c) $t = 10$ hours.

Exercises 3.4

1. **Lunar projectile motion**. A rock thrown vertically upward from the surface of the moon at a velocity of 24 *m/s* reaches a height of $s = 24t - 0.8t^2$ *m* in t sec.

   (a) Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
   (b) How long does it take the rock to reach its highest point?
   (c) How high does the rock go?
   (d) How long does it take the rock to reach half its maximum height?
   (e) How long is the rock aloft?

   Ans: (c) 180 *m*

2. **Vehicular stopping distance**. Based on data from the U.S. Bureau of Public Roads, a model for the total **stopping distance** of a moving car in terms of its speed is

$$s = 1.1v + 0.054v^2,$$

   where s is measured in *ft* and v in *mph*. The linear term $1.1v$ models the distance the car travels during the time the driver perceives a need to stop until the brakes are applied, and the quadratic term $0.054v^2$ models the additional braking distance once they are applied. Find ds/dv at $v = 35$ and $v = 70$ *mph*, and interpret the meaning of the derivative.

3. CAS **Golf Ball Carry**. Using a club of a loft angle θ, a golf player hits a ball with the horizontal head speed v_0. Then, the initial velocity of the ball becomes

$$\langle v_x, v_z\rangle = v_0\cos\theta\langle\cos\theta,\ \sin\theta\rangle = \langle v_0\cos^2\theta,\ v_0\cos\theta\sin\theta\rangle,$$

   where v_x and v_z are the horizontal and vertical velocities, respectively. For simplicity, we assume that the golf field is complete flat. Then, incorporating the gravity acceleration $\mathrm{a} = \langle 0, -g\rangle$, where $g = 9.8\,m/s^2$, the ball location after t seconds can be expressed as

$$L(t) = \langle x(t), z(t)\rangle = \langle 0, -g\rangle\frac{t^2}{2} + \langle v_x, v_z\rangle t = \Big\langle (v_0\cos^2\theta)t,\ -\frac{g}{2}t^2 + (v_0\cos\theta\sin\theta)t\Big\rangle. \tag{3.23}$$

   (a) Find the time for the ball to hit the ground, t_g.
   Hint: The ball will hit the ground when $z(t) = 0$.
   (b) Show that the ball carry, as a function of θ, becomes

$$C(\theta) = \frac{v_0^2\cos^2\theta\sin(2\theta)}{g}. \tag{3.24}$$

   (c) The average club-head speed for male amateur golfers is between 80-90 *mph*. Let the golfer hit the ball with the head speed 90 *mph*, which is the same as 40 *m/s*. Plot $C(\theta)$ over the interval $[0, \pi/2]$.
   (d) At what angle θ does the club make the longest ball carry?

   Although we have ignored the ball spin and air resistance, the analysis in this problem is mostly realistic. Can you see why beginners hit the longest with the **6-Iron** or **7-Iron** among Irons?

3.5. Derivatives of Trigonometric Functions

- Consider the **angle sum identity** for the sine function:

$$\sin(x+h) = \sin x \cos h + \cos x \sin h. \tag{3.25}$$

If $f(x) = \sin x$, then

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\sin(x+h)-\sin x}{h} \\ &= \lim_{h\to 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} \\ &= \lim_{h\to 0} \left[\sin x \cdot \frac{\cos h - 1}{h} + \cos x \cdot \frac{\sin h}{h} \right] \\ &= \sin x \cdot \underbrace{\lim_{h\to 0} \frac{\cos h - 1}{h}}_{=0;\ \text{Example 2.34}} + \cos x \cdot \underbrace{\lim_{h\to 0} \frac{\sin h}{h}}_{=1;\ \text{Theorem 2.33}} \\ &= \cos x \end{aligned} \tag{3.26}$$

- Similarly, we can find $\dfrac{d}{dx}\cos x = -\sin x$, using the **angle sum identity** for the cosine function:

$$\cos(x+h) = \cos x \cos h - \sin x \sin h. \tag{3.27}$$

- Note that $\tan x = \dfrac{\sin x}{\cos x}$. For $\tan x$ and other trigonometric functions, **Quotient Rule** can be used to get their derivatives.

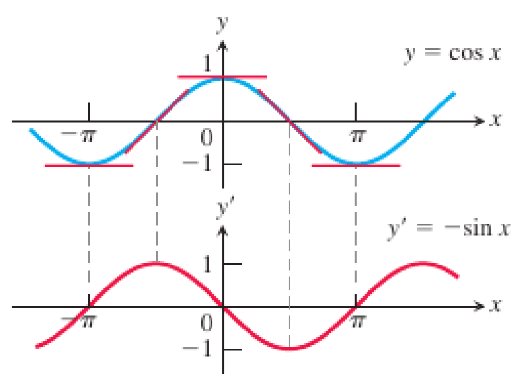


Figure 3.4: Visualization of $\dfrac{d}{dx}\cos x$, by graphing the slopes of the tangent lines.

Derivatives of Trigonometric Functions

$$\begin{aligned} \frac{d}{dx}\sin x &= \cos x & \frac{d}{dx}\csc x &= -\csc x\cot x \\ \frac{d}{dx}\cos x &= -\sin x & \frac{d}{dx}\sec x &= \sec x\tan x \\ \frac{d}{dx}\tan x &= \sec^2 x & \frac{d}{dx}\cot x &= -\csc^2 x \end{aligned} \tag{3.28}$$

Example 3.40. Differentiate the following:

(a) $g(x) = 3e^x + x^2\cos x$

(b) $f(\theta) = \dfrac{\sin\theta}{1-\cos\theta}$

(c) $y = x\sec x\tan x$

(d) $h(t) = \dfrac{t\sin t}{e^t}$

Example 3.41. Find an equation of the tangent line to the curve $y = 2x\sin x$ at the point $(\pi/2, \pi)$.

Solution.

Example 3.42. Differentiate $f(x) = \dfrac{\sec x}{3 + \sec x}$ and find the values of x where the graph of f have a horizontal tangent line.

Solution.

Example 3.43. By computing the first few derivatives and looking for a pattern, find the following.

$$\frac{d^{2022}}{dx^{2022}}(\cos x).$$

Solution.

Example 3.44. Assume that a particle's position on the x-axis is given by $x = 3\cos t + 4\sin t$, where x is measured in feet and t is measured in seconds. Find the particle's velocity when a) $t = 0$ and b) $t = \dfrac{\pi}{2}$.

Solution.

Exercises 3.5

1. Differentiate the following:

   (a) $y = x^2 \cos x$

   (b) $f(x) = \cos x \tan x$

   (c) $s = t^2 - \sec t + 5e^{-t}$

   (d) $g(x) = \dfrac{\tan x}{1 + \tan x}$

   Ans: (d) sympy result: `1/((tan(x) + 1)**2*cos(x)**2)`

sympy_diff.py
```
import sympy as sym

x = sym.symbols('x')
y = sym.tan(x)/(1+sym.tan(x))
print( sym.simplify( y.diff(x) ) )
```

2. *Sketch* the curves over the given intervals, together with their tangent lines at the given values of x. Label each curve and tangent line with its equation.

   (a) $y = \sin x, \quad -3\pi/2 \le x \le 2\pi, \quad x = 0,\ 3\pi/2$

   (b) $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi, \quad x = -\pi/3,\ 3\pi/2$

3. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \ge 0 \end{cases}$$

   continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

   Ans: No value of b for g to be differentiable.

4. Find values of a and b that will make

$$f(x) = \begin{cases} ax + b, & x < 0 \\ 1 - \tan x, & x \ge 0 \end{cases}$$

   differentiable at $x = 0$?

3.6. The Chain Rule

Let's differentiate the function

$$f(x) = (3x^2 + 1)^2. \tag{3.29}$$

- The expanded formula for f reads

$$f(x) = 9x^4 + 2 \cdot (3x^2) + 1 = 9x^4 + 6x^2 + 1.$$

Thus

$$\frac{df}{dx} = \frac{d}{dx}(9x^4 + 6x^2 + 1) = 36x^3 + 12x. \tag{3.30}$$

- Now, we will try a different way. Let

$$u(x) = 3x^2 + 1$$

Then the function f can be formulated in terms of u:

$$f(x) = (3x^2 + 1)^2 = u^2.$$

- The derivative $\frac{df}{dx}$ **is a rate of change of f with respect to x.** Thus it can be written as

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx} \tag{3.31}$$

so that

$$\frac{df}{dx} = 2u \cdot 6x = 12u \cdot x = 12(3x^2 + 1)x = 36x^3 + 12x,$$

which is the same as in (3.30).

Theorem 3.45. The Chain Rule: *If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and*

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \;=\; \frac{d\, f(g(x))}{d\, g(x)} \cdot \frac{dg(x)}{dx}. \tag{3.32}$$

Letting $y = f(u)$ and $u = g(x)$, a simpler form of ***Leibniz's notation*** *reads*

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{3.33}$$

where dy/du is evaluated at $u = g(x)$.

Proof. For $y = f(u) = f(g(x))$, define $\Delta u = g(x+h) - g(x)$. Then $h \to 0$ implies $\Delta u \to 0$ and

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0} \frac{f(g(x+h)) - f(g(x))}{h} = \lim_{h\to 0} \frac{f(g(x)+\Delta u) - f(g(x))}{h} \\
&= \lim_{h\to 0} \frac{f(g(x)+\Delta u) - f(g(x))}{\Delta u} \cdot \frac{\Delta u}{h} \\
&= \lim_{h\to 0} \frac{f(g(x)+\Delta u) - f(g(x))}{\Delta u} \cdot \lim_{h\to 0} \frac{\Delta u}{h} \\
&= \lim_{\Delta u\to 0} \frac{f(g(x)+\Delta u) - f(g(x))}{\Delta u} \cdot \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= f'(g(x)) \cdot g'(x),
\end{aligned}$$

which completes the proof. □

Example 3.46. Differentiate $\sin(x^2 + e^x)$ with respect to x.

Solution.

Example 3.47. Find dy/dx, by first writing the function in the form $y = f(u)$.

(a) $y = (x^3 - 1)^{100}$

(b) $y = \sin(2x^2)$

(c) $g(x) = \tan^3 x$

(d) $y = e^{(4\sqrt{x}+x^2)}$

Note: We sometimes have to use the Chain Rule two or more times to find the derivative and combine with other rules we've previously learned.

Example 3.48. Find the derivative of the following:

(a) $y = \sqrt{e^{\cos x} + 1}$

(b) $y = (2x - 5)^{-2}(x^2 - 5x)^6$

Solution.

Example 3.49. Find the derivative of the following:

(a) $y = e^{\sin^2(\pi t-1)}$ (b) $y = \cos^4(\sec 3t)$ (c) $y = \left(\dfrac{t^2}{t^3-4}\right)^3$

Example 3.50. Find the value of $(f \circ g)'$ at the given value of x. Use this to find an equation of the tangent line to the curve at the given point.

$$f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$$

Solution.

Exercises 3.6

1. Given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

   (a) $y = \sin u, \quad u = 3x + 1$

   (b) $y = -\sec u, \quad u = \dfrac{1}{x} + 7x$

   Ans: (b) $dy/dx = -\sec(1/x + 7x)\tan(1/x + 7x)(-1/x^2 + 7)$

2. Find the derivatives of the functions.

   (a) $y = \left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^4$

   (b) $y = \cos^{-4} x$

   (c) $y = xe^{-x} + e^{x^3}$

   (d) $f(x) = x^2 \sec(1/x)$

   Ans: (b) $-4\cos^{-5} x(-\sin x) = 4\cos^{-5} x \sin x$. (c) $(1-x)e^{-x} + 3x^2 e^{x^3}$

3. Find y''.

   (a) $y = e^{x^2} + (2x+1)^4$

   (b) $y = \sin(e^x)$

   Ans: (b) $\big(-e^x \sin(e^x) + \cos(e^x)\big)e^x$

4. Suppose that functions f and g and their derivatives with respect to x have the values in the table at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

   Find the derivatives with respect to x of the following combinations at the given value of x.

   (a) $f(x) \cdot g(x), \quad x = 3$

   (b) $f(g(x)), \quad x = 2$

   (c) $\sqrt{f(x)}, \quad x = 2$

   (d) $1/g^2(x), \quad x = 3$

   Ans: (b) $f'(g(x))g'(x)\big|_{x=2} = f'(2)g'(2) = -1$. (c) $\sqrt{2}/24$. (d) $(1/g^2)' = -2g'/g^3\big|_{x=3} = 5/32$

5. Let $f(x) = \tan(\pi x/4)$.

   (a) Find an eauation of the tangent line to the curve $y = f(x)$ at $t = 1$.

   (b) **Slopes on a tangent curve**. What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

   Ans: (b) $f'(x) = \sec^2(\pi x/4) \cdot \frac{\pi}{4}$. Since $\min\limits_{-2<x<2} \sec^2(\pi x/4) = 1$, the minimum of $f'(x) = \pi/4$.

6. **Challenge** Verify each of the following statements.

   (a) If f is even, then f' is odd.

   (b) If f is odd, then f' is even.

3.7. Implicit Differentiation

Implicitly Defined Functions

- The functions we have dealt with so far have been defined **explicitly** – for example, $y = \sqrt{x^3+1}$ and $y = x^2 \sin x$.
- Some functions are defined **implicitly** by a relation between x and y such as
$$x^2 + y^2 - 25 = 0 \quad \text{and} \quad x^2 + y^3 - 9xy = 0.$$
- **Implicitly Defined Functions** are in the form of $F(x, y) = 0$, in which ***y* is a function of *x***.

Implicit Differentiation: ***An application of Chain Rule.***

- Differentiate the equation (both sides) with respect to x, treating y as a differentiable function of x.
- Collect the terms with dy/dx on one side of the equation and solve for it.

Example 3.51. If $x^2 + y^2 - 25 = 0$, find the derivative dy/dx.

Solution.

Example 3.52. Find the derivative.

(a) $2xy + y^2 = x + y$ (b) $e^{2x} = \sin(x + 3y)$

Solution.

Example 3.53. Find the derivative:

$$x^3 = \frac{2x - y}{x + 3y}$$

Solution.

Example 3.54. Use implicit differentiation to find equations of the **tangent line** and the **normal line** to the curve at the given point.

$$x^2 + xy - y^2 = 1, \quad (2, 3)$$

Solution. Start with figuring out $y' = -(2x + y)/(x - 2y)$.

Example 3.55. Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

$$2xy + \pi \sin y = 11\pi, \quad (2, 5\pi/2)$$

Solution. Start with figuring out $y' = -2y/(2x + \pi \cos y)$.

Exercises 3.7

1. Use implicit differentiation to find dy/dx.

   (a) $2xy + y^2 = x + y$ (b) $e^{2x} = \sin(x + 3y)$

   (c) $x = \sec y$ (d) $3 + \sin y = y - x^3$

   Ans: (a) $(1 - 2y)/(2x + 2y - 1)$. (b) $2e^{2x}/(3\cos(x + 3y)) - 1/3$.

2. Use implicit differentiation to find dy/dx and then d^2y/dx^2. Write the solutions in terms of x and y only.

   (a) $x^2 + y^2 = 1$ (b) $xy + y^2 = 1$

   Ans: (b) $y'' = 2y(x + y)/(x^3 + 6x^2y + 12xy^2 + 8y^3)$

3. Verify that the given point is on the curve and find the lines that are (1) tangent and (2) normal to the curve at the given point.

   (a) $x^2 + y^2 = 25, \quad (3, -4)$ (b) $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$

   Hint: (b) $y' = -(2y\sin(2x) + \sin(2y))/(2x\cos(2y) - \cos(2x))$

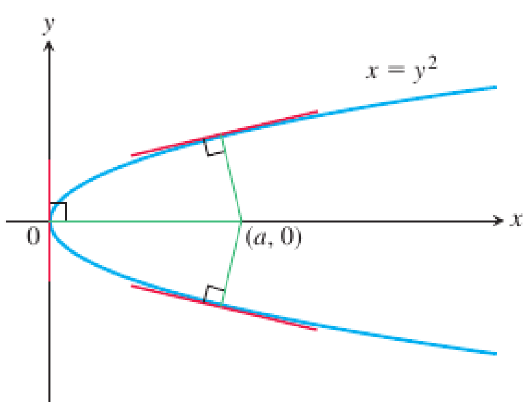
4. Verify that the following pairs of curves meet orthogonally.

$$x^2 + y^2 = 4, \qquad x^2 = 3y^2$$

5. **Challenge** **Normals to a parabola**.

   (a) Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown in the accompanying diagram, then a must be greater than $1/2$.

   (b) One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



Hint: $y' = 1/(2y)$. Thus the normal line at (x_0, y_0) reads $y - y_0 = -2y_0(x - x_0)$. (a) Now, the value a is decided when $y = 0$. (b) The two normal lines will be perpendicular when each of them makes an angle of $45°$ with the coordinate axes, i.e., their slope is either 1 or -1.

3.8. Derivatives of Inverse Functions and Logarithms

3.8.1. Derivatives of inverses of differentiable functions

Let f be differentiable and have inverse f^{-1}.

- Then
$$f^{-1}(f(x)) = x.$$
- Applying Chain Rule results in
$$[f^{-1}]'(f(x)) \cdot f'(x) = 1 \iff [f^{-1}]'(f(x)) = \frac{1}{f'(x)}, \tag{3.34}$$
which can be interpreted geometrically as in Figure 3.5.

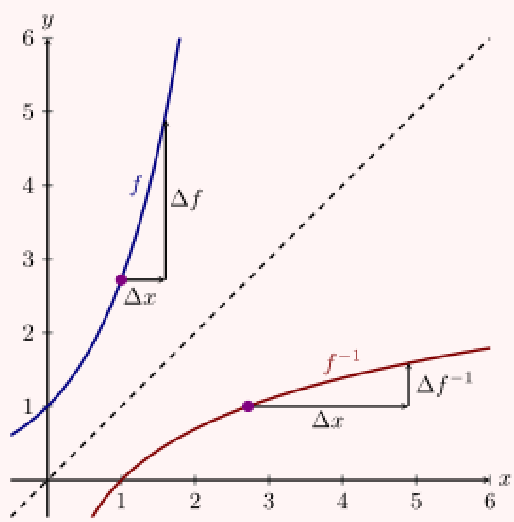


Figure 3.5: The derivative of the inverse function.

- Similarly, starting from $f(f^{-1}(x)) = x$, we obtain
$$[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))}. \tag{3.35}$$

We will see exactly what (3.35) means, solving examples.

Example 3.56. Let $f(x) = x^3 - 2$, $x > 0$. Find $\dfrac{df^{-1}}{dx}(6)$.

Solution.

(a) **Explicit construction of f^{-1}**:
The inverse of f can be found explicitly as

$$f^{-1}(x) = \sqrt[3]{x+2}.$$

(b) **Using Formula (3.35)**:
Note that f^{-1} is defined in the range of f, $f(X)$. The point 6 must be one of them, that is,

$$6 = f(x)$$

for some x. Thus

$$x = f^{-1}(6) = 2.$$

Ans: $1/12$

Example 3.57. Let $f(x) = x^3 - 3x^2 - 1$, $x \geq 2$. Find $\dfrac{df^{-1}}{dx}(-1)$.

Solution.

Example 3.58. Let $f(x) = x^2 - 4x - 5$, $x > 2$. Fund the value of df^{-1}/dx at $x = 0$.

Solution.

3.8.2. Derivatives of Logarithmic Functions

Let $f(x) = \log_a x$.

- It follows from the definition of $\log$ that

$$y = \log_a x \iff x = a^y = e^{\ln a^y} = e^{y \ln a}. \tag{3.36}$$

- Apply implicit differentiation to the right side of (3.36) to get

$$1 = e^{y \ln a} \cdot y' \ln a \quad \Longrightarrow \quad y' = \frac{1}{e^{y \ln a} \cdot \ln a} = \frac{1}{x \ln a}. \tag{3.37}$$

Note: If $x > 0$, then $x = e^{\ln x}$.

Summary 3.59. We may apply above arguments and Chain Rule to get the following formulas.

$$\begin{aligned} \frac{d}{dx}\log_a x &= \frac{1}{x \ln a} & \qquad \frac{d}{dx}\ln x &= \frac{1}{x} \\ \frac{d}{dx}a^x &= a^x \ln a & \qquad \frac{d}{dx}e^x &= e^x \\ \frac{d}{dx}\ln f(x) &= \frac{f'(x)}{f(x)} & \qquad \frac{d}{dx}\ln |x| &= \frac{1}{x} \end{aligned} \tag{3.38}$$

See Example 3.64 below for the formula: $\dfrac{d}{dx}a^x = a^x \ln a$.

Example 3.60. Differentiate $f = x^x$, $x > 0$.
Solution. Note that $x^x = e^{\ln x^x} = e^{x \ln x}$.

Ans: $f'(x) = x^x(\ln x + 1)$

Example 3.61. Find the derivative of the following:

(a) $y = \ln(3x^2)$

(b) $g(x) = \sqrt{\ln x}$

(c) $y = \ln(\cos\theta)$

(d) $y = \ln(3te^{-t})$

Solution.

Example 3.62. Find $\dfrac{d}{dx}\left[\ln\left(\sqrt{\dfrac{(x+1)^{10}}{(2x+1)^5}}\right)\right]$.

Solution.

Note: From the previous example, we see that using the properties of logarithms prior to differentiation may simplify our derivatives overall.

3.8.3. Logarithmic Differentiation

Let's try to use the natural logarithm to manipulate products and powers effectively for a convenient calculation of the derivative.

Algorithm 3.63. Logarithmic Differentiation

1. **Take natural logarithms** of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to conveniently reform the right side.
2. **Differentiate implicitly** with respect to x.
3. **Solve the resulting equation for y'.**
4. **Replace y with $f(x)$.**

Example 3.64. Let $y = a^x$. Use logarithmic differentiation to prove that $y' = a^x \ln a$.

Solution.

Example 3.65. Use logarithmic differentiation to find the derivative of

(a) $y = (x+1)^x$.

(b) $y = (\sin x)^{\ln x}$.

Solution.

Theorem 3.66. **The Euler's number e as a Limit**: *It can be calculated as the limit*

$$e = \lim_{x\to 0}(1+x)^{1/x}. \tag{3.39}$$

Proof. If $f(x) = \ln x$, then $f'(x) = 1/x$ and $f'(1) = 1$. By the definition of derivative,

$$\begin{aligned} f'(1) &= \lim_{h\to 0}\frac{f(1+h)-f(1)}{h} = \lim_{x\to 0}\frac{f(1+x)-f(1)}{x} \\ &= \lim_{x\to 0}\frac{1}{x}\ln(1+x) = \lim_{x\to 0}\ln(1+x)^{1/x} \\ &= \ln\big[\lim_{x\to 0}(1+x)^{1/x}\big] = 1, \end{aligned} \tag{3.40}$$

from which (3.39) follows. □

Remark 3.67. The formula in (3.39) can be rewritten in various forms. A popular form of the **Euler's number** e reads

$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n, \tag{3.41}$$

where the right-side is an increasing sequence, approaching e, as n grows.

Exercises 3.8

1. Suppose that the differentiable function $y = f(x)$ has an inverse and that the graph of f passes through the point $(1, 5)$ and has a slope of $1/2$ there. Can you find each of the following? If yes, find it.

   (a) $\dfrac{df^{-1}}{dx}(1/2)$. (b) $\dfrac{df^{-1}}{dx}(1)$. (c) $\dfrac{df^{-1}}{dx}(5)$.

2. Let $f(x) = x^2 - 4x - 5$, $x > 2$. Find $\dfrac{df^{-1}}{dx}(0)$.
   Hint: $0 = f(5)$

3. Suppose that the function f ans its derivative have the following values at $x = 0, 1, 2, 3, 4$.

x	0	1	2	3	4
$f(x)$	-4	3	-1	2	1
$f'(x)$	3	2	5/4	2/3	1/5

   Assuming the inverse function f^{-1} is differentiable, find the slope of $f^{-1}(x)$ at

   (a) $x = 1$ (b) $x = 2$ (c) $x = 3$

   Ans: (c) $1/2$

4. **Logarithmic Differentiation**. Use logarithmic differentiation to find the derivative of y.

   (a) $y = x(x+1)(x+2)$ (b) $y = \sqrt{x(x+1)}$

   (c) $y = \dfrac{\theta \sin\theta}{\sqrt{\sec\theta}}$ (d) $y = (\tan\theta)\sqrt{2\theta+1}$

5. Use logarithmic differentiation or the method used in Example 3.60 to find dy/dx.

   (a) $y = x^{\sqrt{x}}$ (b) $y = (\sin x)^x$

   (c) $y^x = x^3 y$ (d) $x = y^{xy}$

   Ans: (d) $\dfrac{1 - xy\ln y}{x^2(1+\ln y)}$

3.9. Inverse Trigonometric Functions

Recall from Section 1.6.3

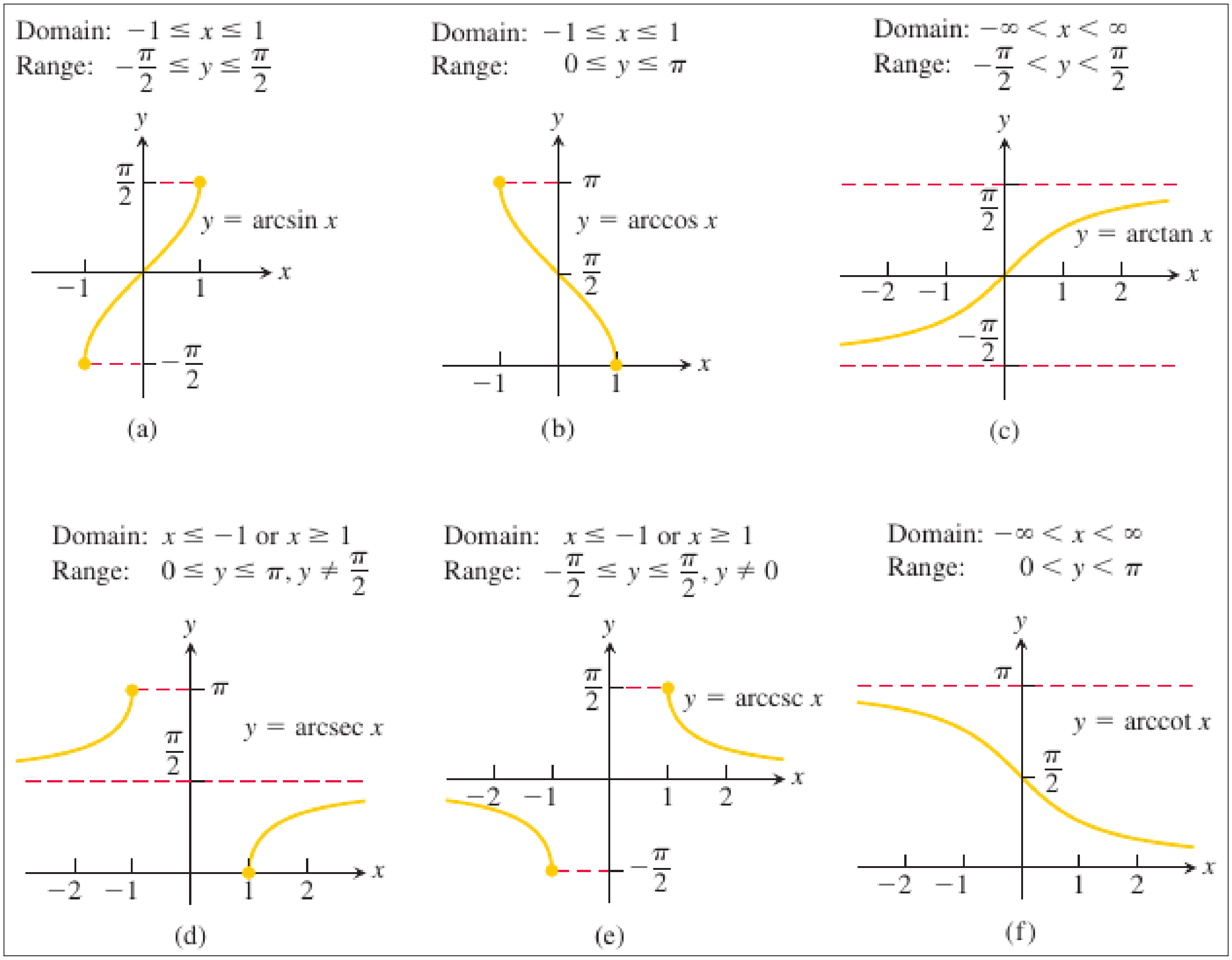


Figure 1.26 (p.55): Graphs of the six basic inverse trigonometric functions.

The trigonometric functions are not one-to-one. If we restrict their domains, we can make them one-to-one and they will have inverses. For example,

Definition 1.70 (p.55). (The Arcsine and Arccosine Functions)

$$y = \textbf{arcsin}\, x \text{ is the number in } [-\pi/2, \pi/2] \text{ for which } \sin y = x$$
$$y = \textbf{arccos}\, x \text{ is the number in } [0, \pi] \text{ for which } \cos y = x$$

Example 3.68. Find the exact value for the expression:

(a) $\cos^{-1}(1/2)$ (b) $\arctan(1)$ (c) $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$

Solution.

Derivatives of the Inverse Trigonometric Functions

The derivatives can be found by using the definition of inverses.

- $\boldsymbol{y = \arcsin x}$: Then, $x = \sin y$. Implicit differentiation gives

$$1 = (\cos y)y' \quad \Longrightarrow \quad y' = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} = \frac{1}{\sqrt{1 - x^2}}$$

- $\boldsymbol{y = \arctan x}$: Then, $x = \tan y$. Implicit differentiation gives

$$1 = (\sec^2 y)y' \quad \Longrightarrow \quad y' = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$

- The derivatives can be found similarly for other inverse trigonometric functions.

Summary 3.69. Here we collect the derivatives of the inverse trigonometric functions.

$$\begin{aligned}
\frac{d}{dx}(\arcsin x) &= \frac{1}{\sqrt{1-x^2}} \quad (|x|<1) & \frac{d}{dx}(\arccos x) &= -\frac{1}{\sqrt{1-x^2}} \quad (|x|<1)\\
\frac{d}{dx}(\arctan x) &= \frac{1}{1+x^2} & \frac{d}{dx}(\operatorname{arccot} x) &= -\frac{1}{1+x^2}\\
\frac{d}{dx}(\operatorname{arcsec} x) &= \frac{1}{|x|\sqrt{x^2-1}} \quad (|x|>1) & \frac{d}{dx}(\operatorname{arccsc} x) &= -\frac{1}{|x|\sqrt{x^2-1}} \quad (|x|>1)
\end{aligned} \tag{3.42}$$

Note: We can also combine these derivatives with ALL the rules we've previously covered.

Example 3.70. Find the derivative of the functions.

(a) $y = \arcsin(\sqrt{2}t)$

(b) $y = \sec^{-1}(5s)$

(c) $y = \ln(\operatorname{arccot}(9x^2))$

(d) $y = x \arcsin x + \sqrt{1-x^2}$

Exercises 3.9

1. Find exact angles:

   (a) $\arcsin(-1/2)$ (b) $\arccos(1/2)$

   (c) $\tan^{-1}\sqrt{3}$ (d) $\csc^{-1}\sqrt{2}$

   Ans: (d) $\pi/4$

2. Find the values.

   (a) $\sin\left(\cos^{-1}\left(\dfrac{1}{\sqrt{2}}\right)\right)$ (b) $\tan\left(\arcsin\left(-\dfrac{1}{2}\right)\right)$

   Ans: (b) $-1/\sqrt{3}$

3. Find the derivative of y.

   (a) $y = \arccos(1/x)$ (b) $y = \ln(\tan^{-1} x)$

   (c) $y = \tan^{-1}(\ln x)$ (d) $y = \cos(x - \arccos x)$

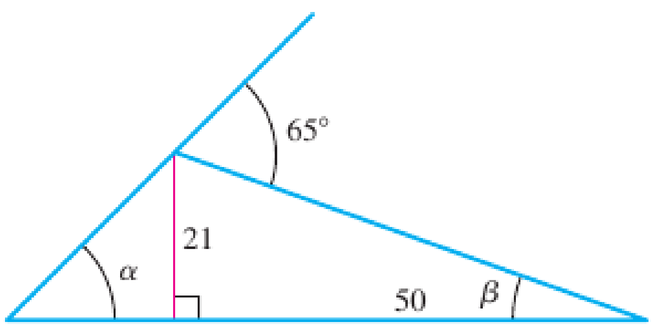
   Ans: (a) $1/(x^2\sqrt{1-1/x^2})$. (d) $-(1+1/\sqrt{1-x^2})\sin(x-\arccos x)$.

4. Use implicit differentiation to find dy/dx and then an equation of the tangent line to the curve at $P(0, 1/2)$.

$$\arcsin(x+y) + \arccos(x-y) = \frac{5\pi}{6}$$

   Ans: $y = 1/2$

5. Find the angle α (in degrees). ***Hint***: $\alpha + \beta = 65°$.



3.10. Related Rates

Note: In this section, we look at questions that arise when two or more related quantities are changing (w.r.t. a common variable, e.g., time). The problem of determining how the rate of change of one of them affects the rate of change of the others is called a **related rates problem**.

Example 3.71. Suppose we are pumping air into a spherical balloon.

- Both the volume and radius of the balloon are increasing over **time**.
- If V is the volume and r is the radius of the balloon at an instant of time, then
$$V = \frac{4}{3}\pi r^3. \tag{3.43}$$
- Using the Chain Rule, we differentiate both sides with respect to t to find an equation relating the rates of change of V and r,
$$\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}. \tag{3.44}$$
- The rates dV/dt and dr/dt are related by (3.44). If we know one, then the other can be computed by the relation.

Strategy 3.72. Related Rates Problem Strategy:

1. Let t denote time, and **name the variables and constants**.
   - We will assume that all variables are differentiable functions of t.
   - For most cases, to **draw a picture** would be helpful.
2. Write **an equation that relates the variables**.
   - You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
3. Differentiate with respect to t to obtain a **related rates equation**.
4. Use **known values (data)** to find **the unknown rate**.

Example 3.73. A spherical balloon is inflated with helium at a rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft?

Solution.

Ans: 1 ft/min

Example 3.74. The length of a rectangle is decreasing at a rate of 2 cm/sec and its width is increasing at a rate of 2 cm/sec. When the length is 12 cm and the width is 5 cm, how fast is the area of the rectangle changing?

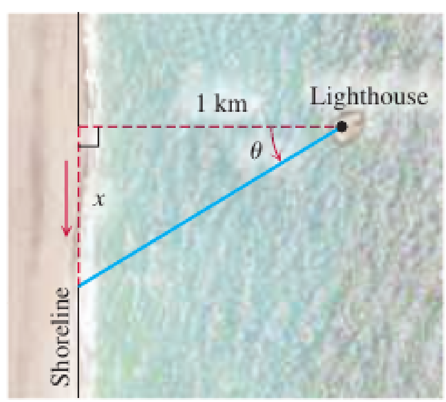
Solution.

Example **3.75.** Sand falls from a conveyor belt at a rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eights of the base diameter. How fast is the height of the pile increasing when the pile is 4 m high?

Solution. Begin with $V = \frac{1}{3}\pi r^2 h$ and $h = \frac{3}{8} \cdot 2r$.

Example **3.76.** A lighthouse sits 1 km offshore, and its beam of light rotates counterclockwise at the constant rate of 3 full circles per minute. At what rate is the image of the beam moving down the shoreline when the image is 1 km from the spot on the shoreline nearest the lighthouse?

Solution. Begin with $x = \tan\theta$.



Ans: 12 km/min

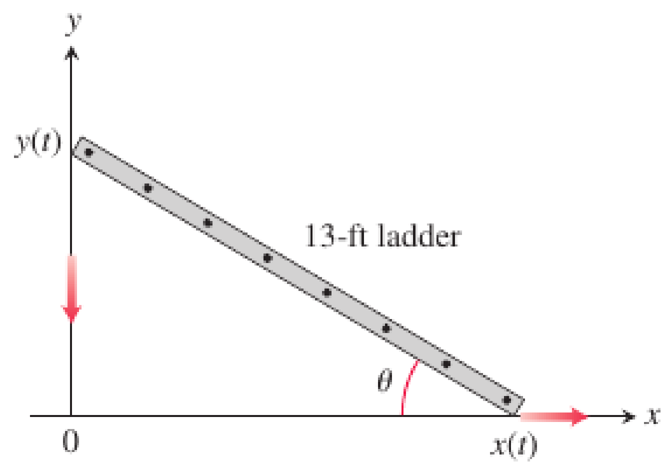
Example 3.77. A sliding ladder. A 13 ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.

(a) How fast is the top of the ladder sliding down the wall then?

(b) At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?

(c) At what rate is the angle θ between the ladder and the ground changing then?

Solution. Begin with
(a) $h^2 + b^2 = 13^2 = 169$,
(b) $A = \dfrac{1}{2}hb$, and
(c) $\tan\theta = \dfrac{h}{b}$.



Ans: (b) -59.5 ft^2/sec. (c) -1 rad/sec.

Exercises 3.10

1. If $x^2 + y^2 = 25$ and $dx/dt = -2$, then what is dy/dt when $x = 3$ and $y = -4$?

2. (a) Assume that $y = 5x$ and $dx/dt = 2$. Find dy/dt.
   (b) Assume that $2x + 3y = 12$ and $dy/dt = -2$. Find dx/dt.

   Ans: (b) 3.

3. Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.

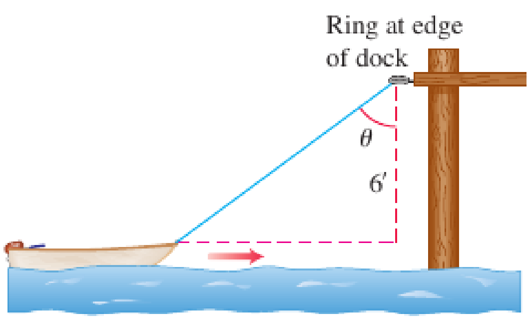
   (a) How is ds/dt related to dx/dt if y is constant?
   (b) How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
   (c) How is dx/dt related to dy/dt if s is constant?

   Ans: (b) $\dfrac{ds}{dt} = \dfrac{x}{\sqrt{x^2+y^2}}\dfrac{dx}{dt} + \dfrac{y}{\sqrt{x^2+y^2}}\dfrac{dy}{dt}$.

Note: Let $s = xy$, for example. Then, we know $\dfrac{ds}{dt} = \dfrac{dx}{dt}y + x\dfrac{dy}{dt}$. Such Product Rule can be rephrased as the following: $\dfrac{ds}{dt} = \dfrac{ds}{dx}\dfrac{dx}{dt} + \dfrac{ds}{dy}\dfrac{dy}{dt} = y\dfrac{dx}{dt} + x\dfrac{dy}{dt}$.

4. **Hauling in a dinghy**. A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. See the figure below. The rope is hauled in at the rate of 2 ft/sec.

   (a) At what rate is the angle θ changing when 10 ft of rope are out?
   (b) How fast is the boat approaching the dock at this instant?



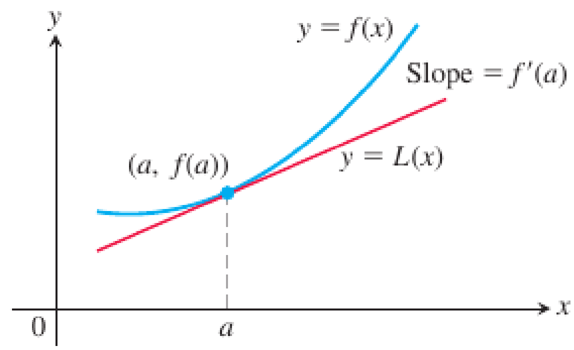
Hint: Begin with (a) $r\cos\theta = 6$ and (b) $\tan\theta = x/6$.

3.11. Linearization and Differentials

3.11.1. Linearization

In general, the **tangent line** to $y = f(x)$ at a point $x = a$, where f is differentiable, passes through the point $(a, f(a))$, so its **point-slope equation** is

$$y = f(a) + f'(a)(x - a).$$



Definition 3.78. If f is differentiable at $x = a$, then the approximating function

$$L(x) := f(a) + f'(a)(x - a) \tag{3.45}$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x) \tag{3.46}$$

of f by L is the **linear approximation** (or, **tangent line approximation**) of f at a. The point $x = a$ is the **center** of the approximation.

Note: As long as this line remains close the graph of f, as we move off the point of tangency, $L(x)$ gives a good approximation to $f(x)$.

Example 3.79. Find the linearization $L(x)$ of $f(x) = x^3 - 3x + 2$ at $a = 0$.

Solution.

Example 3.80. Find the linearization $L(x)$ of $f(x) = \sqrt[3]{x}$ at $a = -8$.
Solution.

Example 3.81. Find the linearization $L(x)$ of $f(x) = 2\tan x$ at $a = \pi$.
Solution.

Example 3.82. Use a linear approximation to estimate the given number: $(1.0002)^{50}$.
Solution.

3.11.2. Differentials

We often use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce **two new variables dx and dy**, **of which their ratio (when exists) is equal to the derivative**.

Definition 3.83. Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is a dependent variable, defined as

$$dy = f'(x)dx. \tag{3.47}$$

Remark 3.84. What is dy?
Often the variable dx is chosen to be Δx, the change in x. Then the differential dy is the change in the linearization of f at $x = a$, $\boldsymbol{\Delta L}$.

Example 3.85. Let $f(x) = x^5 + 37x$.

(a) Find dy.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

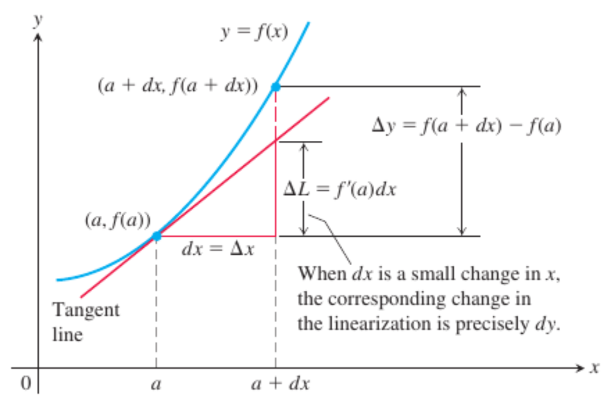


Figure 3.6: Geometrically, the differential $dy = \Delta L$, when $x = a$ and $dx = \Delta x$.

Note: **Differentials**
You may consider differentials as **changes**, small changes. The equation $dy = f'(x)\,dx$ gives a relation between the change in x (dx) and the change in y (dy).

Example 3.86. Find the differential dy of the following functions.

(a) $y = 4e^{x^2}$

(b) $y = \ln\left(\dfrac{x+1}{\sqrt{x-1}}\right)$

Solution.

Example 3.87. Find the differential dy and evaluate it for the given values of x and dx.

(a) $f(x) = 2x^2 + 4x - 3$,
$x = -2,\ dx = 0.1$

(b) $y = x^{-1},\ x = 0.5,\ dx = 0.1$

Solution.

Exercises 3.11

1. **Common linear approximations at $x = 0$.** Find the linearizations of the following functions at $x = 0$.

   (a) $\sin x$
   (b) $\cos x$
   (c) $\tan x$
   (d) e^x

   Ans: (b) $L(x) = 1$. (c) $L(x) = x$.

2. (i) Find $f(a)$. (ii) Find a linearization at a suitably chosen integer near a at which the given function and its derivative are easy to evaluate. (iii) Use the linearization to estimate f at a.

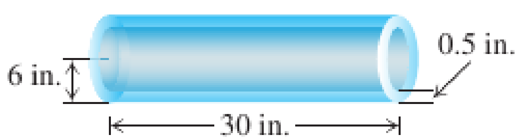
   (a) $f(x) = x^2 + 2x, \quad a = 0.1$
   (b) $f(x) = 2x^2 + 3x - 3, \quad a = -0.9$

   Ans: (b) (i) -4.08. (ii) $L(x) = -x - 5$. (iii) -4.1.

3. Each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$.
   Find $\Delta f = f(x_0 + dx) - f(x_0)$ and $df = f'(x_0)dx$.

   (a) $f(x) = 2x^2 + 4^x - 3, \quad x_0 = -1, \quad dx = 0.1$
   (b) $f(x) = x^3 - x, \quad x_0 = 1, \quad dx = 0.05$

4. **Estimating volume**. Estimate the volume of material in a cylindrical shell with length 30 in., radius 6 in., and shell thickness 0.5 in.



   Hint: The whole volume $V = \pi r^2 \cdot 30 = 30\pi r^2$. Thus $dV = 60\pi r \cdot dr$. Now, what are r and dr?

> **Definition 3.88.** A **quadratic approximation** $Q(x) = b_0 + b_1(x-a) + b_2(x-a)^2$ to $f(x)$ at $x = a$ is defined to satisfy the properties:
>
> (i) $Q(a) = f(a)$
> (ii) $Q'(a) = f'(a)$
> (ii) $Q''(a) = f''(a)$
>
> Thus, $b_0 = f(a)$, $b_1 = f'(a)$, and $b_2 = \dfrac{f''(a)}{2}$.

5. CAS Let $f(x) = \sqrt{1+x}$.

   (a) Find the linear and quadratic approximations of f at $x = 0$.
   (b) Graph f and its linear and quadratic approximations over the interval $[-1, 2]$.

CHAPTER 4

Applications of Derivatives

In this chapter, ...

Contents of Chapter 4

4.1. Extreme Values of Functions on Closed Intervals

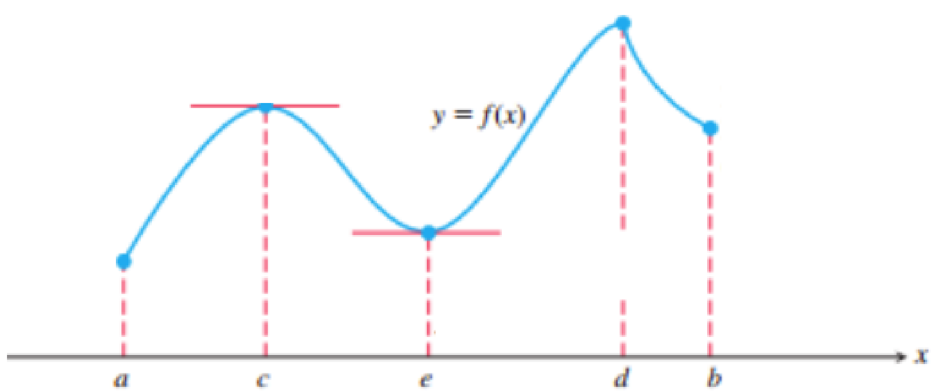
Definition 4.1. Let f be function with domain D. Then f has

- **absolute maximum** value on D at a point c
  if $f(c) \geq f(x)$ for all x in D
- **absolute minimum** value on D at a point c
  if $f(c) \leq f(x)$ for all x in D

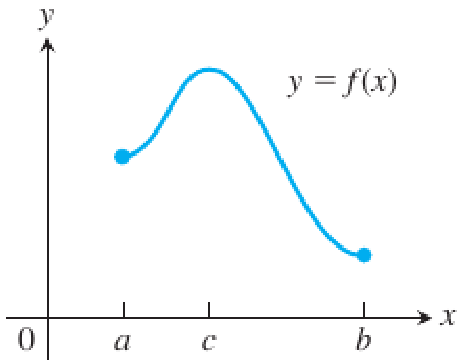
The number $f(c)$ is a

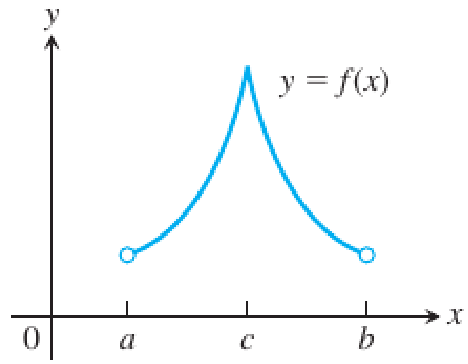
- **local maximum** value at a point c
  if $f(c) \geq f(x)$ when x is in an open interval containing c
- **local minimum** value at a point c
  if $f(c) \leq f(x)$ when x is in an open interval containing c

Example 4.2. For each point, a–e, identify whether it corresponds to a local/absolute maximum or minimum.



Example 4.3. Determine from the graph whether the function has any absolute extreme values on $[a, b]$.

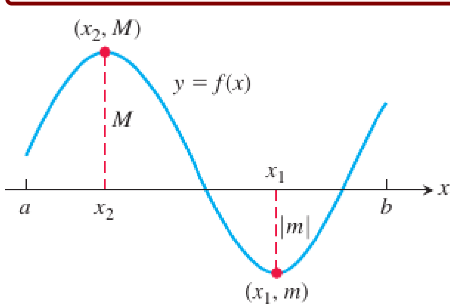




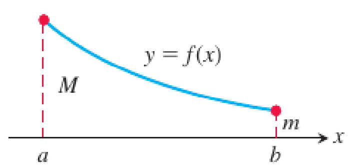
Theorem 4.4. The Extreme Value Theorem. *If f is **continuous** on a **closed interval** $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$, i.e.,*

- *there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and*

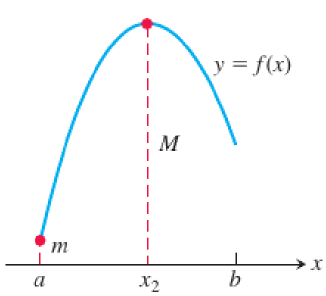
$$m \le f(x) \le M, \quad \text{for every } x \in [a, b]. \tag{4.1}$$



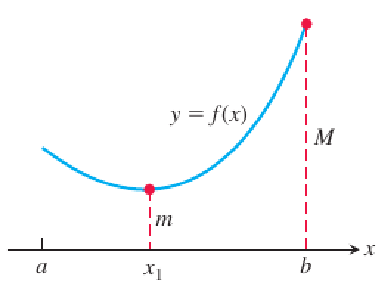
Maximum and minimum at interior points



Maximum and minimum at endpoints



Maximum at interior point, minimum at endpoint



Minimum at interior point, maximum at endpoint

Finding Extrema

How do we know where to look for a function's extrema? The following theorem and definition can help.

Theorem 4.5. The First Derivative Theorem for Local Extreme Values. *If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then*

$$f'(c) = 0.$$

Definition 4.6. An interior point in the domain of a function f where ***f'* is zero or undefined** is a **critical point** of f.

Example 4.7. Determine all critical points for each function.

(a) $f(x) = 6x^2 - x^3$ (b) $f(x) = 4x - \tan x$ (c) $f(x) = x - 3x^{\frac{2}{3}}$

Solution.

Strategy 4.8. Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Find all **critical points** of f on the interval.
2. Evaluate f at all **critical points** and **endpoints**.
3. Take the largest and smallest of these values.

Example 4.9. Find the absolute maximum and minimum values of each function on the given interval.

(a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 1, \quad 0 \le x \le 4$

(b) $f(x) = \csc x, \quad \dfrac{\pi}{3} \le x \le \dfrac{2\pi}{3}$

(c) $f(x) = \dfrac{1}{x} + \ln x, \quad \dfrac{1}{e} \le x \le 4$

Exercises 4.1

1. Sketch the graph of each function and determine whether the function has any absolute extreme values on its domain. Explain how your answer is consistent with the Extreme Value Theorem, p.173.

   (a) $f(x) = |x|, \quad -1 < x < 2$

   (b) $g(x) = \begin{cases} -x, & 0 \le x < 1 \\ x-1, & 1 \le x \le 2 \end{cases}$

   Clue: (b) g is not continuous.

2. Find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

   (a) $f(x) = 4 - x^3, \quad -2 \le x \le 1$

   (b) $g(x) = -\dfrac{1}{x}, \quad -2 \le x \le -1$

   (c) $h(t) = 2 - |t|, \quad -1 \le t \le 3$

   (d) $F(x) = \sec x, \quad -\dfrac{\pi}{3} \le x \le \dfrac{\pi}{6}$

   Ans: (d) absolute maximum: $(-\pi/3, 2)$, absolute minimum: $(0, 1)$.

3. Determine all critical points for each function.

   (a) $f(x) = x(x-3)^3$

   (b) $g(x) = \dfrac{x^2}{x-2}$

   Clue: (b) Is $x = 2$ is a critical point of g? See Definition 4.6.

   Ans: (b) $x = 0, 4$

4. $\boxed{\text{CAS}}$ You will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

   (1) Plot the function over the interval to see its general behavior there.
   (2) Find the interior points where $f' = 0$. (You may want to plot f' as well.)
   (3) Find the interior points where f' does not exist.
   (4) Evaluate the function at all points found in parts (2) and (3) and at the endpoints of the interval.
   (5) Find the function's absolute extreme values on the interval and identify where they occur.

   (a) $f(x) = \sqrt{x} + \cos x, \quad [0,\ 2\pi]$

   (b) $f(x) = \ln(2x + x\sin x), \quad [1,\ 15]$

4.2. The Mean Value Theorem

To arrive at the Mean Value Theorem, we first investigate a special case.

$\boxed{\textbf{Theorem}}$ **4.10.** ***Rolle's Theorem***. *Suppose that $y = f(x)$ is **continuous** over the **closed interval** $[a,b]$, and **differentiable** at every point of its interior (a,b). If $f(a) = f(b)$, then there is at least one number $c \in (a,b)$ at which*

$$f'(c) = 0. \tag{4.2}$$

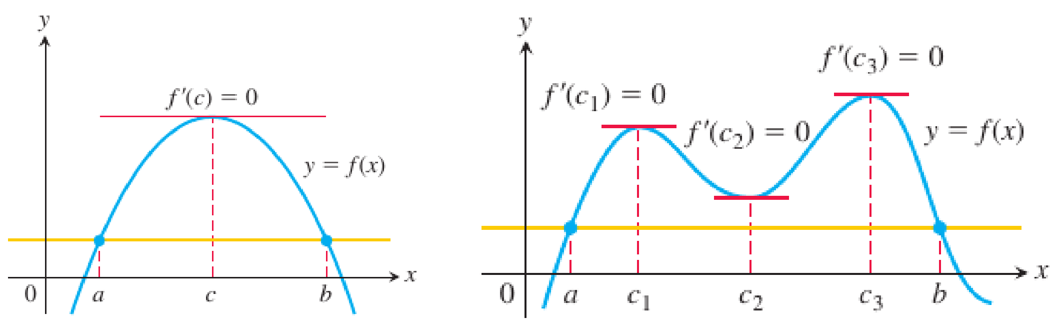


Figure 4.1: Rolle's Theorem

Note: The **Mean Value Theorem**, first stated by **Joseph-Louis Lagrange**, is a slanted version of Rolle's Theorem. The Mean Value Theorem guarantees that there is a point where the tangent line is parallel to the secant line that joins A and B.

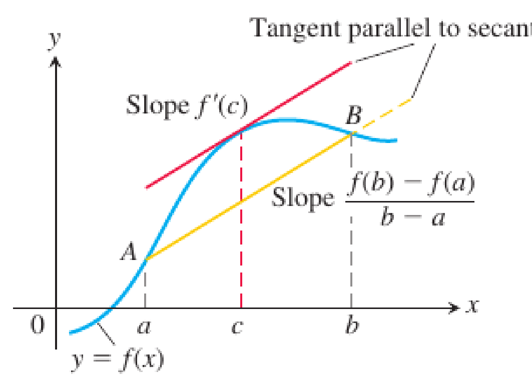


Figure 4.2: The Mean Value Theorem

$\boxed{\textbf{Theorem}}$ **4.11. The Mean Value Theorem (MVT).** *Suppose $y = f(x)$ is **continuous** over a **closed interval** $[a,b]$, and **differentiable** on the interval's interior (a,b). Then there is at least one point $c \in (a,b)$ at which*

$$f'(c) = \frac{f(b) - f(a)}{b - a}. \tag{4.3}$$

Example **4.12.** Verify that the function satisfies the hypotheses of the MVT on the given interval. Find the value or values c that satisfy (4.3).

(a) $f(x) = x^2 + 2x - 1$
$[0, 1]$

(b) $f(x) = x + \dfrac{1}{x}$
$\left[\dfrac{1}{2}, 2\right]$

(c) $f(x) = \ln(x - 1)$
$[2, 4]$

Solution.

Example **4.13.** Suppose that $f'(x) \leq 1$ for $1 \leq x \leq 4$. Show that $f(4) - f(1) \leq 3$.

Solution.

Example 4.14. It took 14 seconds for a mercury thermometer to rise from $-19°$C to $100°$C when it was taken from a freezer and placed in boiling water.

(a) Show that somewhere along the way, the temperature of the mercury was rising at a rate of $8.5°$ C/sec.

(b) Can we find a moment when the temperature of the mercury was rising at a rate of $9°$ C/sec?

Solution.

Corollary 4.15. **Mathematical Consequences of the MVT**

1. *If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $a \le x \le b$, where C is a constant.*
2. *If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $a \le x \le b$. That is, $f - g$ is a constant function on (a, b).*

Example **4.16.** Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution.

Ans: $f(x) = -\cos x + 3$.

Example **4.17.** Find all possible functions with the given derivative.

(a) $y' = 2x - 1$ (b) $y' = \sin 2t + \sqrt{t}$

Solution.

Exercises 4.2

1. Find the value or values of c that satisfy the equation $f'(c) = \dfrac{f(b)-f(a)}{b-a}$.

   (a) $f(x) = x^3 - x^2, \quad [-1, 2]$

   (b) $f(x) = \sqrt{x-1}, \quad [1, 3]$

2. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x? Give reasons for your answer.

3. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

   (a) $f(0) = 0$

   (b) $f(1) = 0$

   (c) $f(-2) = 3$

   Ans: (b) $f(2) = 3$.

4. Find the function with the given derivative whose graph passes through the point P.

   (a) $f'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

   (b) $g'(t) = \sec t \tan t - 1, \quad P(0, 0)$

   Ans: (b) $g(t) = \sec t - t - 1$

5. Show that for any numbers x and y, the sine inequality $|\sin x - \sin y| \le |x - y|$ is true. ***Hint***: Use the MVT.

6. The **arithmetic mean** of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the **Mean Value Theorem** for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

7. **Challenge** Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). ***Hint***: You may assume that f has three *distinct* zeros. The problem is to apply Rolle's Theorem.

4.3. Monotonic Functions and the First Derivative Test

Corollary 4.18. **Another corollary to the Mean Value Theorem.** *Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).*

*(a) If $f'(x) > 0$ at each point $x \in (a, b)$, then f is **increasing** on $[a, b]$.*

*(b) If $f'(x) < 0$ at each point $x \in (a, b)$, then f is **decreasing** on $[a, b]$.*

Remark 4.19. Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$.

- Then the Mean Value Theorem says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1), \tag{4.4}$$

for some $c \in (x_1, x_2)$. Corollary 4.18 follows from (4.4).

- A function that is increasing or decreasing on an interval is said to be **monotonic** on the interval.

Example 4.20. Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the open intervals on which f is increasing and on which f is decreasing.

Solution.

x		-2		2	
Sign of $f'(x)$		0		0	
Value of $f(x)$					
Behavior of f					

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c, except possibly at c itself. Moving across this interval from left to right,

1. if f' changes from negative to positive at c,
   then f has a local minimum at c;
2. if f' changes from positive to negative at c,
   then f has a local maximum at c;
3. if f' does not change sign at c,
   then f has no local extremum at c.

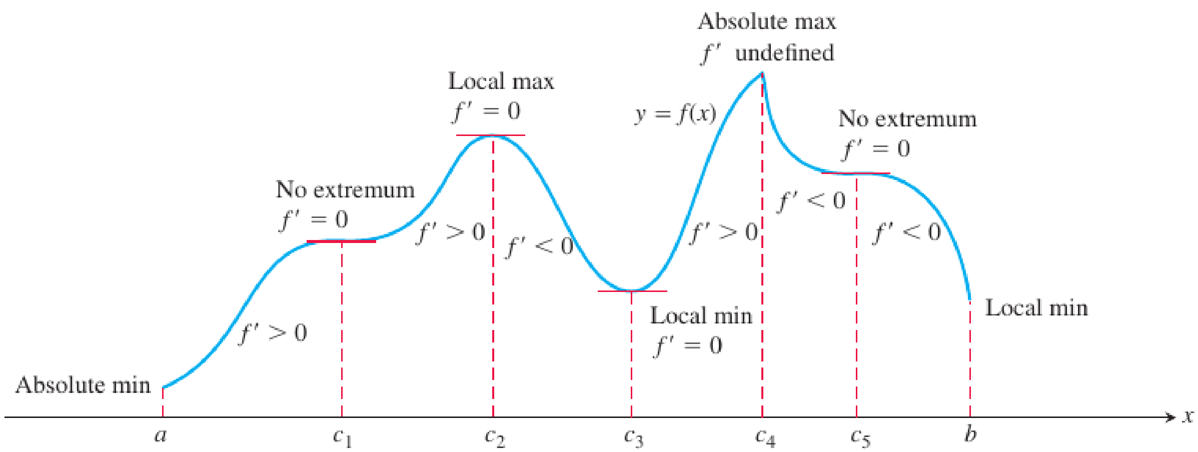


Figure 4.3: First Derivative Test for local extrema.

$\boxed{\textbf{Example}}$ **4.21.** Let $f(x) = (x^2 - 3)e^x$.

(a) Find the critical points of the function.

(b) Identify the open intervals on which f is increasing and on which f is decreasing.

(c) Find the function's local and absolute extreme values.

Exercises 4.3

1. (i) Find the open intervals on which the function is increasing and decreasing. (ii) Identify the function's local and absolute extreme values, if any, saying where they occur.

   (a) $f(x) = -x^3 + 3x^2$ (b) $g(x) = x^4 - 4x^3 + 4x^2$

2. (a) Prove that $f(x) = x - \ln x$ is increasing for $x > 1$.
   (b) Using part (a), show that $\ln x < x$ if $x > 1$.

   Hint: (a) $f'(x) = 1 - 1/x$, which is clearly positive for $x > 1$.

3. Determine the values of constants a and b so that $f(x) = ax^2 + bx$ has an absolute maximum at the point $(1, 2)$. ***Hint***: $f(1) = 2$ and $f'(1) = 0$.

4. CAS (i) Identify the function's local extreme values in the given domain, and say where they occur. (ii) Which of the extreme values, if any, are absolute? (iii) Support your findings with a CAS.

   (a) $f(x) = x^3 - 3x^2, \quad -\infty < x \leq 3$ (b) $g(x) = \dfrac{x-2}{x^2-1}, \quad 0 \leq x < 1$

4.4. Concavity and Curve Sketching

Definition 4.22. The graph of a differentiable function $y = f(x)$ is

(a) **concave up (convex)** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

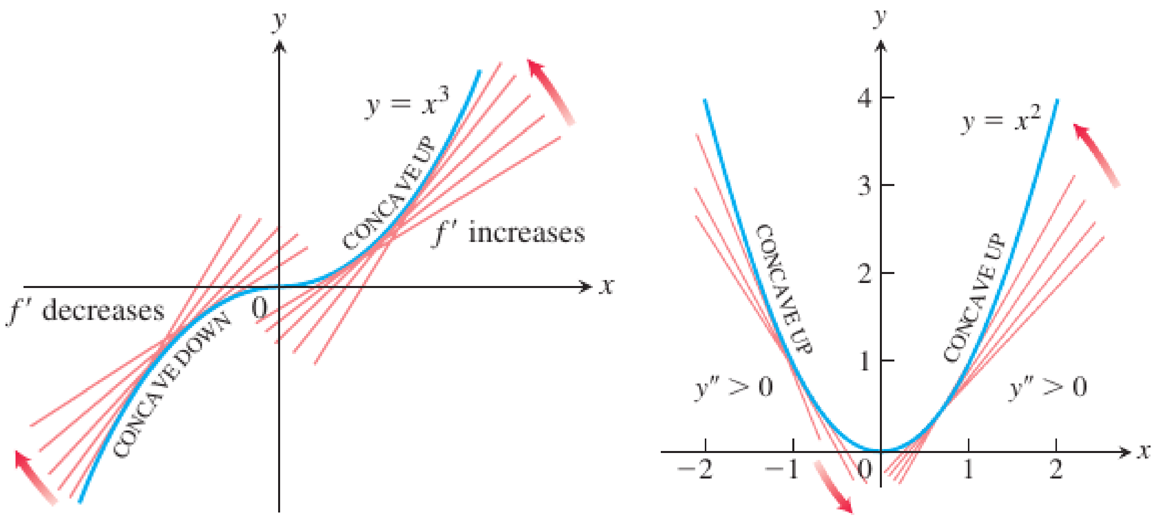


Figure 4.4: Concavity

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

(a) If $f'' > 0$ on I, the graph of f over I is concave up.

(b) If $f'' < 0$ on I, the graph of f over I is concave down.

Definition 4.23. A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

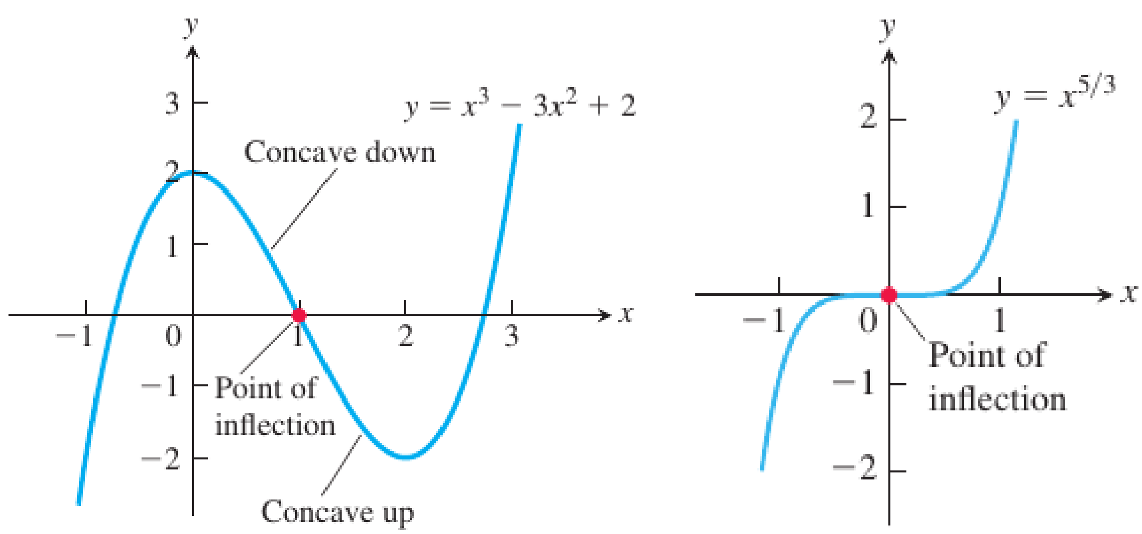
Example 4.24. For $f(x) = x^3$, as in Figure 4.4, $(0, 0)$ is an inflection point.

Remark 4.25. At a point of inflection $(c, f(c))$, either $f''(c) = 0$ or $f''(c)$ fails to exist.

Example 4.26. Determine the concavity and find the inflection points of

(a) $f(x) = x^3 - 3x^2 + 2$ (b) $g(x) = x^{5/3}$

Solution.



Theorem 4.27. Second Derivative Test for Local Extrema
Suppose f'' is continuous on an open interval that contains $x = c$.

(a) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

(b) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Example 4.28. Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$. Identify the coordinates of any local extreme points, inflection points, and concavity.

Solution.

x		0		2		3	
$f'(x)$		0				0	
$f''(x)$		0		0			
$f(x)$							
Behavior of f							

Strategy 4.29. Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives f' and f''.
3. Find the **critical points** of f, if any, and identify the function's behavior at each one.
4. Find where the curve is **increasing** and where it is **decreasing**.
5. Find the **points of inflection**, if any occur, and determine the **concavity** of the curve.
6. **Plot key points**, such as the intercepts and the points found in Steps 3–5, and **sketch the curve** together **with any asymptotes** that exist.

Example **4.30.** Graph the following functions using appropriate methods from the graphing procedures, identify the coordinates of any local extreme points and inflection points.

(a) $y = x^3 - 3x + 3$

(b) $y = 5\sqrt{3}x + 10\cos x, \quad 0 \le x \le 2\pi$

Note: For many functions, the graph can be sketched without computing the second derivative.

Example 4.31. Graph the following functions using appropriate methods from the graphing procedures, identify the coordinates of any local extreme points and inflection points.

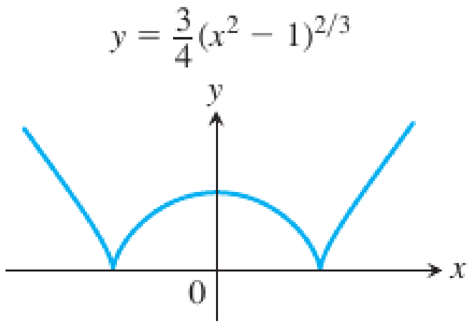
(a) $y = \ln(3 - x^2)$

(b) $y = \dfrac{x}{x^2 - 1}$

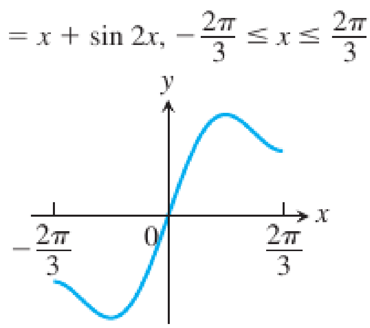
Exercises 4.4

1. For the functions, identify (i) the inflection points, (ii) local maxima and minima, and (iii) the intervals on which the functions are concave up and concave down.

(a)



(b)



Hint: (b) $y' = 1 + 2\cos 2x$, $y'' = -4\sin 2x$, and $\frac{-4\pi}{3} \le 2x \le \frac{4\pi}{3}$. Thus $y' = 0$ implies $2x = \pm\frac{2\pi}{3}$, $\pm\frac{4\pi}{3}$, and $y'' = 0$ implies $2x = 0, \pm\pi$. Thus, for example, the three inflection points are $(0, 0)$ and $\pm(\frac{\pi}{2}, \frac{\pi}{2})$.

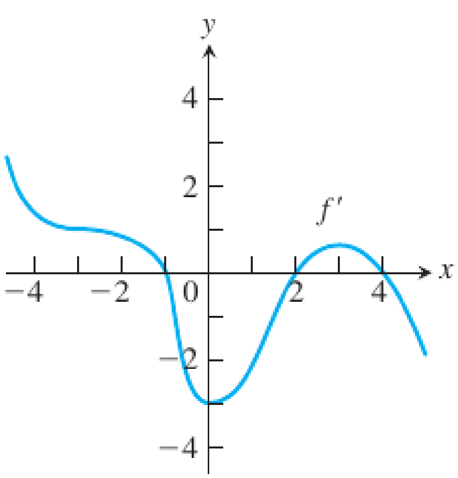
2. Identify the coordinates of any local and absolute extreme points and inflection points. Then, graph the function.

(a) $y = x^2(6 - x^2) - 4$

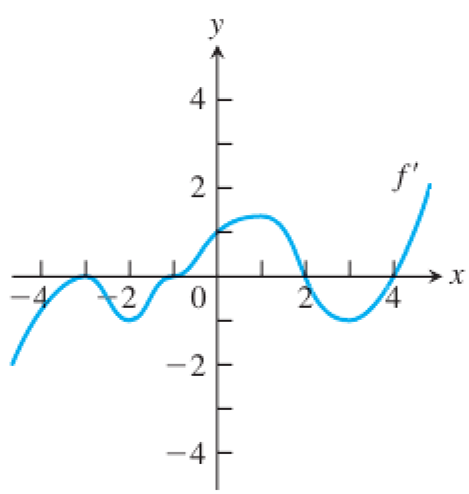
(b) $y = \cos x + \sqrt{3}\sin x, \quad 0 \le x \le 2\pi$

3. The graph of f' is given. Determine x-values corresponding to local minima, local maxima, and inflection points for the graph of f.

(a)



(b)



Hint: (b) Local minima: at $x = -1, 4$. Inflection points: at $x = -3, 1, 3$. Note that $f'' = 0$ at $x = -1$; however the corresponding point is not an inflection point.

4.5. Indeterminate Forms and L'Hôpital's Rule

4.5.1. L'Hôpital's Rule

Definition 4.32. Consider a limit of the form

$$\lim_{x\to a} \frac{f(x)}{g(x)}. \tag{4.5}$$

(a) If both $f(x) \to 0$ and $g(x) \to 0$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** $0/0$.

(b) If both $f(x) \to \infty$ and $g(x) \to \infty$ as $x \to a$, then this limit may or may not exist and is called an **indeterminate form of type** ∞/∞.

Theorem 4.33. L'Hôpital's Rule (Bernoulli's Rule)
Assume that f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \textit{and} \quad \lim_{x\to a} g(x) = 0 \tag{4.6}$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \textit{and} \quad \lim_{x\to a} g(x) = \pm\infty. \tag{4.7}$$

(In other words, we have an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{4.8}$$

Remark 4.34.

- We can continue to differentiate f and g so long as we still get $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$ form.
- L'Hôpital's Rule does not apply when either the numerator or the denominator has a finite nonzero limit.

Example 4.35. Use L'Hôpital's Rule to evaluate the limit. Compare this with the methods studied in Chapter 2.

(a) $\displaystyle\lim_{x\to -2} \frac{x+2}{x^2-4}$

The method from Chap.2 | L'Hôpital's Rule

(b) $\displaystyle\lim_{x\to \infty} \frac{5x^2-3x}{7x^2+1}$

The method from Chap.2 | L'Hôpital's Rule

Example 4.36. Use L'Hôpital's Rule to find the limit.

(a) $\displaystyle\lim_{t\to 0} \frac{\sin 5t}{2t}$

(b) $\displaystyle\lim_{\theta\to 0} \frac{\cos\theta - 1}{e^{\theta} - \theta - 1}$

(c) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$

(d) $\displaystyle\lim_{x\to 0} \frac{x2^x}{2^x - 1}$

4.5.2. Other Indeterminate Forms

Indeterminate Products (Type $0 \cdot \infty$)

Example 4.37. Use L'Hôpital's Rule to find the limit.

(a) $\lim_{x\to\infty} x^3 e^{-x^2}$

(b) $\lim_{x\to 0^+} x^2 \ln x$

Solution.

Indeterminate Differences (Type $\infty - \infty$)

Example 4.38. Find the limit of functions on the $\infty - \infty$ form.

(a) $\lim_{x\to 0} \left[\frac{1}{\sin x} - \frac{1}{x}\right]$

(b) $\lim_{x\to\infty} [\ln 2x - \ln(x+1)]$

Solution.

Ans: (a) 0.

Indeterminate Powers: 1^∞, 0^0, ∞^0

If $\lim_{x\to a} \ln f(x) = L$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} e^{\ln f(x)} = e^L. \tag{4.9}$$

Here a may be either finite or infinite.

Example 4.39. Apply L'Hôpital's Rule to show that

$$\lim_{x\to 0^+} (1+x)^{1/x} = e. \tag{4.10}$$

Solution. Start with $\lim_{x\to 0^+} \ln f(x) = \lim_{x\to 0^+} \dfrac{\ln(1+x)}{x}$.

Example 4.40. Find $\lim_{x\to\infty} x^{1/x}$

Solution.

Ans: 1.

Exercises 4.5

1. Use L'Hôpital's Rule to evaluate the limit.

   (a) $\displaystyle\lim_{x\to 0}\frac{1-\cos x}{x^2}$ (b) $\displaystyle\lim_{x\to\infty}\frac{\ln(1+1/x)}{\sin(1/x)}$ (c) $\displaystyle\lim_{x\to 0^+}\sqrt{x}\ln x$

   Ans: (b) 1. (c) 0.

2. Use L'Hôpital's Rule to evaluate the limit.

   (a) $\displaystyle\lim_{x\to 0}\frac{\sin x^2}{x}$ (b) $\displaystyle\lim_{x\to\infty}\frac{\ln(x+1)}{\log_2 x}$

   (c) $\displaystyle\lim_{x\to 0^+}[\ln x-\ln\sin x]$ (d) $\displaystyle\lim_{h\to 0}\frac{e^h-(1+h)}{h^2}$

   Ans: (c) 0.

3. L'Hôpital's Rule may not help with the limits. Try it first – you will just keep on cycling. Find the limits some other way.

   (a) $\displaystyle\lim_{x\to(\pi/2)^+}\frac{\sec x}{\tan x}$ (b) $\displaystyle\lim_{x\to\infty}\frac{e^{x^2}}{xe^x}$ (c) $\displaystyle\lim_{x\to\infty}\frac{2^x-3^x}{3^x+4^x}$

   Hint: (a) Try to simplify the fraction. (b) Apply the natural logarithm.

   Ans: (b) ∞. (c) 0.

4. **Challenge** For what values of a and b is

$$\lim_{x\to 0}\left[\frac{\sin x}{x^2}+\frac{a}{x}+\frac{\sin bx}{x}\right]=2?$$

5. CAS This problem explores the difference between the limits

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x \quad \text{and} \quad \lim_{x\to\infty}\left(1+\frac{1}{x^2}\right)^x$$

   (a) Use L'Hôpital's Rule to show that $\displaystyle\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x=e$.

   (b) Graph

$$f(x)=\left(1+\frac{1}{x}\right)^x \quad \text{and} \quad g(x)=\left(1+\frac{1}{x^2}\right)^x$$

   together for $x\ge 0$. How does the behavior of g compare with that of f? Estimate the value of $\lim_{x\to\infty} g(x)$.

   (c) Confirm your estimate of $\lim_{x\to\infty} g(x)$ by calculating it with L'Hôpital's Rule.

4.6. Applied Optimization

Strategy 4.41. Solving Applied Optimization Problems

1. *Read the problem*. Read the problem until you ***understand*** it. What is **given**? What is the **unknown quantity to be optimized**?
2. *Introduce variables*. List every relevant relation in the problem as an equation. In most problem it is helpful to draw a picture.
3. *Write an equation for the unknown quantity*.
   Express **the quantity to be optimized** as a function of **a single variable**. This may require considerable manipulation.
4. *Test the critical points and endpoints in the domain of the function found in the previous step*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Note: Mostly, the optimization problem will be formulated with **two quantities and two variables**; **the quantity to be optimized** can be expressed as a function of **a single variable.**

Example 4.42. What is the smallest perimeter possible for a rectangle whose area is 16 in^2, and what are its dimensions?

Solution.

Example 4.43. A 2048 ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. What dimensions of the tank will minimize the weight?

Solution.

Ans: $x = 16$, $y = 8$

Example 4.44. You have been asked to design a one-liter can shaped like a right circular cylinder (Figure 4.5). What dimensions will use the least material?

Solution. Surface area: $A = 2\pi r^2 + 2\pi rh$, with $\pi r^2 h = 1000$.

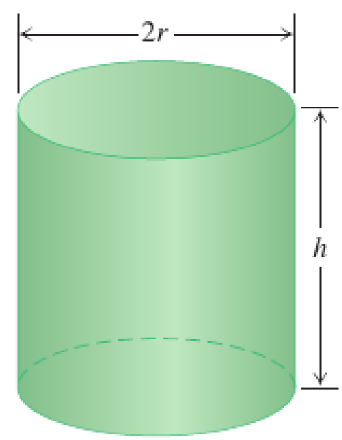


Figure 4.5: The one-liter can.

Ans: $r = \sqrt[3]{500/\pi}$ and $h = 2r$.

Example 4.45. A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800 m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?

Solution.

Example 4.46. A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?

Solution.

Example 4.47. Find a positive number for which the sum of it and its reciprocal is the smallest (least) possible.

Solution.

Exercises 4.6

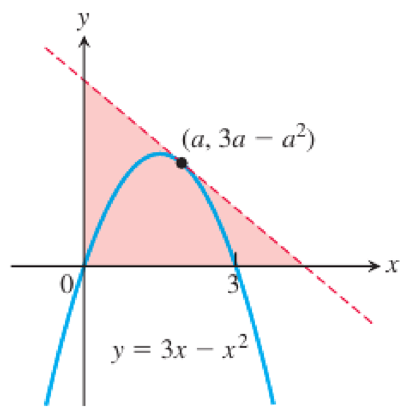
1. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?
   Hint: Using symmetry, let $[-x, x]$ be the base, where $x > 0$. Then the area $A = 2xy = 2x(12-x^2)$.

2. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? What kind of triangle is it?
   Hint: $A = \frac{1}{2}ab\sin\theta$.

   Ans: $\frac{\pi}{2}$.

3. Find the point on the line $\dfrac{x}{a} + \dfrac{y}{b} = 1$ that is closest to the origin. ***Clue***: Let (x, y) be a point on the line. Then the square-distance from the point to the origin is $d^2 = x^2 + y^2$. Note that the equation of the line can be rewritten as $y = b(1 - x/a) = b - \dfrac{b}{a}x$. Thus $d^2 = x^2 + (b - \frac{b}{a}x)^2$, which can be minimized.

4. Among all triangles in the first quadrant formed by the x-axis, the y-axis, and tangent lines to the graph of $y = 3x - x^2$, what is the smallest possible area?
   Clue: First of all, $a \in (3/2, 3)$. The tangent line reads $y = (3 - 2a)(x - a) + 3a - a^2$, of which the x-intercept is $x = \dfrac{a^2}{2a - 3}$. Now, find the y-intercept. Then the area of the triangle is $A = \dfrac{1}{2}xy$, as a function of a.



   Ans: $A = 8$ when $a = 2$.

5. What value of a makes $f(x) = x^2 + \dfrac{a}{x}$ have
   (a) a local minimum at $x = 2$?
   (b) a point of inflection at $x = 1$?

   Ans: (a) 16. (b) -1.

4.7. Newton's Method

In this section we study a numerical method called **Newton's method** or the **Newton-Raphson method**, which is a technique to approximate the solutions to an equation

$$f(x) = 0. \tag{4.11}$$

- Newton's method estimates the solutions using tangent lines of the graph of $y = f(x)$ near the points where f is zero.
- We call a solution of the equation $f(x) = 0$ a **root of the equation** or a **zero of the function** f.

Procedure for Newton's Method

We can derive a formula for generating the successive approximations in the following way.

- Given an **initial approximation x_0**, the point-slope equation for the tangent to the curve at $(x_0, (x_0))$ is

$$y = f'(x_0)(x - x_0) + f(x_0). \tag{4.12}$$

- We can find where it crosses the x-axis by setting $y = 0$:

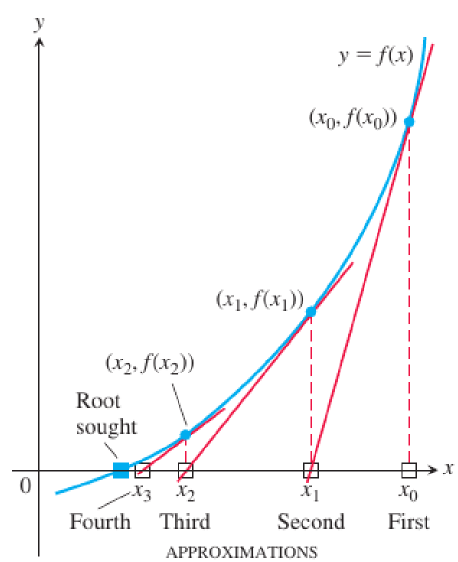
$$f'(x_0)(x - x_0) = -f(x_0),$$

which implies

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{4.13}$$

when $f'(x_0) \neq 0$.

- This value of x is the **next approximation x_1**.
- Repeat the steps to find new approximations.



Algorithm 4.48. Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \tag{4.14}$$

Example 4.49. Approximate the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution. Since $f'(x) = 2x$,

$$\begin{aligned} x_{n+1} &= x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{1}{2}\Big(x_n - \frac{2}{x_n}\Big) \\ &= \frac{1}{2}\Big(x_n + \frac{2}{x_n}\Big) \end{aligned}$$

newton_method.m

```
% Solving f(x) = x^2 - 2 =0

x=1;
for n = 1:6
   x = (x + 2/x)/2;
   fprintf('x_%d = %.15f\n',n,x)
end
```

Output

```
x_1 = 1.500000000000000
x_2 = 1.416666666666667
x_3 = 1.414215686274510
x_4 = 1.414213562374690
x_5 = 1.414213562373095
x_6 = 1.414213562373095
```

Details of the correction:

newton_method2.m

```
% Solving f(x) = x^2 - 2 =0

x=1;
for n = 1:6
   h = (x-2/x)/2;
   x = x -h;
   fprintf('x_%d = %.15f; h = %.15f\n',n,x,h)
end
```

Output
```
x_1 = 1.500000000000000; h = -0.500000000000000
x_2 = 1.416666666666667; h =  0.083333333333333
x_3 = 1.414215686274510; h =  0.002450980392157
x_4 = 1.414213562374690; h =  0.000002123899820
x_5 = 1.414213562373095; h =  0.000000000001595
x_6 = 1.414213562373095; h =  0.000000000000000
```

Example 4.50. Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution. What we should solve is $x^3 - x = 1$. Let

$$f(x) = x^3 - x - 1.$$

Then $f(1) = -1$ and $f(2) = 5 \Rightarrow$ There is a solution in $(1, 2)$, by the IVT. Since $f'(x) = 3x^2 - 1$, Newton's method reads

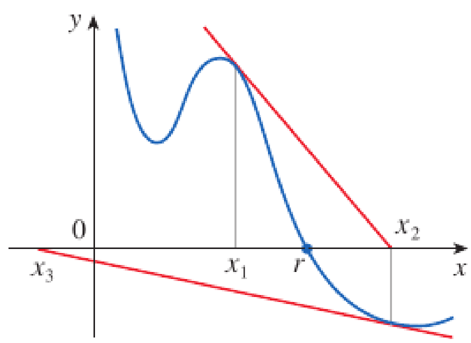
$$x_{n+1} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}.$$

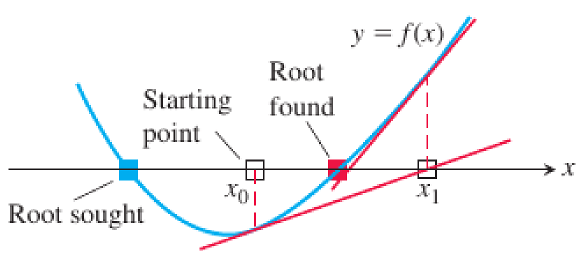
newton_method3.m
```
% Solving f(x) = x^3 - x -1 =0

x=1.5;
for n = 1:6
   fx= x^3-x-1;
   h = fx/(3*x^2-1);
   x = x -h;
   fprintf('x_%d = %.15f; h = %.15f; f(xn)= %.15f\n',n,x,h,fx)
end
```

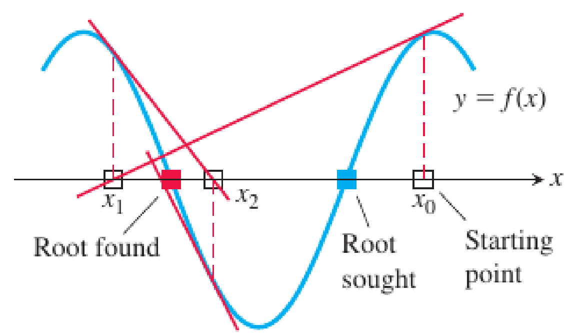
Output
```
x_1 = 1.347826086956522; h = 0.152173913043478; f(xn)= 0.875000000000000
x_2 = 1.325200398950907; h = 0.022625688005615; f(xn)= 0.100682173091148
x_3 = 1.324718173999054; h = 0.000482224951853; f(xn)= 0.002058361916663
x_4 = 1.324717957244790; h = 0.000000216754264; f(xn)= 0.000000924377760
x_5 = 1.324717957244746; h = 0.000000000000044; f(xn)= 0.000000000000187
x_6 = 1.324717957244746; h = 0.000000000000000; f(xn)= 0.000000000000000
```

Convergence of the Approximations

- Newton's method does not always converge, particularly *when the initial approximation is not accurate enough.*
- If Newton's method does converge, it converges to a root.
- When Newton's method converges to a root, it may not be the root you have in mind.







$\boxed{\textbf{Theorem}}$ **4.51.** **(Convergence of the Newton's method)**: *Let the second derivative of $f(x)$ be continuous, $f(\widehat{x}) = 0$, and* $\boldsymbol{f'(\widehat{x}) \neq 0}$*. When* $\boldsymbol{x_0}$ **is chosen near to** $\boldsymbol{\widehat{x}}$, *the Newton's method generates a convergent sequence $\{x_n\}$ satisfying*

$$|x_{n+1} - \widehat{x}| < C\,|x_n - \widehat{x}|^2, \tag{4.15}$$

for a positive constant C.
Such a convergence is called a ***quadratic convergence****.*

Example 4.52. Let

$$f(x) = \arctan(x).$$

Then $\widehat{x} = 0$ is the only root of $f(x)$. Use the Newton's method to find a root, starting with

(a) $x_0 = \pi/2$ (b) $x_0 = \pi/4$

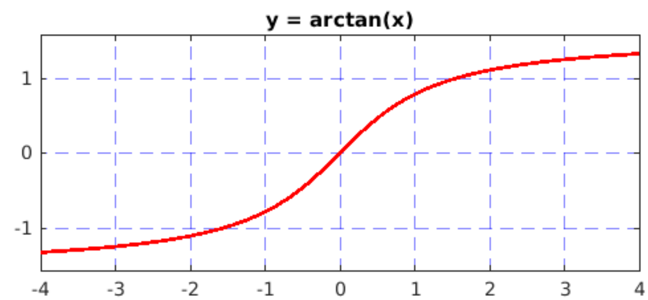


Figure 4.6: The graph of $y = \arctan(x)$.

Solution.

newton_method_convergence.m

```
% Newton's Method: for f(x) = atan(x) = 0

f = @(x) atan(x); df= @(x) 1/(1+x^2);

x=pi/4
for n = 1:5
    h = f(x) / df(x);
    x = x -h;
    fprintf('x_%d = %g; h = %g; f(xn) = %g\n',n,x,h,f(x))
end
```

Output

```
x = 1.5708  %x=pi/2
x_1 =     -1.91008; h =      3.48087; f(xn) = -1.08849
x_2 =      3.14967; h =     -5.05974; f(xn) =  1.26337
x_3 =    -10.6468;  h =     13.7965;  f(xn) = -1.47715
x_4 =    158.273;   h =   -168.92;    f(xn) =  1.56448
x_5 = -39033.9;     h =  39192.1;     f(xn) = -1.57077

x = 0.7854  %x=pi/4
x_1 = -0.291058;    h =  1.07646;     f(xn) = -0.283233
x_2 =  0.0161691;   h = -0.307227;    f(xn) =  0.0161677
x_3 = -2.81804e-06; h =  0.016172;    f(xn) = -2.81804e-06
x_4 =  1.49192e-17; h = -2.81804e-06; f(xn) =  1.49192e-17
x_5 =  0;           h =  1.49192e-17; f(xn) =  0
```

Exercises 4.7

1. Using your calculator (or pencil-and-paper), run two iterations of Newton's method to find x_2 for given f and x_0.

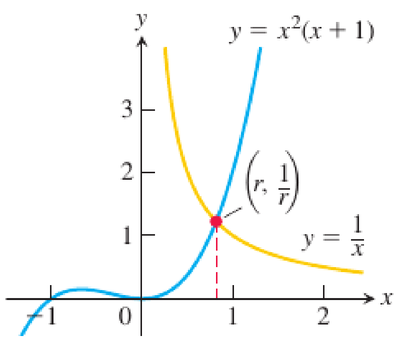
   (a) $f(x) = x^4 - 2$, $x_0 = 1$
   (b) $f(x) = xe^x - 1$, $x_0 = 0.5$

   Ans: (b) $x_2 = 0.56715557$

2. CAS Implement Newton's method to solve $f(x) = \arctan(x) = 0$,

   (a) with $x_0 = \pi/2$.
   (b) with $x_0 = \pi/4$.

3. CAS The graphs of $y = x^2(x+1)$ and $y = 1/x$ $(x > 0)$ intersect at one point $x = r$. Use Newton's method to estimate the value of r to eight decimal places.



4.8. Antiderivatives

Many problems require **to recover a function from its derivative**.

- For instance, the laws of physics tell us the **acceleration of an object falling** from an initial height, and we can use this to compute its **velocity** and its **height** at any time.
- More generally, starting with a function f, we want
  to find a function F whose derivative is f.

4.8.1. Finding Antiderivatives

Definition 4.53. A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Example 4.54. Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$ (b) $g(x) = \cos x$ (c) $h(x) = \sec^2 x + \dfrac{1}{2\sqrt{x}}$

Solution.

Note: If F is an antiderivative of f, so is $F + C$, for all constants C.

Theorem 4.55. *Let F and G be antiderivatives of f on an interval I. Then*

$$F(x) = G(x) + C, \quad \textit{for some } C. \tag{4.16}$$

Table 4.1: Antiderivative formulas, k a nonzero constant.

Function	General antiderivative
x^n	$\dfrac{1}{n+1}x^{n+1}+C, \quad n \neq -1$
$1/x$	$\ln\lvert x\rvert + C$
$\sin kx$	$-\dfrac{1}{k}\cos kx + C$
$\cos kx$	$\dfrac{1}{k}\sin kx + C$
$\sec^2 kx$	$\dfrac{1}{k}\tan kx + C$
$\csc^2 kx$	$-\dfrac{1}{k}\cot kx + C$
$\sec kx \tan kx$	$\dfrac{1}{k}\sec kx + C$
$\csc kx \cot kx$	$-\dfrac{1}{k}\csc kx + C$

Example **4.56.** Find the general antiderivative of each of the following functions.

(a) $f(x) = \dfrac{1}{\sqrt{x}}$ (b) $g(x) = \cos\dfrac{x}{2} + \sec^2 2x$ (c) $h(x) = \dfrac{3}{x^3} + \dfrac{1}{x^2}$

Solution.

4.8.2. Initial Value Problems and Differential Equations

Example 4.57. Find an antiderivative of $f(x) = 3x^2$ that satisfies $F(1) = -1$.

Solution.

Remark 4.58. Antiderivatives play important roles in mathematics and its applications. Methods and techniques for finding them are a major part of calculus, and we take up that study in Chapter 8.

- **Finding an antiderivative for a function $f(x)$** is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x), \tag{4.17}$$

  which is called a **differential equation**.

- We can **fix the arbitrary constant C**, arising in the **antidifferentiation process**, by specifying an **initial value**

$$y(x_0) = y_0. \tag{4.18}$$

- The combination of a differential equation and an initial condition, (4.17) and (4.18), is called an **initial value problem**.
- Such problems play important roles in **all branches of science.**

Example 4.59. Solve the initial value problems.

(a) $\dfrac{df}{dx} = \cos x + \sin x, \quad f(\pi) = 1$

(b) $\dfrac{d^2y}{dx^2} = 2 - 6x, \quad y'(0) = 4, \, y(0) = 1.$

Solution.

4.8.3. Indefinite Integrals

Definition 4.60. The collection of all antiderivatives of *f* is called the **indefinite integral of *f*** with respect to x, and is denoted by

$$\int f(x)\, dx. \tag{4.19}$$

The symbol $\displaystyle\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Example 4.61. Evaluate $\displaystyle\int \left(x^2 - \frac{2}{x} + 5\right) dx.$

Solution.

Exercises 4.8

1. Find an antiderivative for each function. Check your answers by differentiation.

   (a) $x^3 - \dfrac{1}{x^3}$ (b) $\sqrt{x} + \dfrac{1}{\sqrt{x}} - \dfrac{1}{3}x^{-2/3}$ (c) $\dfrac{1}{1+4x^2}$

   Ans: (c) $\dfrac{1}{2}\arctan 2x$

2. Find the most general antiderivative or indefinite integral. You may need to try a solution and then adjust your guess. Check your answers by differentiation.

   (a) $\displaystyle\int 3\cos 5\theta\, d\theta$ (b) $\displaystyle\int (3e^{-2x} + 2^x)dx$

   (c) $\displaystyle\int (4\sec x \tan x - 2\sec^2 x)dx$ (d) $\displaystyle\int (2 + \tan^2 x)dx$

   Ans: (d) $x + \tan x + C$.

3. Which is right? Give a brief reason for your answer.

   (a) $\displaystyle\int x\sin x\, dx = \frac{x^2}{2}\sin x + C$

   (b) $\displaystyle\int x\sin x\, dx = -x\cos x + C$

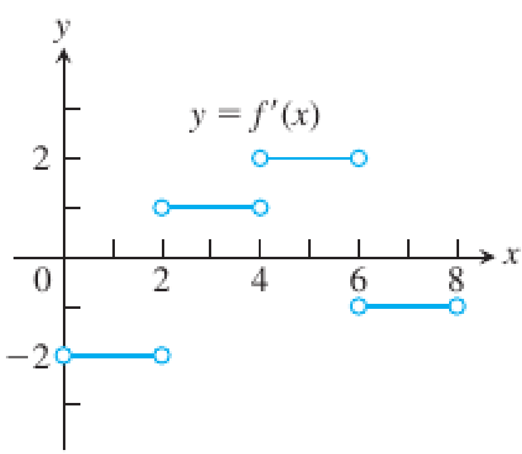
   (c) $\displaystyle\int x\sin x\, dx = -x\cos x + \sin x + C$

4. Solve the initial value problems.

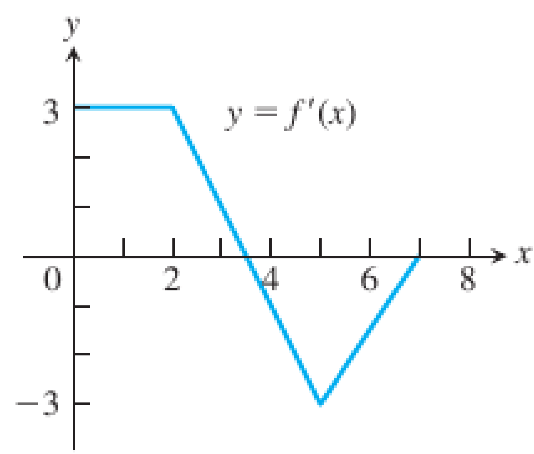
   (a) $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$ (b) $y'' = -\cos x + 8\sin 2x$
   $y'(0) = -2$, $y(0) = 3$

   Ans: (b) $y = \cos x - 2\sin 2x + 2x + 2$

5. The graph of f' is given. Assume that $f(0) = 1$ and sketch a possible continuous graph of f.



(a)



(b)

CALCULUS II

Integration

CHAPTER **5**

Integrals

A great achievement of classical geometry was obtaining formulas for the areas and volumes of triangles, spheres, and cones. In this chapter:

- We first develop a method, called **integration**, to calculate the areas and volumes of more general shapes.
- The **definite integral** is the key tool in calculus for defining and calculating areas and volumes.
- We also show that the process of computing these definite integrals is closely connected to finding **antiderivatives**. This connection is captured in the **Fundamental Theorem of Calculus**.

Contents of Chapter 5

5.1. Area and Estimating with Finite Sums

5.1.1. The Area Problem

In this subsection, we attempt to solve the **area problem**. It isn't so easy to find the **area under a graph**, a region with a curved side. We can use rectangles to approximate this area.

Example 5.1. Suppose we want to find the area of the shaded region R that lies above the x-axis, below the graph of $y = f(x) = 1 - x^2$, and between the vertical lines $x = 0$ and $x = 1$ (see Figure 5.1).

- The area **cannot be found by a simple formula**.
- However, we can approximate it with a **finite sum of rectangle areas**.

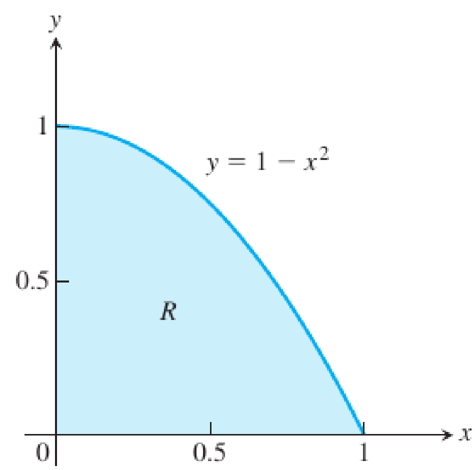


Figure 5.1: The area of the shaded region R cannot be found by a simple formula.

Using left-endpoint values

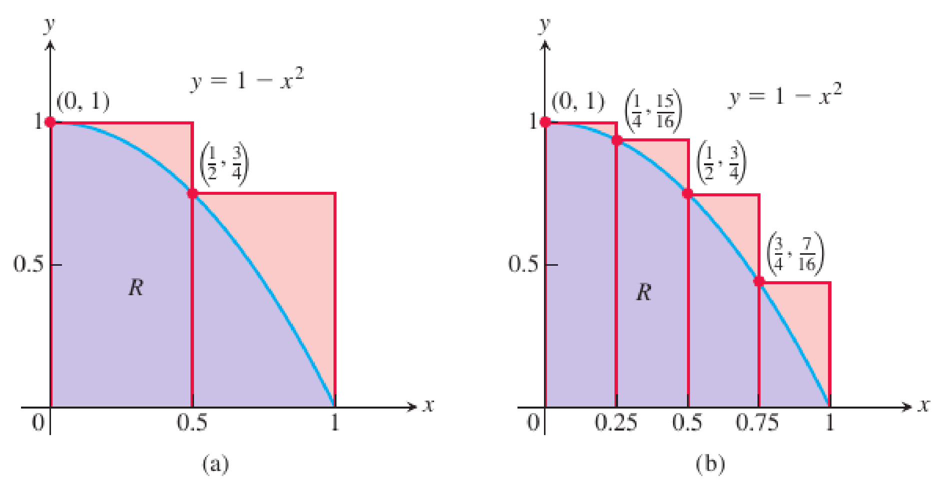


Figure 5.2: Rectangles using **left-endpoint** values, with (a) 2 subintervals and (b) 4 subintervals. **The larger the number of subintervals is, the better the approximation becomes.**

With four subintervals, the area can be approximated as

$$\begin{aligned} \text{area} &\approx \frac{1}{4}\cdot f(0) + \frac{1}{4}\cdot f\Big(\frac{1}{4}\Big) + \frac{1}{4}\cdot f\Big(\frac{2}{4}\Big) + \frac{1}{4}\cdot f\Big(\frac{3}{4}\Big) \\ &= \frac{1}{4}\Big[1 + \frac{15}{16} + \frac{12}{16} + \frac{7}{16}\Big] = \frac{25}{32} = \mathbf{0.78125}. \end{aligned} \tag{5.1}$$

area_approx_left_sum.m

```
f = @(x) 1-x.^2;
a = 0; b = 1;

for i = 1:10
    n = 2^i; dx = (b-a)/n;
    X = (0:n)*dx; Y = f(X);
    left_sum = sum(Y(1:end-1))*dx;
    fprintf('n = %4d; left_sum = %.10f\n',n,left_sum)
end
```

Output

```
n =    2; left_sum = 0.8750000000
n =    4; left_sum = 0.7812500000
n =    8; left_sum = 0.7265625000
n =   16; left_sum = 0.6972656250
n =   32; left_sum = 0.6821289062
n =   64; left_sum = 0.6744384766
n =  128; left_sum = 0.6705627441
n =  256; left_sum = 0.6686172485
n =  512; left_sum = 0.6676425934
n = 1024; left_sum = 0.6671547890
```

Note: The exact value for the area is $2/3$.

Using right-endpoint values & midpoint values

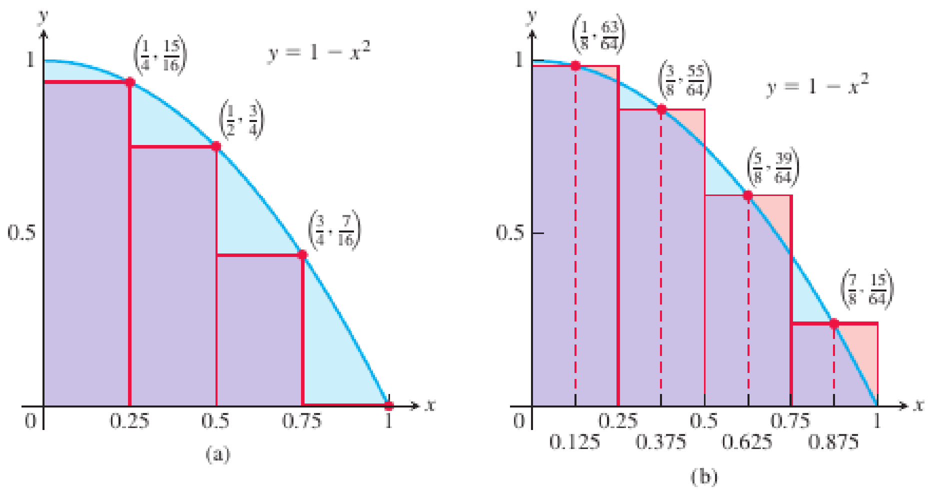


Figure 5.3: Rectangles using **right-endpoint** values and **midpoint** values.

- **Using right-endpoint values**:

$$\begin{aligned} \text{area} &\approx \frac{1}{4}\cdot f\left(\frac{1}{4}\right) + \frac{1}{4}\cdot f\left(\frac{2}{4}\right) + \frac{1}{4}\cdot f\left(\frac{3}{4}\right) + \frac{1}{4}\cdot f(1) \\ &= \frac{1}{4}\left[\frac{15}{16} + \frac{12}{16} + \frac{7}{16} + 0\right] = \frac{17}{32} = \mathbf{0.53125}. \end{aligned} \tag{5.2}$$

- **Using midpoint values**:

$$\begin{aligned} \text{area} &\approx \frac{1}{4}\cdot f\left(\frac{1}{8}\right) + \frac{1}{4}\cdot f\left(\frac{3}{8}\right) + \frac{1}{4}\cdot f\left(\frac{5}{8}\right) + \frac{1}{4}\cdot f\left(\frac{7}{8}\right) \\ &= \frac{1}{4}\left[\frac{63}{64} + \frac{55}{64} + \frac{39}{64} + \frac{15}{64}\right] = \frac{43}{64} = \mathbf{0.671875}. \end{aligned} \tag{5.3}$$

A Comparison

area_approx_left_right_mid.m

```
f = @(x) 1-x.^2;
a = 0; b = 1;

for i = 1:10
    n = 2^i; dx = (b-a)/n;
    X = (0:n)*dx; Y = f(X);
    left_sum  = sum(Y(1:end-1))*dx;
    right_sum = sum(Y(2:end))*dx;
    M = X + dx/2; Y = f(M);
    mid_sum   = sum(Y(1:end-1))*dx;
    fprintf('n = %4d; (left,right,mid)-sum = (%.10f, %.10f, %.10f)\n',...
              n,left_sum,right_sum,mid_sum)
end
```

Output

```
n =    2; (left,right,mid)-sum = (0.8750000000, 0.3750000000, 0.6875000000)
n =    4; (left,right,mid)-sum = (0.7812500000, 0.5312500000, 0.6718750000)
n =    8; (left,right,mid)-sum = (0.7265625000, 0.6015625000, 0.6679687500)
n =   16; (left,right,mid)-sum = (0.6972656250, 0.6347656250, 0.6669921875)
n =   32; (left,right,mid)-sum = (0.6821289062, 0.6508789062, 0.6667480469)
n =   64; (left,right,mid)-sum = (0.6744384766, 0.6588134766, 0.6666870117)
n =  128; (left,right,mid)-sum = (0.6705627441, 0.6627502441, 0.6666717529)
n =  256; (left,right,mid)-sum = (0.6686172485, 0.6647109985, 0.6666679382)
n =  512; (left,right,mid)-sum = (0.6676425934, 0.6656894684, 0.6666669846)
n = 1024; (left,right,mid)-sum = (0.6671547890, 0.6661782265, 0.6666667461)
```

Remark 5.2.

- The sum by left-endpoint values is the **upper sum** approximation.
- The sum by right-endpoint values is the **lower sum** approximation.
- The sum by midpoint values converges, faster than the others.
- However, they converge to the same limit as n (the number of subintervals) approaches infinity.

Algebraic Manipulation, for the Exact Area

Example 5.3. Use rectangles of right-endpoint values to estimate the area under the parabola $y = x^2$ from 0 to 1. (See Figure 5.4, where rectangles of eight subintervals are depicted.)

The exact value of the area is $1/3$.

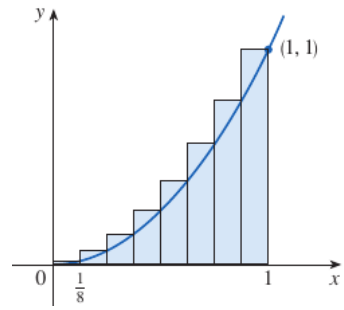


Figure 5.4

Solution. Let the interval $[0, 1]$ be partitioned in to n subintervals. Then, the right-endpoints are

$$\frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}.$$

Thus the sum of areas of rectangles reads

$$\begin{aligned}\text{right-sum} &= \frac{1}{n}\left[\left(\frac{1}{n}\right)^2 + \left(\frac{2}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right] \\ &= \frac{1}{n^3}(\mathbf{1^2 + 2^2 + \cdots + n^2}) \\ &= \frac{1}{n^3}\frac{\mathbf{n(n+1)(2n+1)}}{\mathbf{6}} = \frac{2}{6} + \frac{3}{6n} + \frac{1}{6n^2} \searrow \frac{1}{3},\end{aligned} \tag{5.4}$$

as $n \to \infty$. For the summation formula, see Formula 5.11, p. 226.

area_approx_right_x2.py

```python
import numpy as np
def area(n): return (1/n) * sum( (np.array(range(1,n+1))/n)**2 )
for i in range(1,11): print('%-4d: %.15g' %(2**i,area(2**i)))
```

Output

```
2   : 0.625
4   : 0.46875
8   : 0.3984375
16  : 0.365234375
32  : 0.34912109375
64  : 0.3411865234375
128 : 0.337249755859375
256 : 0.335289001464844
512 : 0.334310531616211
1024: 0.333821773529053
```

5.1.2. The Distance Problem

Note: The **distance traveled** is **the velocity times the time**, when the velocity is constant.

Example 5.4. The following table shows the velocity of a model train engine moving along a track for 10 seconds. Estimate the **distance traveled** by the engine using 10 subintervals of length 1 using

(a) left-endpoints

(b) right-endpoints

(c) midpoints (assuming the velocity varies **locally linearly**)

t (sec)	0	1	2	3	4	5	6	7	8	9	10
v (cm/sec)	0	30	56	25	38	33	28	15	5	15	12

Solution.

distance_traveled.m

```
v = [0 30 56 25 38 33 28 15 5 15 12];
dt= 1;

vL = v(1:end-1);
vR = v(2:end);
vM = (vL+vR)/2;
distL = sum(vL)*dt;
distR = sum(vR)*dt;
distM = sum(vM)*dt;

    fprintf('(distL,distR,distM) = (%g, %g, %g)\n',distL,distR,distM)
```

Output

```
(distL,distR,distM) = (245, 257, 251)
```

Example 5.5. An object is dropped straight down from a helicopter. The object falls faster and faster, but its acceleration decreases over time because of air resistance. The acceleration is measured in ft/sec^2 and recorded every second after the drop for 5 seconds, as shown:

t (sec)	0	1	2	3	4	5
a (ft/sec^2)	32.00	19.41	11.77	7.14	4.33	2.63

(a) Find an upper estimate for the speed when $t = 5$.

(b) Find a lower estimate for the speed when $t = 5$.

(c) Find an upper estimate for the distance fallen when $t = 3$.

Solution.

t (sec)	0	1	2	3	4	5
a (ft/sec^2)	32.00	19.41	11.77	7.14	4.33	2.63
v-upper (ft/sec)	0	32.00				
v-lower (ft/sec)	0	19.41				
s-upper (ft)	0	32.00				

object_fallen.m

```
a = [32.00 19.41 11.77 7.14 4.33 2.63];
n = length(a);
vU = zeros(1,n); vL = zeros(1,n); sU = zeros(1,n);

for i=2:n
    vU(i) = vU(i-1) +max(a(i-1:i));
    vL(i) = vL(i-1) +min(a(i-1:i));
end
for i=2:n
    sU(i) = sU(i-1) +max(vU(i-1:i));
end

    fprintf('\nvU:'); fprintf(' %7.2f',vU)
    fprintf('\nvL:'); fprintf(' %7.2f',vL)
    fprintf('\nsU:'); fprintf(' %7.2f',sU)
```

Output

```
vU:    0.00   32.00   51.41   63.18   70.32   74.65
vL:    0.00   19.41   31.18   38.32   42.65   45.28
sU:    0.00   32.00   83.41  146.59  216.91  291.56
```

5.1.3. Average Value of a Nonnegative Continuous Function

Note: The average value of a collection of n numbers $x_1, x_2, \cdots, x_n$ is obtained by adding them together and dividing by n.

- But what is the average value of a continuous function f on an interval $[a, b]$?
- The **average value** of f on $[a, b]$ can be defined as the **average height** of its graph, which reads

$$\text{av}(\mathbf{f}) \cdot (b - a) = \text{The area under the graph of } f \text{ over } [a, b]. \tag{5.5}$$

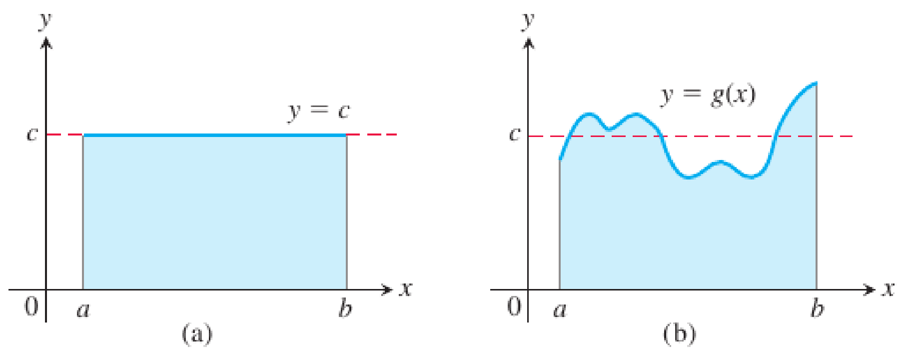


Figure 5.5

  - For $f(x) = c$: av$(f) = c$.
  - For $g(x)$: av(g) is the area below its graph divided by $(b - a)$.

Example 5.6. Let $f(x) = 1/x$ defined on $[1, 9]$. Use a finite sum to estimate the average value of f on the given interval by partitioning the interval into four subintervals of equal length and evaluating f at the subinterval midpoints.

Solution.

Ans: av$(f) \approx \frac{1}{8} \cdot \frac{25}{12} = 0.26041666....$

The true average value is $\dfrac{1}{8} \cdot \ln 9 = 0.274653072167.$

Exercises 5.1

1. Let $f(x) = 1/x$ defined on $[1, 5]$. Use finite approximations to estimate the area under the graph of the function, using

   (a) a lower sum with two rectangles of equal width.
   (b) a lower sum with four rectangles of equal width.
   (c) an upper sum with two rectangles of equal width.
   (d) an upper sum with four rectangles of equal width.

   Ans: (d) $25/12$

2. Using rectangles each of whose height is given by the value of the function at the midpoint of the rectangle's base (the **midpoint rule**), estimate the area under the graphs of the following functions, using first two and then four rectangles.

   (a) $f(x) = 1/x$ defined on $[1, 5]$.
   (b) $f(x) = x^2$ defined on $[0, 1]$.

   Ans: (a) 1.5 & 1.5746031746. (The true area is $\ln 5 = 1.6094379$.)

3. **Length of a road**. You and a companion are about to drive a twisty stretch of dirt road in a car whose speedometer works but whose odometer (mileage counter) is broken. To find out how long this particular stretch of road is, you record the car's velocity at 10-sec intervals, with the results shown in the accompanying table. Estimate the length of the road using

   (a) left-endpoint values.
   (b) right-endpoint values.

Time (sec)	0	10	20	30	40	50	60	70	80	90	100	110	120
v (ft/sec)	0	44	15	35	30	44	35	15	22	35	44	30	35

   Ans: (b) 3840 ft

4. Use a finite sum to estimate the average value of f on the given interval by partitioning the interval into **four subintervals** of equal length and evaluating f at the subinterval **midpoints**.

   (a) $f(x) = x^3$ on $[0, 2]$.
   (b) $f(t) = 1 + \sin^2 t$ on $[0, 2\pi]$

   Ans: (a) $31/16$. (b) $3/2$.

5. Challenge Inscribe a regular n-sided polygon inside a circle of radius 1.

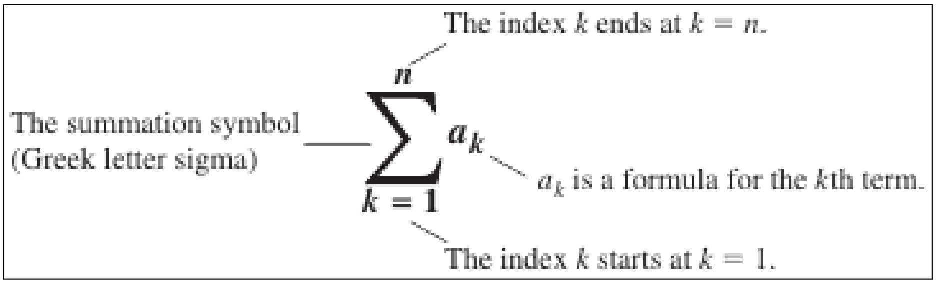
   (a) Compute the area of the polygon for $n = 4, 8, 16$.
   (b) Compute the limit of the area of the inscribed polygon as $n \to \infty$.

   Hint: (b) The area of one of the n congruent triangles is $\frac{1}{2} \cdot 1 \cdot \sin\left(\frac{2\pi}{n}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{n}\right)$. Now, you may try to use (2.25), page 78.

   Ans: (b) π.

5.2. Sigma Notation and Limits of Finite Sums

Finite Sums and Sigma Notation



Example 5.7.

- We can write the squares of the numbers 1 through 10 as

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2 + 10^2 = \sum_{k=1}^{10} k^2. \tag{5.6}$$

- The sum of $f(i)$ for integers i from 1 to 100:

$$f(1) + f(2) + \cdots + f(100) = \sum_{i=1}^{100} f(i). \tag{5.7}$$

Example 5.8. Express the sum $1+3+5+7+9+11$ in sigma notation. You can express it with various starting k: e.g., $k = 0, 1, 2$.

Solution.

Starting with $k = 0$: $\quad 1 + 3 + 5 + 7 + 9 + 11 = \sum_{k=0}^{5}(2k+1)$

Starting with $k = 1$:

Starting with $k = 2$:

Formula 5.9. Algebra rules for Finite Sums:

$$
\begin{aligned}
&\text{Sum Rule :} && \sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k \\
&\text{Difference Rule :} && \sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k \\
&\text{Constant Multiple Rule :} && \sum_{k=1}^{n} ca_k = c \cdot \sum_{k=1}^{n} a_k \\
&\text{Constant Value Rule :} && \sum_{k=1}^{n} c = n \cdot c
\end{aligned}
$$

Example 5.10. Show that the sum of the first n positive integers is

$$1 + 2 + \cdots + n = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

Solution.

Formula 5.11. Summation Formulas:

$$
\begin{aligned}
1 + 2 + \cdots + n &= \sum_{k=1}^{n} k = \frac{n(n+1)}{2} \\
1^2 + 2^2 + \cdots + n^2 &= \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \\
1^3 + 2^3 + \cdots + n^3 &= \sum_{k=1}^{n} k^3 = \left[\frac{n(n+1)}{2}\right]^2
\end{aligned}
\tag{5.8}
$$

Remark 5.12. To estimate an area, in Section 5.1, we have used a summation of the form

$$S_P = \sum_{k=1}^{n} f(c_k)\Delta x_k = f(c_1)\Delta x_1 + f(c_2)\Delta x_2 + \cdots + f(c_n)\Delta x_n, \tag{5.9}$$

where

$$P = \{x_0,\, x_1,\, \cdots,\, x_{n-1},\, x_n\} \quad \text{with} \quad \Delta x_k = x_k - x_{k-1},$$

and c_k chosen from the k-th subinterval $[x_{k-1}, x_k]$:

$$c_k \in [x_{k-1}, x_k].$$

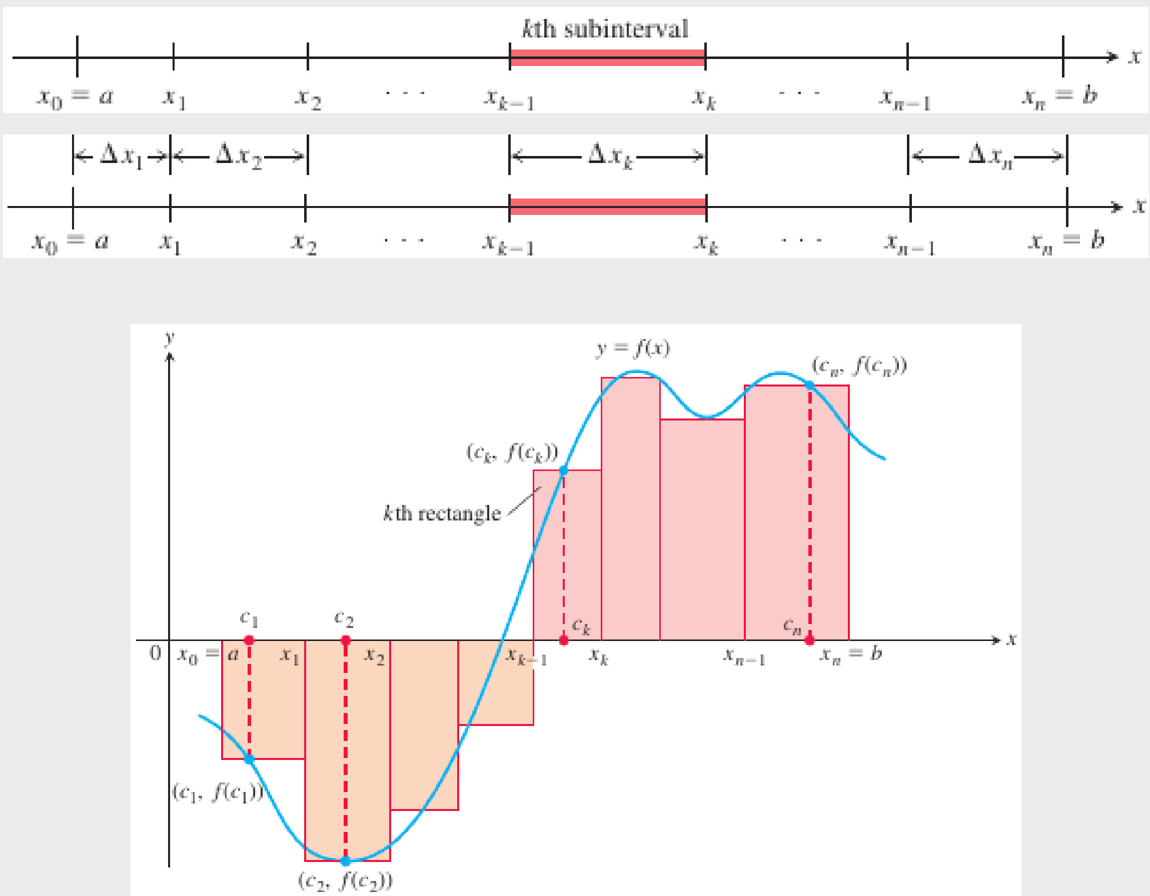


Figure 5.6: A Riemann sum, estimating an area.

Definition 5.13. The sum S_P is called a **Riemann sum** for f on the interval $[a, b]$ associated with a **partition** P.

For convenience, we will use **uniform partitions**, of which each subinterval has an equal-length

$$\Delta x_k = \Delta x = \frac{b-a}{n}.$$

Example 5.14. Find the **limiting value of lower sum approximations** to the area of the region R below the graph of $y = 1 - x^2$ and above the interval $[0, 1]$ on the x-axis using equal-width rectangles whose widths approach zero and whose number approaches infinity. (Recall Figure 5.3, page 218.)

Solution.

Example 5.15. For the function given below, find a formula for the Riemann sum obtained by dividing the interval $[0, 4]$ into n equal subintervals for each c_k (the lower sum and the upper sum). Then take a limit of this sum as $n \to \infty$ to calculate the area under the curve over $[0, 4]$.

$$f(x) = x^2 + 2$$

Solution.

Exercises 5.2

1. Write the sums without sigma notation. Then evaluate them.

   (a) $\displaystyle\sum_{k=1}^{3} \frac{k-1}{k}$ (b) $\displaystyle\sum_{k=1}^{4} \cos k\pi$ (c) $\displaystyle\sum_{k=1}^{3} (-1)^{k+1} \sin \frac{\pi}{k}$

   Ans: (a) $5/6$. (c) $\sqrt{3}/2 - 1$.

2. Find all that express $1 - 2 + 4 - 8 + 16 - 32$ in sigma notation.

   (a) $\displaystyle\sum_{k=1}^{6} (-2)^{k-1}$ (b) $\displaystyle\sum_{k=0}^{5} (-1)^{k+1} 2^k$ (c) $\displaystyle\sum_{k=-2}^{3} (-1)^k 2^{k+2}$

3. Express the sums in sigma notation. The form of your answer will depend on your choice for the starting index.

   (a) $1 + 4 + 9 + 16 + 25$ (b) $\dfrac{1}{2} + \dfrac{1}{4} + \dfrac{1}{8} + \dfrac{1}{16}$ (c) $-\dfrac{1}{5} + \dfrac{2}{5} - \dfrac{3}{5} + \dfrac{4}{5} - \dfrac{5}{5}$

   Ans: (b) $\displaystyle\sum_{k=1}^{4} \frac{1}{2^k}$.

4. Evaluate the sums.

   (a) $\displaystyle\sum_{k=1}^{10} k^3$ (b) $\displaystyle\sum_{k=1}^{6} (k^2 - 5)$

   (c) $\displaystyle\sum_{k=1}^{5} k(3k + 5)$ (d) $\displaystyle\sum_{k=6}^{10} k(k - 1)$

   Ans: (d) 290.

5. For the following functions, find a formula for the **Riemann sum** obtained by dividing the interval $[a, b]$ into n equal subintervals and using the right-hand endpoint for each c_k. Then take a limit of these sums as $n \to \infty$ to calculate the area under the curve over $[a, b]$.

   (a) $f(x) = x + x^2$, over the interval $[0, 1]$ (b) $f(x) = 1 - x^3$, over the interval $[0, 1]$

5.3. The Definite Integral

5.3.1. The Limit of Riemann Sums

Definition 5.16. Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral** of f over $[a, b]$ and that J is **the limit of the Riemann sums**:

$$J = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k \tag{5.10}$$

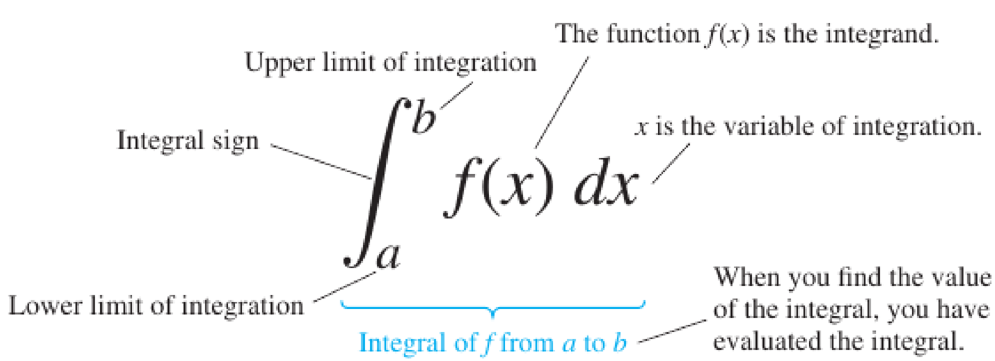
if the following condition is satisfied:

> Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for **every partition** $P = \{x_0, x_1, \cdots, x_n\}$ with $\|P\| < \delta$ and **any choice** of $c_k \in [x_{k-1}, x_k]$, we have
>
> $$\left| \sum_{k=1}^{n} f(c_k) \Delta x_k - J \right| < \varepsilon. \tag{5.11}$$

The Definite Integral as the Limit of Riemann Sums

If the definite integral exists, then instead of writing J we write

$$\int_a^b f(x)\, dx = \lim_{||P|| \to 0} \sum_{k=1}^{n} f(c_k) \Delta x_k = \lim_{n \to \infty} \sum_{k=1}^{n} f(c_k) \Delta x_k \tag{5.12}$$



Definition 5.17. When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x)dx$ and that f is **integrable** over $[a, b]$.

Theorem 5.18. Integrability of Continuous Functions: *If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.*

Example 5.19. Show that the function

$$f(x) = \begin{cases} 1, & \text{if } x \text{ is rational}, \\ 0, & \text{if } x \text{ is irrational}, \end{cases} \tag{5.13}$$

is ***not integrable*** over $[0, 1]$.

Proof. Let P be a partition of $[0, 1]$. Then in each subinterval $[x_{k-1}, x_k]$, there is at least a rational point, say c_k. Thus the **upper sum approximation** for this choice of c_k's is

$$U = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} (1)\Delta x_k = 1. \tag{5.14}$$

On the other hand, each subinterval $[x_{k-1}, x_k]$ include an irrational point, say c_k again. Thus the **lower sum approximation** for this choice of c_k's is

$$L = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} (0)\Delta x_k = 0. \tag{5.15}$$

Thus making different choices for the points c_k results in different limits for the corresponding Riemann sums. We conclude that the definite integral of f over the interval $[0, 1]$ does not exist, and that f is not integrable over $[0, 1]$. □

Note: For the integral, you may write

$$\int_a^b f(t)\,dt \quad \text{or} \quad \int_a^b f(u)\,du \quad \text{instead of} \quad \int_a^b f(x)\,dx.$$

No matter how we write the integral, it is still the same number, the limit of the Riemann sums as the norm of the partition approaches zero. Since it does not matter what letter we use, the variable of integration is called a **dummy variable**.

Example 5.20. Express the limit as a definite integral.

$$\lim_{||P||\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\Delta x_k,$$

where P is a partition of $[-7, 5]$.

Solution.

Recall: **The Definite Integral as the Limit of Riemann Sums** (5.12):

$$\int_a^b f(\mathbf{x})\,\mathbf{dx} = \lim_{||P||\to 0} \sum_{k=1}^{n} f(\mathbf{c_k})\mathbf{\Delta x_k} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\mathbf{c_k})\mathbf{\Delta x_k}$$

Equal-Width Subintervals

Remark 5.21. In the cases where the subintervals all have equal width $\Delta x = (b-a)/n$, the Riemann sums have the form

$$S_n = \sum_{k=1}^{n} f(c_k)\Delta x_k = \sum_{k=1}^{n} f(c_k)\Big(\frac{b-a}{n}\Big), \quad c_k \in [x_{k-1}, x_k]. \tag{5.16}$$

The Definite Integral with Equal-Width Subintervals

If we pick the point c_k to be the right endpoint of the k-th subinterval, then the formula for the definite integral becomes

$$\int_a^b f(\mathbf{x})\,\mathbf{dx} = \lim_{n\to\infty}\sum_{k=1}^{n} f(\mathbf{c_k})\mathbf{\Delta x} = \lim_{n\to\infty}\sum_{k=1}^{n} f(a+k\Delta x)\Delta x, \tag{5.17}$$

where $\Delta x = (b-a)/n$.

Example 5.22. Express the limit as a definite integral.

$$\lim_{n\to\infty}\sum_{k=1}^{n}\Big[\Big(3+\frac{2k}{n}\Big)^2 + 4\Big(3+\frac{2k}{n}\Big)\Big]\Big(\frac{2}{n}\Big)$$

Note that the expression is not unique!

Solution.

Ans: $\displaystyle\int_0^2 [(3+x)^2 + 4(3+x)]\,dx = \int_3^5 (x^2+4x)\,dx.$

5.3.2. Properties of Definite Integrals

Proposition 5.23. Suppose f and g are integrable over the interval $[a, b]$. Let $c \in [a, b]$ and k a constant. Then

1. *Order of Integration* : $\displaystyle\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx$
2. *Zero Width Interval* : $\displaystyle\int_a^a f(x)\,dx = 0$
3. *Constant Multiple* : $\displaystyle\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$
4. *Sum and Difference* : $\displaystyle\int_a^b [f(x) \pm g(x)]\,dx = \int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx$ (5.18)
5. *Additivity* : $\displaystyle\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$
6. *Max-Min Inequality* : $\displaystyle(\min f)\cdot(b-a) \le \int_a^b f(x)\,dx \le (\max f)\cdot(b-a)$
7. *Domination* : $\displaystyle\int_a^b f(x)\,dx \ge \int_a^b g(x)\,dx,$ when $f(x) \ge g(x)\ \ \forall x \in [a, b]$

Example 5.24. Suppose f and g are integrable and that

$$\int_1^9 f(x)\,dx = -1, \qquad \int_7^9 f(x)\,dx = 5, \qquad \int_7^9 g(x)\,dx = -4$$

Find

(a) $\displaystyle\int_7^9 [2f(x) - 3g(x)]\,dx$

(b) $\displaystyle\int_1^7 f(x)\,dx$

(c) $\displaystyle\int_9^7 [g(x) - f(x)]\,dx$

Area under the Graph of a Nonnegative Function

Definition 5.25. If $y = f(x)$ is nonnegative and integrable over a closed interval $[a, b]$, then the **area under the curve $y = f(x)$ over $[a, b]$** is the integral of f from a to b,

$$A = \int_a^b f(x)\,dx. \tag{5.19}$$

Example 5.26. Evaluate the integral by interpreting it in terms of areas.

(a) $\displaystyle\int_0^b x\,dx$

(b) $\displaystyle\int_a^b c\,dx$

(c) $\displaystyle\int_{-1}^3 |2x|\,dx$

(d) $\displaystyle\int_{-3}^3 5 + \sqrt{9 - x^2}\,dx$

Example 5.27. Compute $\displaystyle\int_0^b x^2\,dx$, by finding the limit of Riemann sums.

Solution. For n equal subintervals, the right Riemann sum reads

$$\sum_{k=1}^{n}\left(k\frac{b}{n}\right)^2\cdot\frac{b}{n}$$

Average Value of a Continuous Function: Revisited

In Section 5.1.3, we informally introduced the average value of a **nonnegative continuous** function over an interval $[a, b]$.

Definition 5.28. If f is integrable on $[a, b]$, then its **average value** on $[a, b]$, which is also called its **mean**, is

$$\text{av}(f) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{5.20}$$

Example 5.29. Find the average value of $\boldsymbol{f(x) = -\sqrt{4-x^2}}$ on $[-2, 2]$.

Solution. (Note that $f \le 0$.)

Exercises 5.3

1. Express the limits as definite integrals.

   (a) $\displaystyle\lim_{||P||\to 0}\sum_{k=1}^{n} 2c_k^3\Delta x_k$, where P is a partition of $[-1,0]$.

   (b) $\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\left[3\left(\frac{2k}{n}\right)^2+\left(4+\frac{2k}{n}\right)\right]\left(\frac{2}{n}\right)$

2. Suppose that f is integrable and that $\int_0^3 f(x)\,dx = 3$ and $\int_0^4 f(x)\,dx = 7$. Find

   (a) $\displaystyle\int_3^4 f(x)\,dx$ (b) $\displaystyle\int_4^3 f(t)\,dt$

3. Use the method of Example 5.27 to evaluate the definite integrals.

   (a) $\displaystyle\int_0^2 (2x+1)\,dx$ (b) $\displaystyle\int_a^b x^2\,dx,\quad a<b$

   (c) $\displaystyle\int_0^b x^3\,dx$ (d) $\displaystyle\int_{-1}^1 x^3\,dx$

   Ans: (b) $\frac{b^3-a^3}{3}$. (c) $b^4/4$.

4. Challenge Let $a<b$.

   (a) What values of a and b maximize the value of $\displaystyle\int_a^b (x-x^2)\,dx$?

   Hint: Where is the integrand positive?

   (b) What values of a and b minimize the value of $\displaystyle\int_a^b (x^4-2x^2)\,dx$?

5. CAS Let $f(x)=\sin x$ and $g(x)=\sin^2 x$. For each of the functions, use a CAS to perform the following steps:

   (a) Plot the function over the interval $[0,\pi]$.
   (b) Partition the interval into n = 10, 100, and 1000 subintervals of equal length, and evaluate the function at the midpoint of each subinterval.
   (c) Compute the average value of the function values generated in part (b).
   (d) Use the result in part (c) to estimate $\displaystyle\int_0^\pi f(x)\,dx$ and $\displaystyle\int_0^\pi g(x)\,dx$.

   Repeat the above to estimate $\displaystyle\int_0^1 x\ln x\,dx$.

   You may use the following.

average_value.m

```
%f = @(x) (sin(x));    a=0; b=pi;
 f = @(x) (sin(x)).^2; a=0; b=pi;
%f = @(x) (x.*log(x)); a=0; b=1;

%% Add plot

for n =[10 100 1000]
    P = linspace(a,b,n+1);
    M = (P(1:end-1)+P(2:end))/2; % mid points
    fsum = sum(f(M));

    %% Here, add the required
end
```

5.4. The Fundamental Theorem of Calculus

5.4.1. Fundamental Theorem of Calculus, Part 1

Theorem 5.30. **(FTC1)** *If f is continuous on $[a, b]$, then*

$$F(x) = \int_a^x f(t)\,dt \tag{5.21}$$

is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{5.22}$$

That is, F is an ***antiderivative*** *of f.*

Example 5.31. Use FTC1 to find the derivative of the function.

(a) $g(x) = \displaystyle\int_0^x \sin(1+t^2)\,dt$

(b) $h(x) = \displaystyle\int_2^{x^3} e^{-t}\,dt$

(c) $y = \displaystyle\int_x^5 3t\sin t\,dt$

(d) $y = \displaystyle\int_{\sin x}^1 \frac{1}{\sqrt{1-t^2}}\,dt$

Note:

$$\begin{aligned}\frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt &= \frac{d}{dx}\left[F(t)\Big|_{h(x)}^{g(x)}\right] = \frac{d}{dx}\left[F(g(x)) - F(h(x))\right] \\ &= f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x).\end{aligned} \tag{5.23}$$

5.4.2. Fundamental Theorem of Calculus, Part 2

Theorem 5.32. **(FTC2)** *If f is continuous over $[a,b]$ and F is any antiderivative of f on $[a,b]$, then:*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{5.24}$$

Remark 5.33. Theorem 5.32 is also called the **Evaluation Theorem**. The theorem is important because it says that **to calculate the definite integral of f** over an interval $[a,b]$, we need do only two things:

1. Find an antiderivative F of f, and
2. Calculate the number $F(b) - F(a)$.

Note: Theorem 5.32 shows a connection between **antiderivatives** and **definite integrals**. This is the reason that in Definition 4.60, p. 211, the integral sign $\int$ is used to denote *the collection of all antiderivatives*.

Example 5.34. Evaluate the integral.

(a) $\displaystyle\int_1^4 \left(3x^2 - \frac{x^3}{4}\right) dx$ (b) $\displaystyle\int_1^8 \frac{x^{2/3}+1}{x^{1/3}}\,dx$ (c) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1+\cos 2t}{2}\,dt$

Solution.

Ans: (b) $63/4$

Example 5.35. Evaluate the integral.

(a) $\displaystyle\int_1^3 \left(\frac{1}{x} + e^{-3x}\right) dx$ (b) $\displaystyle\int_0^{\frac{1}{2\sqrt{3}}} \frac{1}{1+4x^2}\, dx$

Solution.

Ans: (b) $\pi/12$

Mean Value Theorem for Definite Integrals

Theorem 5.36. The Mean Value Theorem for Definite Integrals. *If f is continuous on $[a,b]$, then at some point $c \in [a,b]$,*

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\, dx. \tag{5.25}$$

Proof. It follows from the Max-Min Inequality in (5.18) that

$$\min f \le \frac{1}{b-a}\int_a^b f(x)\, dx \le \max f. \tag{5.26}$$

Since f is continuous, the **Intermediate Value Theorem** for Continuous Functions (Theorem 2.52) says that f must assume every value between $\min f$ and $\max f$. It must therefore assume the value $\frac{1}{b-a}\int_a^b f(x)\, dx$ at some point $c \in [a,b]$. □

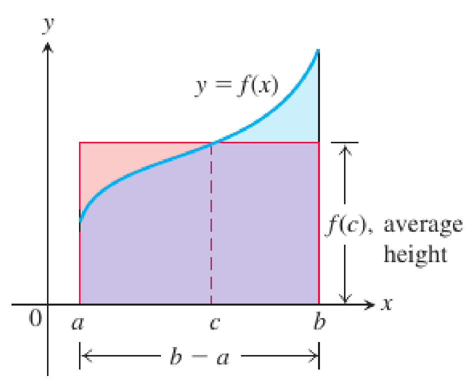


Figure 5.7: The value $f(c)$ in the Mean Value Theorem is the **average height** (*mean*) of f.

$\boxed{\textbf{Example}}$ **5.37.** Find the average value of $f(x) = x(1-x)$ over $[1,3]$.
Solution.

$\boxed{\textbf{Example}}$ **5.38.** Show that if f is continuous on $[a,b]$, $a \neq b$, and if

$$\int_a^b f(x)\,dx = 0,$$

then f has a zero in $[a,b]$.
Solution.

$\boxed{\textbf{Theorem}}$ **5.39. The Net Change Theorem**
The net change in a differentiable function $F(x)$ over an interval $[a,b]$ is the integral of its rate of change:

$$F(b) - F(a) = \int_a^b F'(x)\,dx. \tag{5.27}$$

$\boxed{\textbf{Example}}$ **5.40.** If $c(x)$ is the cost of producing x units of a certain commodity, then $c'(x)$ is the **marginal cost**. From the Net Change Theorem,

$$c(x_2) \;=\; c(x_1) + \int_{x_1}^{x_2} c'(x)\,dx,$$

which says that the final cost $c(x_2)$ is the same as the initial cost $c(x_1)$ plus the net change for the production increase from x_1 units to x_2 units.

5.4.3. The Total Area, between the Graph and the x-axis

Position, Velocity, Displacement, and Distance

The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.

(a) The **velocity** of an object at time t is $v(t) = s'(t)$.

(b) The Net Change Theorem says that

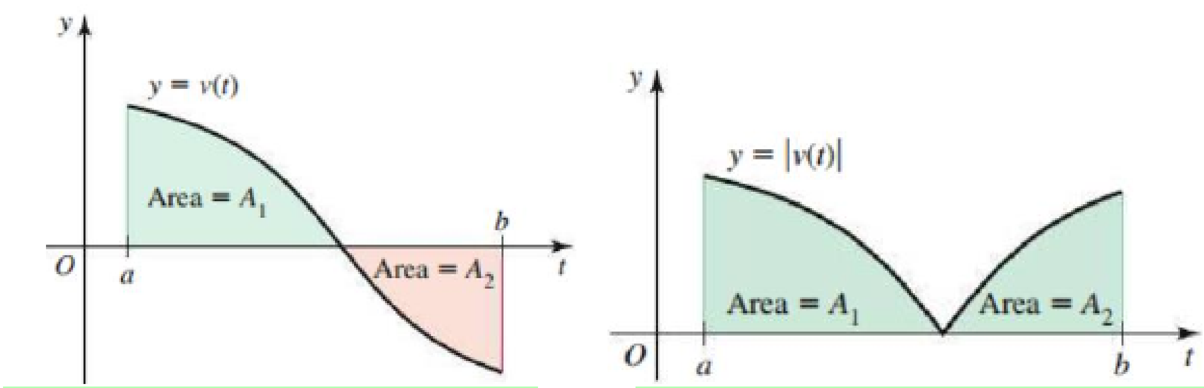
$$s(b) - s(a) = \int_a^b v(t)\,dt, \tag{5.28}$$

so the integral of velocity is the **displacement** of the object over the time interval $[a, b]$.

(c) The **distance traveled** over the time interval $[a, b]$ is

$$\text{Distance traveled} = \int_a^b |v(t)|\,dt, \tag{5.29}$$

where $|v(t)|$ is the **speed** of the object at time t.



Displacement $= A_1 - A_2 = \int_a^b v(t)\,dt$

Distance traveled $= A_1 + A_2 = \int_a^b |v(t)|\,dt$

Figure 5.8: Interpretations of the displacement and the distance traveled.

Example 5.41. The velocity function (in meters per second) is given for a particle moving along a line, as

$$v(t) = t^2 - 2t - 3, \qquad t \in [2, 4].$$

(a) Find the displacement. (b) Find the distance traveled.

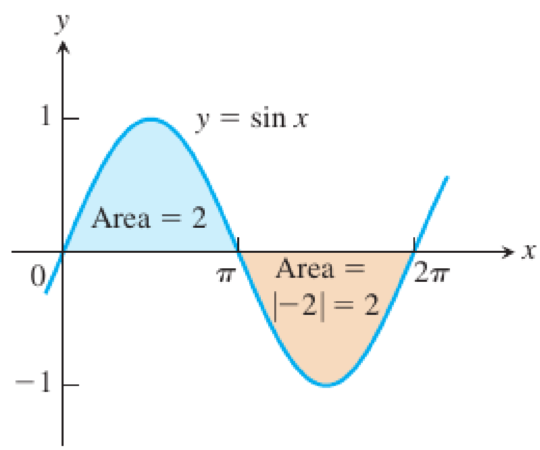
Solution.

Total Area

Example 5.42. The figure shows the graph of the function $f(x) = \sin x$ between $x = 0$ and $x = 2\pi$. Compute

(a) the definite integral of $f(x)$ over $[0, 2\pi]$.

(b) the area between $y = \sin x$ and the x-axis over $[0, 2\pi]$.



Solution.

Strategy 5.43. To find the area between the graph of $y = f(x)$ and the x-axis over the interval $[a, b]$:

(a) Subdivide $[a, b]$ at the zeros of f.

(b) Integrate f over each subinterval.

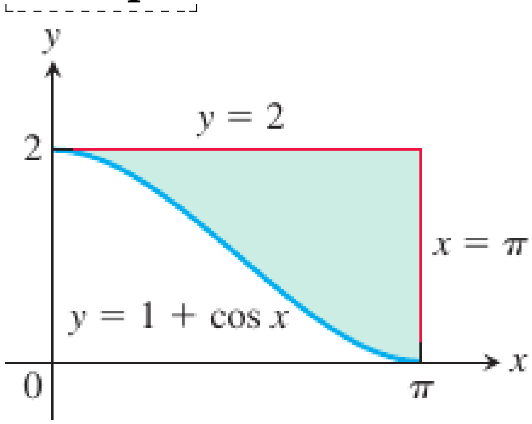
(c) Add the absolute values of the integrals.

Example 5.44. Find the area of the region between the x-axis and the graph of $f(x) = x^3 - x^2 - 2x$, $-1 \le x \le 2$.

Solution.

Ans: Total enclosed area $= \dfrac{37}{12}$.

Example 5.45. Find the area of the shaded region.



Exercises 5.4

1. Evaluate the integrals.

(a) $\displaystyle\int_{-1}^{1}(x^{999}-2x+1)\,dx$ (b) $\displaystyle\int_{0}^{\pi/4}\tan^2 x\,dx$

(c) $\displaystyle\int_{-4}^{4}|x|\,dx$ (d) $\displaystyle\int_{0}^{\pi/2}\sin^2 x\,dx$

(e) $\displaystyle\int_{0}^{\pi/3}(\cos x+\sec x)^2\,dx$ (f) $\displaystyle\int_{1}^{\sqrt{2}}\frac{s^2+\sqrt{s}}{s^2}\,ds$

Hint: (b) $1+\tan^2 x=\sec^2 x$. (d-e) $\sin^2 x=\frac{1-\cos 2x}{2}$ and $\cos^2 x=\frac{1+\cos 2x}{2}$.

Ans: (b) $1-\dfrac{\pi}{4}$. (f) $\sqrt{2}-\sqrt[4]{8}+1$.

2. Find the derivatives in two different ways and compare them:

(i) by evaluating the integral and differentiating the result.
(ii) by differentiating the integral directly (as in FTC1).

(a) $\displaystyle\frac{d}{dx}\int_{1}^{\sin x}3t^2\,dt$ (b) $\displaystyle\frac{d}{dt}\int_{0}^{t^4}\sqrt{u}\,du$

3. Find dy/dx.

(a) $\displaystyle y=\int_{\tan x}^{0}\frac{dt}{1+t^2}$ (b) $\displaystyle y=\left(\int_{0}^{x}(t^3+1)^{10}\,dt\right)^3$

Ans: (a) -1

4. Find the total area between the graph of $y=f(x)$ and the x-axis.

(a) $y=x^3-3x^2+2x,\quad x\in[0,2]$. (b) $y=3x^2-3,\quad x\in[-2,2]$.

Ans: (a) $1/2$.

5. **Challenge** Find

(a) $\displaystyle\lim_{x\to 1}\frac{1}{x-1}\int_{1}^{x}\sqrt{t}\,dt$. (b) $\displaystyle\lim_{x\to a}\frac{1}{x-a}\int_{a}^{x}f(t)\,dt$.

Hint: (b) Use FTC2 to generalize what you would learn from (a).

Ans: (a) 1.

5.5. Indefinite Integrals and the Substitution Method

Summary 5.46.

- The **Fundamental Theorem of Calculus** says that a **definite integral** $\int_a^b f(x)\,dx$ of a continuous function f can be computed directly, if we can find an **antiderivative** of the function.
- We defined the **indefinite integral** of the function f with respect to x as the set of all antiderivatives of f, symbolized by $\int f(x)\,dx$.
- Since any two antiderivatives of f differ by a constant, for any antiderivative F of f,
$$\int f(x)\,dx = F(x) + C, \tag{5.30}$$
where C is an arbitrary constant.
- **The connection between antiderivatives and the definite integral** (stated in the FTC) now explains the use of notation $\int$:
$$\begin{aligned}\int_a^b f(x)\,dx &= F(b) - F(a) = [F(b) + C] - [F(a) + C] \\ &= [F(x) + C]_a^b = \Big[\int f(x)\,dx\Big]_a^b\end{aligned} \tag{5.31}$$

Example 5.47. Find $\displaystyle\int_1^\pi \left(3x^2 + \cos x + \frac{1}{x}\right) dx.$

The Substitution Rule

Theorem 5.48. *If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then*

$$\int f(g(x)) \cdot \boldsymbol{g'(x)\, dx} = \int f(u)\, \boldsymbol{du}. \tag{5.32}$$

Remark 5.49. As with differentials, when computing integrals we have

$$du = \frac{du}{dx} dx. \tag{5.33}$$

which leads to the **substitution method** for computing integrals with

$$\boldsymbol{g'(x)dx \;=\; du}. \tag{5.34}$$

The idea here is to replace a complicated integral by a much simpler integral.

Strategy 5.50. **Substitution Method to evaluate $\int f(g(x))\cdot g'(x)\, dx$**

(a) Substitute $u = g(x)$ and $du = \left(\frac{du}{dx}\right)dx = g'(x)dx$ to obtain $\int f(u)\, du$.

(b) Integrate it with respect to u.

(c) Replace u by $g(x)$.

Example 5.51. Evaluate $\displaystyle\int \sqrt{2x+1}\, dx$, using a **smart guess** (**method of undetermined coefficients**) and the **substitution method**.

Solution. smart guess substitution method

Example 5.52. Evaluate the integrals.

(a) $\displaystyle\int 2x(x^2+5)^{-4}\,dx$

(b) $\displaystyle\int \frac{9x^2}{\sqrt{1-x^3}}\,dx$

(c) $\displaystyle\int \frac{\cos(1/x)}{x^2}\,dx$

(d) $\displaystyle\int \tan^2 x \sec^2 x\,dx$

Example 5.53. Evaluate the integrals.

(a) $\displaystyle\int (\cos x)e^{\sin x}\,dx$

(b) $\displaystyle\int \frac{\arcsin^2 x}{\sqrt{1-x^2}}\,dx$

(c) $\displaystyle\int x^3\sqrt{1+x^2}\,dx$

(d) $\displaystyle\int \frac{x}{(2x-1)^2}\,dx$

Formula 5.54. Integrals of Some Trigonometric Functions:
Sometimes we can use trigonometric identities to transform integrals we do not know how to evaluate using the Substitution Rule.

(a) $\displaystyle\int \sin^2 x\,dx = \int \frac{1-\cos 2x}{2}\,dx = \frac{x}{2} - \frac{\sin 2x}{4} + C$

(b) $\displaystyle\int \cos^2 x\,dx = \int \frac{1+\cos 2x}{2}\,dx = \frac{x}{2} + \frac{\sin 2x}{4} + C$

(c) $\displaystyle\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

(d) $\displaystyle\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx \overset{u=\cos x}{=\!=\!=} \int \frac{-du}{u} = -\ln|u| + C = \ln|\sec x| + C$

(e)
$$\begin{aligned} \int \sec x\,dx &= \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\,dx = \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\,dx \\ &= \ln|\sec x + \tan x| + C \end{aligned}$$

Similarly, we can derive the following

(f) $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$

(g) $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$

Exercises 5.5

1. Evaluate the integrals.

(a) $\displaystyle\int \frac{4x^3}{(x^4+1)^2}\,dx$ (b) $\displaystyle\int x\sin(2x^2)\,dx$

(c) $\displaystyle\int \sqrt{\sin x}\cos^3 x\,dx$ (d) $\displaystyle\int \frac{dx}{x\ln x}$

Hint: (c) You may use $u = \sin x$ and $\cos^2 x = 1 - \sin^2 x$.

Ans: (d) $\ln|\ln x| + C$.

2. Evaluate the integrals, using two different substitutions.

(a) $\displaystyle\int \csc^2 2\theta \cot 2\theta\,d\theta \qquad u = \cot 2\theta$ and $u = \csc 2\theta$

(b) $\displaystyle\int \frac{dx}{\sqrt{5x+3}}\,dx \qquad u = 5x+3$ and $u = \sqrt{5x+3}$

Note: If you do not know what substitution to make, try to reduce the integral **step-by-step**, using a trial substitution to simplify the integral a bit and then another to simplify it some more. You will see what we mean if you try the sequences of substitutions.

3. Evaluate the integral $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$, trying different sequences of substitutions.

(a) $u = \tan x$, followed by $v = u^3$, then by $w = 2 + v$.
(b) $u = \tan^3 x$, followed by $v = 2 + u$.
(c) $u = 2 + \tan^3 x$.

4. Solve the **initial value problems**.

(a) $\displaystyle\frac{dx}{dt} = 12t(3t^2-1)^3, \quad x(1) = 3$

(b) $\displaystyle\frac{d^2y}{dx^2} = 2\sec^2 x\tan x, \quad y'(0) = 4, \quad y(0) = -3.$

Ans: (a) $x(t) = \frac{1}{2}(3t^2-1)^4 - 5$. (b) $y = \tan x + 3x - 3$.

5.6. Definite Integral Substitutions and the Area between Curves

5.6.1. Substitution in Definite Integrals

Theorem 5.55. *If $g'(x)$ is continuous on the interval $[a,b]$ and $f(x)$ is continuous on the range of $g(x)=u$, then*

$$\int_a^b f(g(x))\cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du. \tag{5.35}$$

Proof. Let F be any antiderivative of f. Then

$$\begin{aligned}\int_a^b f(g(x))\cdot g'(x)\,dx &= F(g(x))\Big|_{x=a}^{x=b} = F(g(b)) - F(g(a)) \\ &= F(u)\Big|_{u=g(a)}^{u=g(b)} = \int_{g(a)}^{g(b)} f(u)\,du,\end{aligned}$$

which completes the proof. □

Note:

- The **new limits $g(a)$ and $g(b)$** are the values of $u=g(x)$ that correspond to $x=a$ and $x=b$.
- For an antiderivative of $f(u)$, there is no need to replace u by $g(x)$.

Example 5.56. Evaluate definite integrals.

(a) $\displaystyle\int_0^1 (5x^4+2)\sqrt{x^5+2x}\,dx.$

(b) $\displaystyle\int_0^\pi 3\cos^2 x\sin x\,dx.$

Remark 5.57. Substitution in Definite Integrals. Rewrite (5.35):

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

- Due to **the change in the interval length**, it requires a certain scaling. The reciprocal of $g'(x)$ can be viewed as a **scaling factor**.
- This subject will be detailed in terms of **change of variables** when you study multi-variable calculus (Section 15.9).

Example 5.58. Let $f(x) = 2x$. For the definite integral of f over $[0, 1]$, we use the substitution $u = 2x$. Then

$$\int_0^1 (2x)\, dx \;\overset{u=2x}{\underset{du=2dx}{=\!=\!=}}\; \int_0^2 (u)\,\frac{1}{2}du \;=\; \int_0^2 \frac{u}{2}\, du \;=\; \int_0^2 \frac{x}{2}\, dx.$$

Evaluate the integral by interpreting it in terms of areas.

Solution.

Recall: **Definition 1.14, p.10.** A function $y = f(x)$ is an

$$\begin{array}{lll} \textbf{even} \text{ function of } x & \text{if} & f(-x) = f(x) \\ \textbf{odd} \text{ function of } x & \text{if} & f(-x) = -f(x) \end{array} \tag{5.36}$$

for every x in the function's domain.

Theorem 5.59. *Let f be continuous on the symmetric interval $[-a, a]$.*

*(a) If f is **even**, then* $\displaystyle\int_{-a}^{a} f(x)\,dx = 2\int_{0}^{a} f(x)\,dx$.

*(b) If f is **odd**, then* $\displaystyle\int_{-a}^{a} f(x)\,dx = 0$.

Example 5.60. Evaluate integrals.

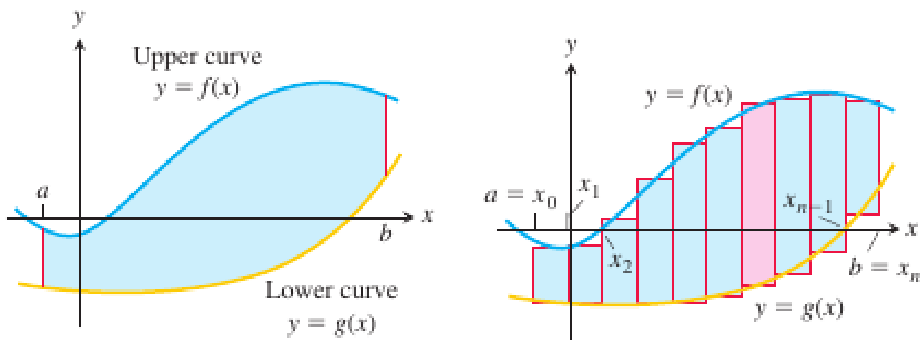
(a) $\displaystyle\int_{-2}^{2} (x^4 - 4x^3 + x^2 + 12x + 1)\,dx$

(b) $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{2\cos\theta}{1+\sin^2\theta}\,d\theta$

(c) $\displaystyle\int_{-\pi/3}^{\pi/3} x^4 \sin x\,dx$

5.6.2. Areas between curves

- Suppose we want to find the area of a region that is bounded above by the curve $y = f(x)$, below by the curve $y = g(x)$, and on the left and right by the lines $x = a$ and $x = b$.
- To see what the area should be, we first approximate the region with n vertical rectangles based on a partition $P = \{x_0, x_1, \cdots, x_n\}$ on $[a, b]$.



- The area of the k-th rectangle reads

$$\Delta A_k = \text{height} \times \text{width} = [f(c_k) - g(c_k)]\Delta x_k. \tag{5.37}$$

- We then approximate the area of the region by adding the areas of the n rectangles:

$$A \approx \sum_{k=1}^{n} \Delta A_k = \sum_{k=1}^{n} [f(c_k) - g(c_k)]\Delta x_k. \tag{5.38}$$

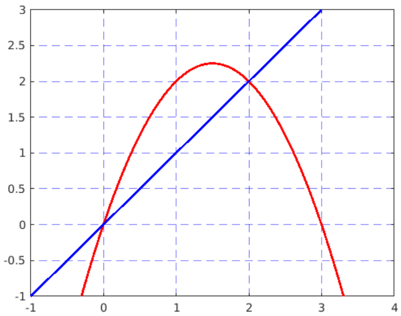
- As $||P|| \to 0$, the sums on the right approach the limit $\int_a^b [f(x) - g(x)]dx$, because f and g are continuous.

Definition 5.61. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from over $[a, b]$ is the integral of $(f - g)$ from a to b:

$$A = \lim_{||P|| \to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)]\Delta x_k = \int_a^b [f(x) - g(x)]dx. \tag{5.39}$$

Example 5.62. Find the area of the region enclosed by the parabola $y = 3x - x^2$ and the line $y = x$.

Solution.



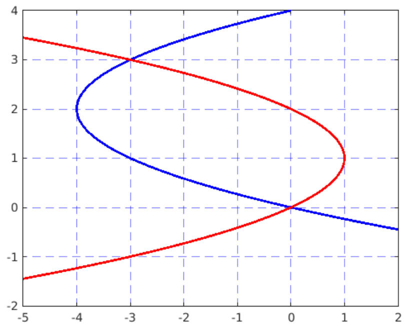
Example 5.63. Find the *total area* of the region between the curves $y = 4 - x^2$ and $y = -x + 2$, between $x = -2$ and $x = 3$.

Solution.

Integration with Respect to y

Example 5.64. Find the area of the region enclosed by the curves $x = y^2 - 4y$ and $x = 2y - y^2$.

Solution.



Example 5.65. Find the area of the region in the first quadrant that is bounded above by $y = \sqrt{x}$ and below by the x-axis and the line $y = x - 2$.

Solution.

Ans: $\frac{10}{3}$.

Exercises 5.6

1. Use the Substitution Formula in Theorem 5.55 to evaluate the integrals.

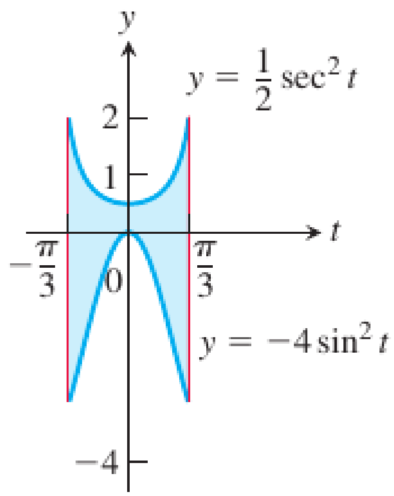
   (a) $\displaystyle\int_0^{\pi/4} \tan x \sec^2 x\, dx$ (b) $\displaystyle\int_1^9 t\sqrt{4+5t}\, dt$

   (c) $\displaystyle\int_2^4 \frac{dx}{x(\ln x)^2}$ (d) $\displaystyle\int_0^{\ln\sqrt{3}} \frac{e^x dx}{1+e^{2x}}$

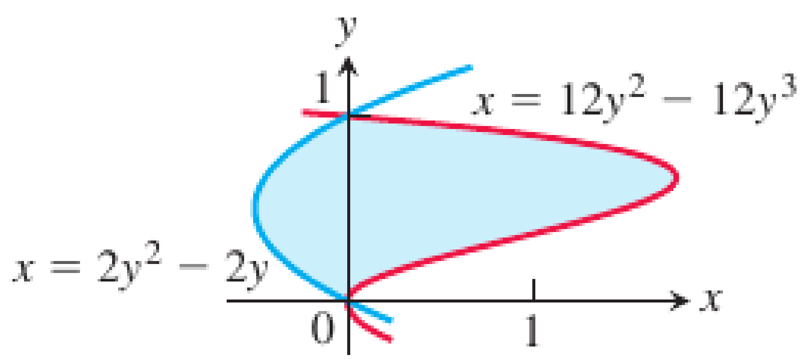
   Ans: (a) $1/2$. (b) $86744/375 = 231.3173333$. (c) $1/\ln 4$. (d) $\pi/12$.

2. Find the total areas of the shaded regions.

   (a)



   (b)



   Ans: (b) $4/3$.

3. Find the areas of the regions enclosed by the lines and curves.

   (a) $y = \sec^2 x$, $y = \tan^2 x$, $x = -\pi/4$, and $x = \pi/4$.
   (b) $x + y^2 = 0$ and $x + 3y^2 = 2$

   Ans: (a) $\pi/2$. (b) $8/3$.

4. **Challenge**

   (a) Show that if f is continuous, then $\displaystyle\int_0^1 f(x)\, dx = \int_0^1 f(1-x)\, dx$.

   (b) By using a substitution, prove that for all positive numbers a and x,

$$\int_a^{ax} \frac{1}{t}\, dt = \int_1^x \frac{1}{t}\, dt$$

CHAPTER **6**

Applications of Definite Integrals

In this chapter, ...

Contents of Chapter 6

6.1. Volume Using Cross-Sections

Proposition 6.1. Suppose that we want to find the volume of a solid S like the one pictured in Figure 6.1.

- At each point $x \in [a, b]$, we form a **cross-section** $S(x)$ by intersecting S with a plane perpendicular to the x-axis through the point x, which gives a planar region whose area is $A(x)$.

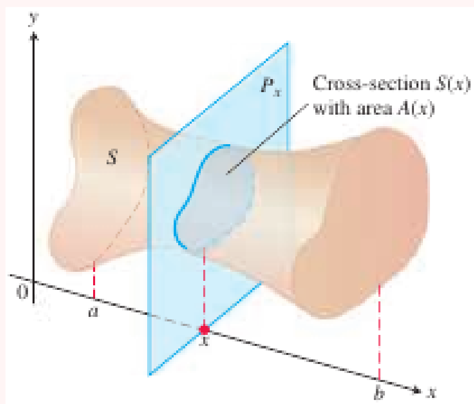


Figure 6.1: A cross-section $S(x)$ of the solid S.

- We will show that if A is a continuous function of x, then the volume of the solid S is the definite integral of $A(x)$. That is,

$$V = \int_a^b A(x)\,dx. \tag{6.1}$$

- This method is known as the **method of slicing** for computing volumes.

6.1.1. Slicing by Parallel Planes

- We partition $[a, b]$ into subintervals of width (length) Δx_k.
- Slice the solid, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$.
- These planes slice S into "**thin slabs**", as shown in Figure 6.2.

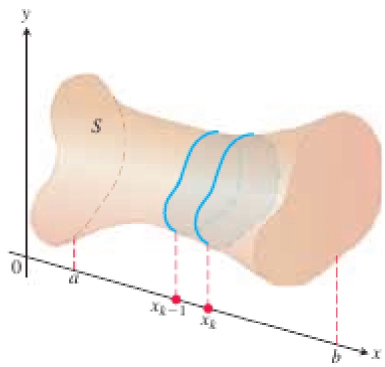


Figure 6.2: A thin slab in the solid S.

- We approximate **the slab** between the plane at x_{k-1} and the plane at x_k by a **cylindrical solid** with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.
- The volume V_k of this cylindrical solid is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{-th slab} \approx V_k = A(x_k)\Delta x_k. \tag{6.2}$$

- The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes: with $c_k = x_k$,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\Delta x_k, \tag{6.3}$$

  which is a **Riemann sum** for the function $A(x)$ on $[a, b]$.

Definition 6.2. The **volume** of a solid of integrable cross-sectional area $A(x)$ for $x \in [a, b]$ is the integral of A over $[a, b]$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\Delta x_k = \int_a^b A(x)\,dx. \tag{6.4}$$

Strategy 6.3. Calculating the Volume of a Solid

(a) **Sketch the solid** to understand.

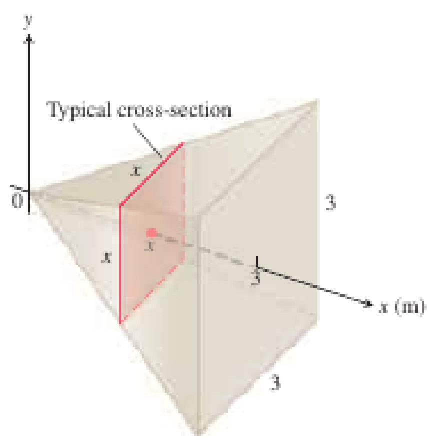
(b) Find **the limits of integration,** $[a, b]$**.**

(c) For each $x \in [a, b]$, find a **formula for** $A(x)$, the area of a typical cross-section.

(d) **Integrate** $A(x)$ to find the volume.

Example 6.4. A pyramid 3 m high has a square base that is 3 m on a side. Find the volume of the pyramid.

Solution.



(a) *Sketch*.

(b) *The limits of integration*: $x \in [0, 3]$.

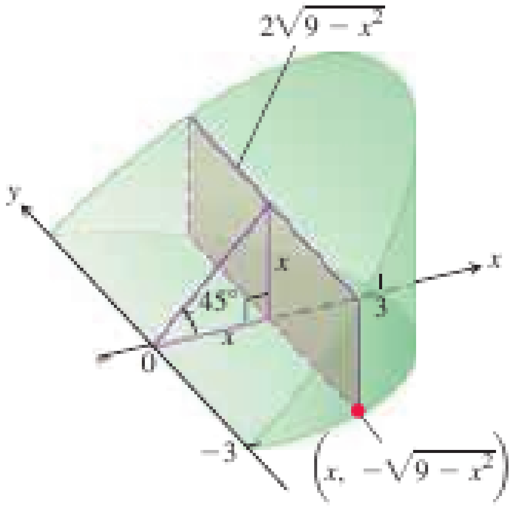
(c) *Formula for* $A(x)$. $A(x) = x^2$.

(d) *Integrate to find the volume*:

$$\int_0^3 A(x)\,dx = \int_0^3 x^2\,dx = \frac{x^3}{3}\Big]_0^3 = 9\,\mathrm{m}^3.$$

Example 6.5. A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

Solution. Note that $A(x) = x \cdot 2\sqrt{9-x^2}$.

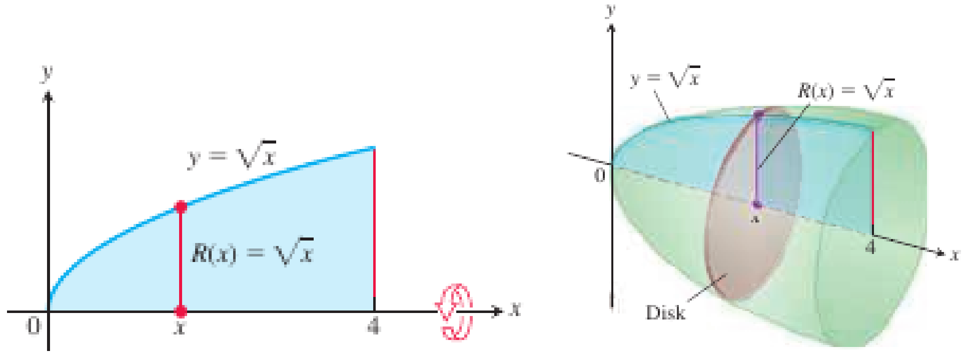


Ans: 18.

6.1.2. Solids of Revolution: the Disk Method

A **solid of revolution** is the solid generated by rotating (or revolving) a planar region about an axis in its plane.

Example 6.6. The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.



Ans: 8π.

Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx, \tag{6.5}$$

where $R(x)$ is the radius.

Note: This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

Example 6.7. Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

$$y = \frac{1}{2}x + 1, \quad y = 0, \qquad x = 0, \quad x = 5.$$

Solution.

Example 6.8. Find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis.

$$xy = 4, \quad x = 0, \quad y = 1, \quad y = 4.$$

Solution.

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx, \tag{6.6}$$

where $R(x)$ is the outer radius and $r(x)$ is the inner radius.

Example 6.9. Find the volume of the solid obtained by rotating the region bounded by the given curves about the x-axis.

$$y = \frac{1}{2}x + 1, \quad y = \frac{\sqrt{x}}{2}, \qquad x = 0, \quad x = 5.$$

Solution.

Example 6.10. The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution.

Ans: $\frac{8}{3}\pi$.

Exercises 6.1

In Problems 1–2, find the volumes of the solids.

1. The base of a solid is the region bounded by the graphs of $y = 3x$, $y = 6$, and $x = 0$. The cross-sections perpendicular to the x-axis are rectangles of height 10.

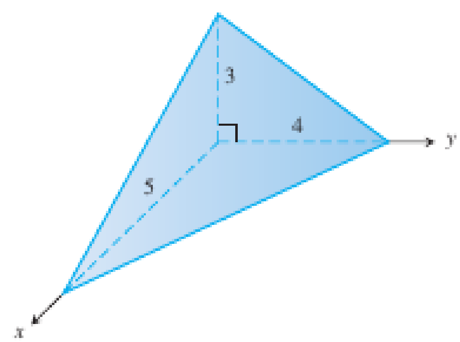
   Ans: 60

2. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

   Ans: 8π

3. Find the volume of the given right tetrahedron. ***Hint***: Consider slices perpendicular to one of the labeled edges.

   Ans: 10.



4. **Volumes by the Disk Method**: Find the volumes of the solids generated by revolving the regions bounded by the lines and curves about the x-axis.

   (a) $y = \sqrt{9 - x^2}, \;\; y = 0$
   (b) $y = e^{-x}, \;\; y = 0, \;\; x = 0, \;\; x = 1$

   Ans: (b) $\dfrac{\pi}{2}(1 - e^{-2})$.

5. **Volumes by the Washer Method**: Find the volumes of the solids generated by revolving the regions bounded by the lines and curves.

   (a) $y = \sec x, \;\; y = \sqrt{2}, \;\; -\pi/4 \le x \le \pi/4;$ revolving about the x-axis.
   (b) The triangle with vertices $(0, 1)$, $(1, 0)$, and $(1, 1)$; revolving about the y-axis.

   Ans: (a) $\pi(\pi - 2)$.

6. Challenge **The Volume of a Torus**: The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b(b > a)$ to generate a solid shaped like a doughnut and called a **torus**. Find its volume.

   Hint: (1) Consider slices by planes perpendicular to the y-axis, with which $-a \le y \le a$. (2) You may use $\displaystyle\int_{-a}^{a} \sqrt{a^2 - y^2}\, dy = \pi a^2/2$, which can be obtained through the observation that the integral is the area of a semicircle of radius a.

   Ans: $V = 2a^2 b\pi^2$.

6.2. Volumes Using Cylindrical Shells

Note: Some volumes cannot be determined easily using the previous methods (disk or washer). Here we will explore another method, called the **Method of Cylindrical Shells** or the **Shell Method**.

The Shell Method

- Consider the region bounded by the graph of a function $y = f(x)$ and the x-axis over the closed interval $[a, b]$; see Figure 6.3(a).
- We assume $L \le a$. We generate a solid S by rotating the region about the **vertical line $x = L$**.

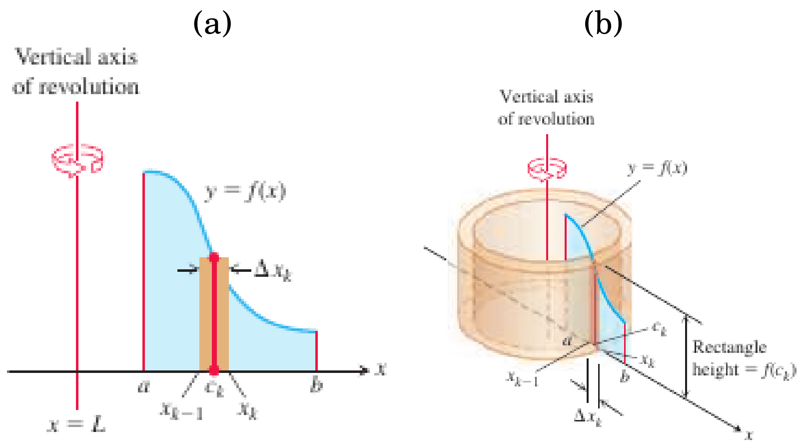


Figure 6.3: A region is resolved about the vertical line $y = L$.

- **Partitioning**. Let P be a partition of the interval $[a, b]$ by the points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

- **Approximation**. **We approximate the region in Figure 6.3(a) with the collection of rectangles based on this partition.**
- **Rectangles**. As usual, choose a point $c_k \in [x_{k-1}, x_k]$, e.g., the midpoint of the subinterval. **A typical approximating rectangle** has

$$\text{height} = f(c_k) \ \text{ and } \ \text{width} = \Delta x_k = x_k - x_{k-1}.$$

- **Rotation: Cylindrical shells**. If **such a rectangle** is rotated about the vertical line $y = L$, then **a shell** is swept out, as in Figure 6.3(b).

- **The volume of the shell** (swept out by the rectangle):

$$\begin{aligned}
\Delta V_k &= \pi[(x_k - L)^2 - (x_{k-1} - L)^2] \cdot \text{(shell height)} \\
&= \pi[(x_k + x_{k-1} - 2L)(x_k - x_{k-1})] \cdot \text{(shell height)} \\
&= 2\pi\Big[\Big(\frac{x_k + x_{k-1}}{2} - L\Big)(x_k - x_{k-1})\Big] \cdot \text{(shell height)} \qquad (6.7) \\
&= 2\pi \cdot \text{(average shell radius)} \cdot \text{(thickness)} \cdot \text{(shell height)} \\
&= 2\pi \cdot (c_k - L) \cdot \Delta x_k \cdot f(c_k).
\end{aligned}$$

- **Riemann sum**. We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles:

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \qquad (6.8)$$

- **The limit of this Riemann sum**. As each $\Delta x_k \to 0$ and $n \to \infty$, it gives the volume of the solid as a definite integral:

$$\begin{aligned}
V = \lim_{n\to\infty} \sum_{k=1}^{n} \Delta V_k &= \int_a^b 2\pi \cdot \text{(shell radius)} \cdot \text{(shell height)}\, dx \\
&= \int_a^b \underbrace{2\pi(x - L) f(x)}_{\text{Area of the thin shell}}\, dx.
\end{aligned} \qquad (6.9)$$

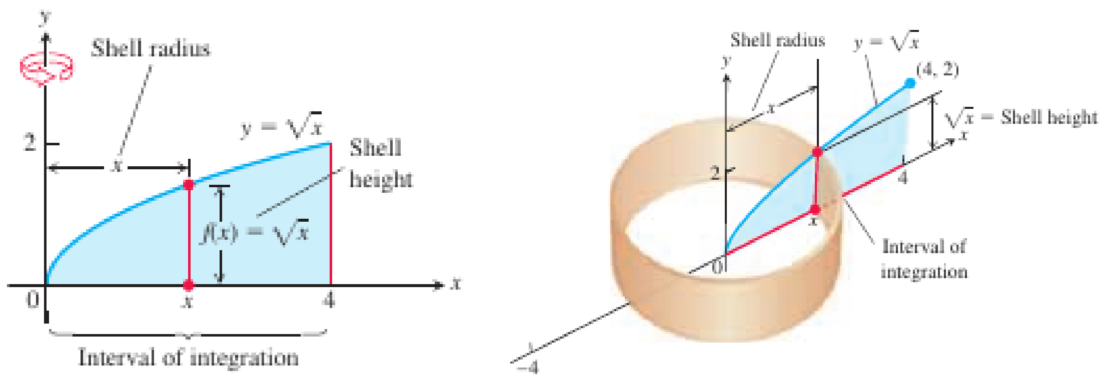
Formula 6.11. **Shell Formula for Revolution About a Vertical Line**. The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0,\ L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b \underbrace{2\pi\text{(shell-radius)} \cdot \text{(shell-height)}}_{\text{Area of the thin shell}}\, dx. \qquad (6.10)$$

Note: **§ 5.3. The Definite Integral as the Limit of Riemann Sums**

The limit of Riemann sums will be explored again and again, **to form definite integrals** for various applications.

Example 6.12. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.



Solution. $V = \displaystyle\int_0^4 2\pi(x) \cdot (\sqrt{x})\, dx$

Ans: $\frac{128\pi}{5}$.

Example 6.13. For the problem in the previous example, Example 6.12, find the volume using the **washer method**.

Solution. $R(y) = 4$ and $r(y) = x = y^2$.

Example 6.14. Use the **method of cylindrical shells** to find the volume of the solid obtained by rotating the region bounded by the given curves about the ***y*-axis**.

$$y = -x^2 + 5x - 4, \quad y = 0$$

Solution. Zeros of f are 1, 4.

Example 6.15. Use the **shell method** to find the volume of the solid obtained by rotating the region bounded by the given curves about the y-axis.

$$y = 2\sqrt{4-x}, \;\; y = 0, \;\; x = 0$$

Solution.

Ans: $\dfrac{512\pi}{15}$.

Example 6.16. Use the shell method to find the volume of the solid obtained by rotating the region bounded by the given curves about the ***x*-axis**.

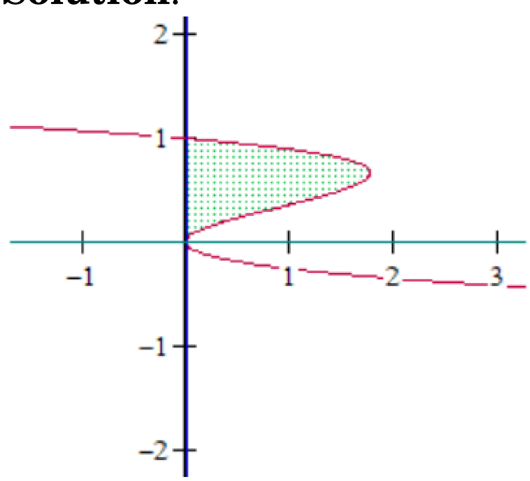
$$x = 2y - y^2, \;\; x = y$$

Solution. $V = \displaystyle\int_0^1 2\pi y \cdot [(2y - y^2) - y]\, dy$

Example 6.17. Use the shell method to find the volume of the solid obtained by rotating the region bounded by the given curves about the **line $\boldsymbol{y = 2}$**.

$$x = 12(y^2 - y^3), \;\; x = 0$$

Solution.



Ans: $\dfrac{14\pi}{5}$.

Exercises 6.2

1. Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about the ***y*-axis**.

   (a) $y = 2x - 1,\ \ y = \sqrt{x},\ \ x = 0$
   (b) $y = x^2,\ \ y = 2 - x,\ \ x = 0$, for $x \geq 0$

   Ans: (b) $5\pi/6$.

2. Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines about the ***x*-axis**.

   (a) $x = 2y - y^2,\ \ x = 0$
   (b) $y = \sqrt{x},\ \ y = 0,\ \ y = x - 2$

   Ans: (b) $16\pi/3$.

3. Use the shell method to find the volumes of the solid generated by revolving the regions bounded by $y = x$ and $y = x^2$ about the following lines.

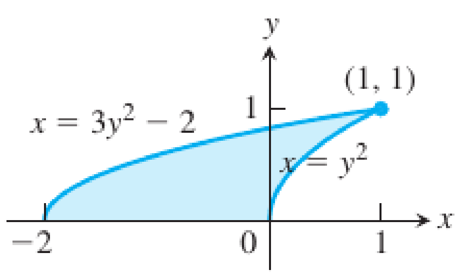
   (a) $x = 1$
   (b) $x = -1$
   (c) The x-axis
   (d) $y = 2$

   Ans: (d) $V = \int_0^1 2\pi(2 - y)(\sqrt{y} - y)\,dy = \frac{8\pi}{15}$.

4. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

   (a) the washer method.
   (b) the shell method.

5. The region shown in the figure is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



6. The volume of a right circular cone of height h and radius r is known as $\dfrac{1}{3}\pi r^2 h$. Derive the formula using an appropriate solid of revolution.

6.3. Arc Length

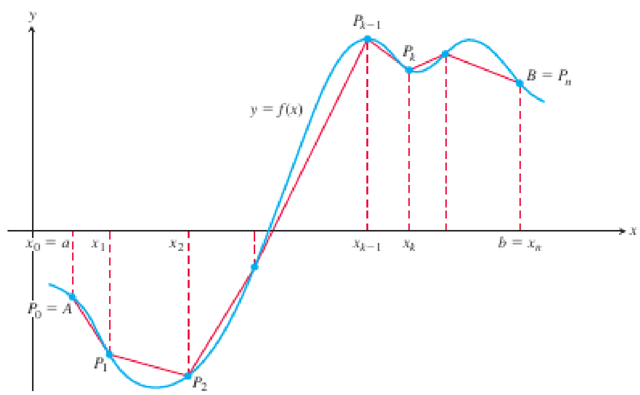
Length of a Curve $y = f(x)$

Suppose we want to find the length of the curve in the graph of $y = f(x)$, from $x = a$ to $x = b$.

- Partition the interval $[a, b]$ into n subintervals with

$$a = x_0 < x_1 < \cdots < x_n = b.$$

- Let $y_k = f(x_k)$. Then the corresponding point $P_k(x_k, y_k)$ lies on the curve.
- Next we connect successive points P_{k-1} and P_k with straight-line segments.



- Then the k-th line segment has length: for $\Delta y_k = y_k - y_{k-1}$,

$$L_k = \sqrt{(x_k - x_{k-1})^2 + (y_k - y_{k-1})^2} = \sqrt{\Delta x_k^2 + \Delta y_k^2}. \tag{6.11}$$

- Since from the **Mean Value Theorem**, we have

$$\Delta y_k = f'(c_k)\Delta x_k, \quad x_{k-1} < c_k < x_k,$$

  L_k can be rewritten as

$$\begin{aligned} L_k &= \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sqrt{\Delta x_k^2 + [f'(c_k)\Delta x_k]^2} \\ &= \sqrt{1 + [f'(c_k)]^2}\Delta x_k. \end{aligned} \tag{6.12}$$

- The **Riemann sum** which approximates the arc length reads

$$\sum_{k=1}^{n} L_k = \sum_{k=1}^{n} \sqrt{\Delta x_k^2 + \Delta y_k^2} = \sum_{k=1}^{n} \sqrt{1 + [f'(c_k)]^2}\Delta x_k. \tag{6.13}$$

Definition 6.18. **Arc Length Formula for $y = f(x)$**
If f' is continuous on $[a, b]$, then the **length (arc length)** of the curve $y = f(x)$ from the point $A = (a, f(a))$ to the point $B = (b, f(b))$ is the value of the integral

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{6.14}$$

Definition 6.19. **Arc Length Formula for $x = g(y)$**
If g' is continuous on $[c, d]$, then the **length (arc length)** of the curve $x = g(y)$ from the point $A = (c, g(c))$ to the point $B = (d, g(d))$ is the value of the integral

$$L = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{6.15}$$

Example 6.20. Find the length of the curve $y = \dfrac{2}{3}x^{3/2},\ 0 \le x \le 2.$

Solution.

Example 6.21. Find the length of the curve.

(a) $y = \dfrac{x^3}{3} + \dfrac{1}{4x}, \quad 1 \le x \le 2$

(b) $x = \dfrac{1}{3}\sqrt{y}(y-3), \quad 1 \le y \le 9$

(c) $x = \dfrac{2}{3}(y-1)^{3/2}, \quad 16 \le y \le 25$

Arc Length Function

Definition 6.22. Let $y = f(x)$ with f' being continuous on $[a, b]$. Then

$$s(x) = \int_a^x \sqrt{1 + [f'(t)]^2}\, dt \tag{6.16}$$

is called the **arc length function** for $y = f(x)$, which measures the length along the curve $y = f(x)$ from the initial point $P_0(a, f(a))$ to the point $Q(x, f(x))$, $x \in [a, b]$.

Remark 6.23. From the **Fundamental Theorem of Calculus**, the function s is differentiable on (a, b) and

$$\frac{ds}{dx} = \sqrt{1 + [f'(x)]^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}.$$

- Then the **differential of arc length** is

$$ds = \sqrt{1 + [f'(x)]^2}\, dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{6.17}$$

- A useful way to remember (6.17) is to write

$$ds = \sqrt{dx^2 + dy^2}. \tag{6.18}$$

Example 6.24. Find the arc length function for the curve $y = \dfrac{x^3}{12} + \dfrac{1}{x}$, taking $A = (1, 13/12)$ as the starting point.

Solution.

Ans: $s(x) = \dfrac{x^3}{12} - \dfrac{1}{x} + \dfrac{11}{12}$.

Exercises 6.3

1. Find the lengths of the curves.

   (a) $y = \dfrac{1}{3}(x^2+2)^{3/2}, \quad 0 \le x \le 2$

   (b) $x = \displaystyle\int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\dfrac{\pi}{4} \le y \le \dfrac{\pi}{4}$

   Ans: (a) $14/3$. (b) 2.

2. CAS Do the following.

   (i) Set up an integral for the length of the curve.

   (ii) Implement a code to approximate the curve's length *numerically*.

   (a) $y = x^2, \quad -1 \le x \le 2$

   (b) $y = \sin x - x\cos x, \quad 0 \le x \le \pi$

   (c) $x = \displaystyle\int_0^y \tan t\, dt, \quad 0 \le y \le \pi/6$

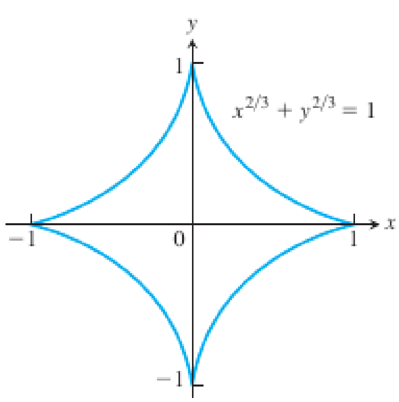
   Ans: (a) (i) $\displaystyle\int_{-1}^{2} \sqrt{1+4x^2}\, dx$. (ii) ≈ 6.1229. (c) (ii) ≈ 0.55.

   For (a), for example, you may use the following code, executable in Matlab/Octave.

   numerical_integration.m

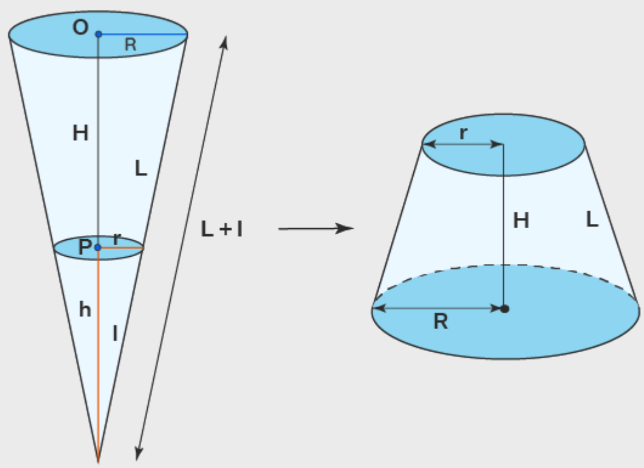
```
a = -1; b = 2;
g = @(x) sqrt(1+4*x.^2);

n = 1000; dx = (b-a)/n;
X = linspace(a,b,n+1); gX = g(X);
length = sum(gX(1:n))*dx
```

3. **The length of an astroid**. The graph of the equation $x^{2/3}+y^{2/3}=1$ is one of a family of curves called **astroids** (not "asteroids"), because of their starlike appearance. Find the length of this particular astroid by finding the length of half the first-quadrant portion, $y=(1-x^{2/3})^{3/2}$, $\sqrt{2}/4 \le x \le 1$, and multiplying by 8.



6.4. Areas of Surfaces of Revolution

Recall: Let's begin with a formula for **curved surface area** (CSA) of a **frustum of a cone**.



- From the figure, we have

$$\frac{L+\ell}{\ell} = \frac{R}{r} \quad \Rightarrow \quad ① \ L+\ell = \frac{R\ell}{r} \quad \text{and} \quad ② \ \ell = \frac{Lr}{R-r}.$$

- Thus

$$\begin{aligned}
&\textbf{The CSA of the frustum} \\
&\quad = \text{(the CSA of the full cone)} - \text{(the CSA of the cut)} \\
&\quad = \left(\frac{1}{2}\cdot(L+\ell)\cdot 2\pi R\right) - \left(\frac{1}{2}\cdot\ell\cdot 2\pi r\right) \\
&\quad = \pi R(L+\ell) - \pi r\ell \ = \pi R\frac{R\ell}{r} - \pi r\ell \qquad [\Leftarrow ①] \\
&\quad = \pi\frac{R^2}{r}\frac{Lr}{R-r} - \pi r\frac{Lr}{R-r} \qquad [\Leftarrow ②] \\
&\quad = \pi L\left(\frac{R^2}{R-r} - \frac{r^2}{R-r}\right) = \pi L(R+r).
\end{aligned} \tag{6.19}$$

Formula 6.25. Curved Surface Area of a Frustum

$$\textbf{Frustum Surface Area} = \boxed{2\pi\cdot\frac{R+r}{2}\cdot L} \tag{6.20}$$

Surface Area for Revolution about the x-axis

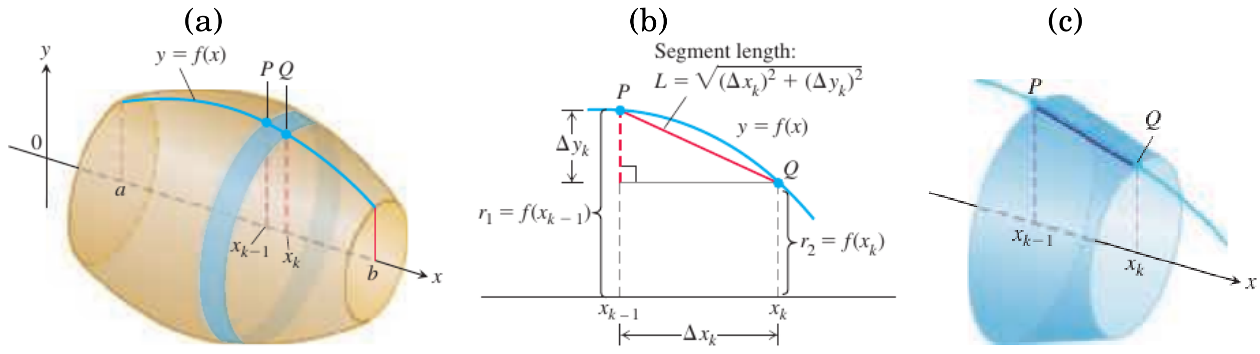


Figure 6.4: Surface area for revolution about the x-axis.

We will **find the area of the surface** generated by revolving the graph of $y = f(x)$, $a \le x \le b$, about the x-axis, as in Figure 6.4(a).

- **Partitioning**: Partition $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.4(a) shows a typical arc PQ and the band it sweeps out.
- **Approximation**: **The surface area of the band swept out by the arc PQ** can be approximated by **the surface area of the frustum of a cone**, shown in Figure 6.4(c).
- It follows from (6.20) that the **surface area of the frustum** reads

$$2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{6.21}$$

  Recall (6.12), p.276:

$$\sqrt{\Delta x_k^2 + \Delta y_k^2} = \sqrt{1 + [f'(c_k)]^2}\Delta x_k,$$

  which is a consequence of the **Mean Value Theorem**.
- The terms in (6.21) are summed to get a **Riemann sum**

$$\begin{aligned} &\sum_{k=1}^{n} 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{1 + [f'(c_k)]^2}\Delta x_k \\ &\quad \approx \sum_{k=1}^{n} 2\pi f(c_k) \cdot \sqrt{1 + [f'(c_k)]^2}\Delta x_k. \end{aligned} \tag{6.22}$$

Definition 6.26. Surface Area for Revolution about the x-axis
If the function $\boldsymbol{y = f(x) \geq 0}$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ **about the x-axis** is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx. \qquad (6.23)$$

Note: In (6.23), $y = f(x)$ plays the role of **radius**.

Definition 6.27. Surface Area for Revolution about the y-axis
If the function $\boldsymbol{x = g(y) \geq 0}$ is continuously differentiable on $[c, d]$, the **area of the surface** generated by revolving the graph of $x = g(y)$ **about the y-axis** is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy. \qquad (6.24)$$

Remark 6.28. Recall the **differential of arc length** in (6.18), $ds = \sqrt{dx^2 + dy^2}$. Integrals in (6.23) and (6.24) can be expressed respectively as

$$S = \int_a^b 2\pi y\, ds \quad \text{and} \quad S = \int_c^d 2\pi x\, ds. \qquad (6.25)$$

Example 6.29. Find the area of the surface obtained by rotating the curve about the $\boldsymbol{x}$**-axis**.

$y = x^3, \quad 0 \leq x \leq 2$

Solution.

Example 6.30. Find the area of the surface obtained by rotating the curve about the ***y*-axis**.

(a) $x = \dfrac{e^y + e^{-y}}{2}, \quad 0 \le y \le \ln 2$

(b) $x = \sqrt{a^2 - y^2}, \quad 0 \le y \le \dfrac{a}{2}$

Claim 6.31. **Curves, Revolved About the Other Axis**

1. **Surface: $y = f(x)$, $a \le x \le b$, revolved about the y-axis**
   - **radius $= x$**.
   - Thus the area of the surface becomes

$$\underbrace{S = \int 2\pi x \, ds}_{(6.25)} = \int_a^b 2\pi x \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx. \tag{6.26}$$

2. **Surface: $x = g(y)$, $c \le y \le d$, revolved about the x-axis**
   - **radius $= y$**.
   - Thus the area of the surface becomes

$$\underbrace{S = \int 2\pi y \, ds}_{(6.25)} = \int_c^d 2\pi y \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{6.27}$$

Example 6.32. Find the area of the surface obtained by rotating the curve about the x**-axis**.

$x = \dfrac{1}{3}(y^2 + 2)^{3/2}, \quad 1 \le y \le 2$

Solution.

Ans: $21\pi/2$.

Claim 6.33. Curves, Revolved About the Other Axis (2)

- Recall the **differential of arc length** in (6.18):

$$\underbrace{ds = \sqrt{dx^2 + dy^2}}_{(6.18)} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{6.28}$$

- Let $y = f(x)$ be a monotone function defined on $[a, b]$, $a \geq 0$, such that

$$[a, b] \quad \underset{g=f^{-1}}{\overset{f}{\rightleftarrows}} \quad [c, d]. \tag{6.29}$$

Then, for example, **the area of the surface** obtained by rotating the curve $y = f(x)$ about the ***y*-axis** reads, with **(radius = x)**,

$$S = \int 2\pi r\, ds = \int_a^b 2\pi x \sqrt{1 + [f'(x)]^2}\, dx = \int_c^d 2\pi g(y) \sqrt{1 + [g'(y)]^2}\, dy \tag{6.30}$$

Example 6.34. Find the area of the surface obtained by rotating the curve $x = \sqrt{4 - y}$, $0 \leq y \leq 4$, about the ***y*-axis**.

Solution.

(a) ***x*-integration**:

(b) ***y*-integration**:

Ans: $\frac{\pi}{6}(17\sqrt{17} - 1)$

Exercises 6.4

1. Find the areas of the surfaces generated by revolving the curves about the indicated axes.

   (a) $y = \sqrt{x}, \quad 0 \le x \le 2; \quad x$-axis
   (b) $x = \sqrt{2y-1}, \quad 1/2 \le y \le 2; \quad y$-axis

   Ans: (b) $14\pi/3$.

2. CAS Do the following.

   (i) Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.
   (ii) Implement a code to approximate the surface area *numerically*.

   (a) $xy = 1, \quad 1 \le y \le 2; \quad y$-axis
   (b) $x^{1/1} + y^{1/2} = 3, \quad 1 \le x \le 4, y > 0; \quad x$-axis
   (c) $y = \tan x, \quad 0 \le x \le \pi/4; \quad x$-axis

   Hint: You may use `numerical_integration.m` given for Exercise 2, Section 6.3.

   Ans: (a) (ii) ≈ 5.02. (b) (ii) ≈ 63.37.

3. Find the area of the surface obtained by rotating the curve about the ***y*-axis**.
   $$y = \frac{x^4}{4} + \frac{1}{8x^2}, \quad 1 \le x \le 2$$
   Hint: Use the formula in (6.26).

   Ans: $253\pi/20$.

4. **The surface of an astroid**. Find the area of the surface generated by revolving about the x-axis an astroid $x^{2/3} + y^{2/3} = 1$, considered in Exercise 3, Section 6.3.

   Hint: Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}$, $0 \le x \le 1$, about the x-axis and double your result.

6.5. Work Done by a Variable Force

Work Done by a Constant Force

When an object moves a **distance d** along a straight line as a result of being acted on by a **constant force F** in the direction of motion, we define the **work** W done by the force on the object with the formula

$$W = Fd. \tag{6.31}$$

Work Done by a Variable Force Along a Line

Suppose that the force acts on **an object moving along a straight line** (say, the x-axis). We assume that **the force F is a continuous function of x**, the object's position.

We want **to find the work done over the interval $x \in [a, b]$**.

- **Partitioning**: Partition the interval $[a, b]$ in the usual way and choose an arbitrary point $c_k \in [x_{k-1}, x_k]$.
- **Approximation**: If the subinterval is short enough, the continuous function **F will not vary much on each of subintervals $[x_{k-1}, x_k]$**. Thus the amount of work done across the subinterval $[x_{k-1}, x_k]$ will be about $F(c_k)\Delta x_k$.
- The total work done from a to b is therefore approximated by the **Riemann sum**

$$\text{Work} \approx \sum_{k=1}^{n} F(c_k)\Delta x_k. \tag{6.32}$$

Definition 6.35. The **work** done by a variable force $F(x)$ in moving an object along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\,dx. \tag{6.33}$$

Example 6.36. The units of the integral are **joules** (=**newton-meters**) if F is in **newtons** and x is in meters, and **foot-pounds** if F is in pounds and x is in feet.

- So the work done by a force of $F(x) = \dfrac{1}{x^2}$ newtons in moving an object along the x-axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2}\,dx = -\frac{1}{x}\bigg]_1^{10} = 0.9\,\mathbf{J}.$$

Hooke's Law for Springs: $F = kx$

Hooke's law says that the force required to hold a stretched or compressed spring x units from its natural (unstressed) length is proportional to x.

$$F = kx. \tag{6.34}$$

- The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring.
- In reality, Hooke's Law gives good results as long as the force doesn't distort the metal in the spring.

Example 6.37. Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution. $F = 16x$ and x is from 0 to 0.25.

Ans: 0.5 ft-lb.

Example 6.38. A spring has a natural length of 1 m. A force of 24 N holds the spring stretched to a total length of 1.8 m.

(a) Find the force constant k.

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?

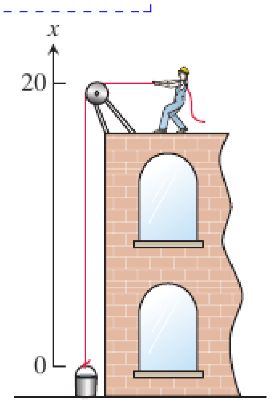
Solution.

(a) $24\,\mathbf{N} = k \cdot 0.8\,\mathbf{m} \Rightarrow k = 24/0.8 = 30\,\mathbf{N/m}. \Rightarrow F = 30x$

(b) $\displaystyle\int_0^2 30x\,ds =$ $= 60\,\mathbf{J}.$

(c)

Lifting objects and Pumping Liquids from Containers



Example 6.39. A 5-kg bucket is lifted from the ground into the air by pulling in 20 m of rope at a constant speed. The rope weighs 0.08 kg/m. How much work was spent lifting the bucket and rope?

> **Note**: "kg" is an *SI unit of mass*.
> (weight) = (mass)·g = (force).

Solution.

(a) **Work for the Bucket** W_b:
(Weight of the bucket) = (5-kg)·$(9.8\,\mathrm{m/s^2})$ = 49 N

(b) **Work for the Rope** W_r:
(Weight of the rope) $= 0.08(20-x)\cdot 9.8 = 0.784(20-x)$ N

Total Work: *Ans*: 1136.8 J.

Exercises 6.5

1. If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?
   Hint: Consider the arguments used in Example 6.38.

2. A mountain climber is about to haul up a 50-m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

   Ans: 780 J.

3. **Kinetic energy**. If a variable force $F(x)$ moves an object of mass m along the x-axis from x_1 to x_2, the object's velocity y can be written as dx/xt. Use **Newton's second law of motion** $F = m(dv/dt)$ and the **Chain Rule**

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = \frac{dv}{dx}v = v\,\frac{dv}{dx}. \tag{6.35}$$

   This shows that the net work done by the force in moving the object from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2, \tag{6.36}$$

   where v_1 and v_2 are the object's velocities at x_1 and x_2. In physics, the expression $\dfrac{1}{2}mv^2$ is called the **kinetic energy** of an object of mass m moving with velocity v.

   Claim 6.40. The work done by the force equals the change in the object's kinetic energy, and we can find the work by calculating this change.

   (a) **Baseball**. How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz, or 0.3125 lb.
   (b) **Golf**. A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done on the ball getting it into the air?

   Hint: (a) 90 mph = 132 ft/s. Try to find the mass from $mg = 0.3125$, where $g = 9.8$ m/s^2 $= 32.1522$ ft/s^2.

   Ans: (a) 84.68.

CHAPTER **7**

Integrals and Transcendental Functions

In this chapter, ...

Contents of Chapter 7

7.1. The Logarithm Defined as an Integral

Definition 7.1. The **natural logarithm** is the function given by

$$\ln x = \int_1^x \frac{1}{t}\,dt, \quad x > 0. \tag{7.1}$$

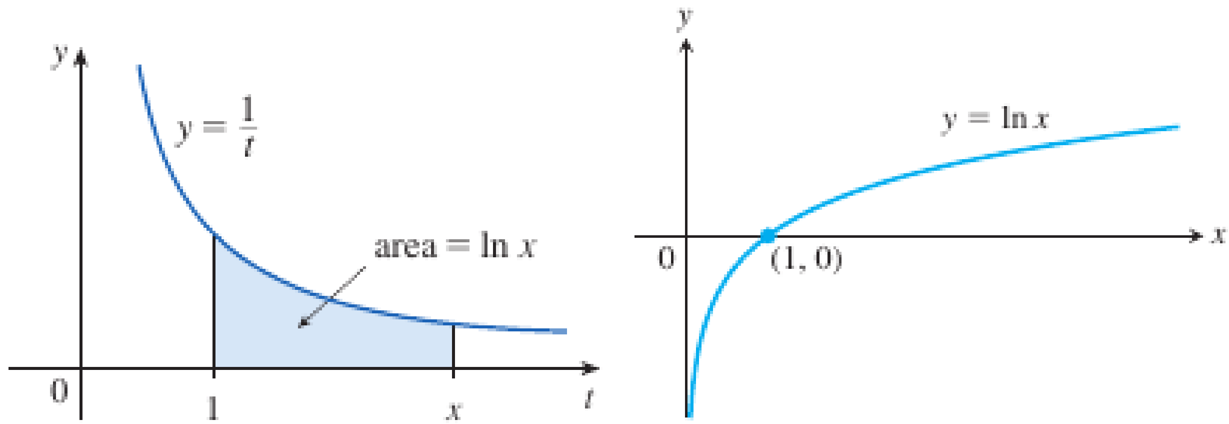


Figure 7.1: The natural logarithm defined as an integral.

Note: From the **Fundamental Theorem of Calculus**, p. 240, $\ln x$ is a continuous and differentiable function. Its derivative is

$$\frac{d}{dx}\ln x = \frac{d}{dx}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}, \quad x > 0. \tag{7.2}$$

Definition 7.2. The number ***e*** is the number in the domain of the natural logarithm that satisfies

$$\ln(e) = \int_1^e \frac{1}{t}\,dt = 1. \tag{7.3}$$

Recall: (Section 4.8).

$$\int \frac{1}{x}\, dx = \ln|x| + C.$$

- If u is a differentiable function and $u \neq 0$, then

$$\int \frac{1}{u}\, du = \ln|u| + C. \tag{7.4}$$

- Whenever $u = f(x)$ is a differentiable function $f(x) \neq 0$, we have that $du = f'(x)\, dx$ and

$$\int \frac{f'(x)}{f(x)}\, dx = \ln|f(x)| + C. \tag{7.5}$$

Example 7.3. Find the integrals.

(a) $\displaystyle\int \frac{y}{y^2 - 25}\, dy$

(b) $\displaystyle\int \frac{\sec x \tan x}{2 + 3\sec x}\, dx$

Substitution Methods

Example 7.4. Find the integrals.

(a) $\displaystyle\int \tan x \ln(\cos x)\, dx$ (b) $\displaystyle\int \frac{\ln(\ln x)}{x \ln x}\, dx$

Solution.

Example 7.5. Find the integrals.

(a) $\displaystyle\int \frac{e^{-1/x^2}}{x^3}\, dx$ (b) $\displaystyle\int \frac{e^{4x}}{1+e^{4x}}\, dx$

Solution.

Recall: **(Summary 3.59, Section 3.8)**

$$\frac{d}{dx}a^x = a^x \ln a \quad\Longrightarrow\quad \int a^x\,dx = \frac{a^x}{\ln a} + C$$
$$\log_a x = \frac{\ln x}{\ln a} \ \text{(Eqn (1.41), p. 53)} \quad\Longrightarrow\quad \frac{d}{dx}\log_a x = \frac{1}{x\ln a} \tag{7.6}$$

Example 7.6. Find the integrals.

(a) $\displaystyle\int_0^1 2^x\,dx$

(b) $\displaystyle\int_1^4 \frac{3^{\sqrt{x}}}{\sqrt{x}}\,dx$

(c) $\displaystyle\int_{1/2}^2 \frac{\log_2(2x)}{x}\,dx$

Ans: (c) $2\ln 2$

Exercises 7.1

1. Find the integrals.

   (a) $\displaystyle\int \frac{3\sec^2 t}{6+3\tan t}\,dt$ (b) $\displaystyle\int_{-3}^{-2} \frac{1}{x}\,dx$

   (c) $\displaystyle\int_0^2 \frac{\log_2(x+2)}{x+2}\,dx$ (d) $\displaystyle\int \frac{dx}{x\log_{10} x}$

   Ans: (b) $\ln(2/3)$. (c) $\frac{3}{2}\ln 2$.

2. Solve the **initial value problems**.

   (a) $\dfrac{dy}{dt} = e^t \sin(e^2 - 2), \quad y(\ln 2) = 1$ (b) $\dfrac{d^2y}{dx^2} = \sec^2 x, \quad y(0) = 0$ and $y'(0) = 1$.

   Ans: (a) $y = 2 - \cos(e^t - 2)$.

3. The region between the curve $y = 1/x^2$ and the x-axis from $x = 1/2$ to $x = 2$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

   Ans: $2\pi \ln 4$.

4. In this problem, answer the question **without using a calculator**.

   Which is bigger, e^π or π^e?

   Hint: You may start with the comparison between $\pi \ln e = \pi$ and $e \ln \pi$. More specifically, consider $f(x) = x - e\ln x$. Then $f(e) = e - e\ln e = 0$. Now what can you say about $f'(x)$ for $x > e$?

5. $\boxed{\text{Challenge}}$ Prove that

$$\frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} < \ln n < 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}. \tag{7.7}$$

   Hint: Recall (7.1): $\ln x = \displaystyle\int_1^x \frac{1}{t}\,dt$, $x > 0$.

7.2. Exponential Change and Separable Differential Equations

In many real-world situations, **the rate of change of a quantity y** is proportional to **its size** at a given time t.

$$\frac{dy}{dt} \sim y \implies \frac{dy}{dt} = ky. \tag{7.8}$$

- Examples of such quantities include the size of a population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium.
- Such quantities are said to undergo **exponential change**.

Example 7.7. Solve the differential equation (7.8).

Solution. Divide (7.8) by y to get

$$\begin{aligned} &\frac{1}{y}\cdot\frac{dy}{dt} = k \\ \Rightarrow\ &\int \frac{\mathbf{1}}{\mathbf{y}}\cdot\frac{\mathbf{dy}}{\mathbf{dt}}\,\mathbf{dt} = \int k\,\mathbf{dt} \\ \Rightarrow\ &\ln|y| = kt + C \quad\Rightarrow\quad |y| = e^{kt+C} \\ \Rightarrow\ &y = \pm e^{C}\cdot e^{kt} = Ae^{kt} \end{aligned} \tag{7.9}$$

The solution of the **initial value problem**

$$\frac{dy}{dt} = ky, \quad y(0) = y_0, \tag{7.10}$$

is

$$y = y_0\, e^{kt}. \tag{7.11}$$

Note: $\displaystyle\int \frac{1}{y}\cdot\frac{\mathrm{dy}}{\mathrm{dt}}\,\mathrm{dt} = \int \frac{1}{y}\,\mathrm{dy}$, the integration of $\dfrac{1}{y}$ with respect to y. That is, **the first line in (7.9) and (7.10)** can be written as

$$\boldsymbol{\frac{1}{y}\,dy = k\,dt} \tag{7.12}$$

Separable Differential Equations

- More general differential equations are of the form

$$\frac{dy}{dx} = f(x,y) = f(x, y(x)). \tag{7.13}$$

- The differential equation (7.13) is **separable** if f can be expressed as a product of a function of x and a function of y:

$$\frac{dy}{dx} = g(x)\,h(y). \tag{7.14}$$

- We can solve (7.14), using the same arguments introduced in Example 7.7 (***separate the variables and integrate***):

$$\frac{1}{h(y)}\,dy = g(x)\,dx \quad \Rightarrow \quad \int \frac{1}{h(y)}\,dy = \int g(x)\,dx. \tag{7.15}$$

Example 7.8. Solve the equation $y(x+1)\dfrac{dy}{dx} = x(y^2+1)$.

Solution.

Ans: $\dfrac{1}{2}\ln(1+y^2) = x - \ln|x+1| + C$, in implicit form.

Population Growth

When **the number of individuals** becomes large enough, the population can **be approximated by a continuous function**.

- **Differentiability** of the approximating function is another reasonable hypothesis in many settings, allowing **for the use of calculus** to model and predict population sizes.
- **Further assumptions**:
  (a) If we assume that the proportion of reproducing individuals remains constant and assume a constant fertility, then at any instant t **the birth rate is proportional to the number** $y(t)$ **of individuals**.
  (b) Let's assume, too, that the death rate of the population is stable and proportional to $y(t)$.
  (c) Further, we neglect departures and arrivals.
- **Modeling**: Then the growth rate dy/dt is the birth rate minus the death rate:

$$\frac{dy}{dt} = (b-d)y = k\,y. \tag{7.16}$$

Example 7.9. The biomass of a yeast culture in an experiment is initially 29 grams. After 30 minutes the mass is 37 grams. Assuming that the equation for unlimited population growth gives a good model for the growth of the yeast when the mass is below 100 grams, how long will it take for the mass to double from its initial value?

Solution. Begin with $y = y_0\,e^{kt}$.

Ans: $k = \frac{1}{30}\ln(\frac{37}{29}) \approx 0.008118$; $t = \frac{\ln 2}{k} \approx 85.38$.

Radioactivity

- **Some atoms are unstable** and can spontaneously emit mass or radiation. This process is called **radioactive decay**.
- **Experiments** have shown that at any given time the rate at which a radioactive element decays is approximately proportional to the number of radioactive nuclei present: proportional to the number of radioactive nuclei present.
- **Modeling**: Thus the decay of a radioactive element is described by the equation
$$\frac{dy}{dt} = -ky, \quad k > 0, \tag{7.17}$$
of which the solution reads
$$y = y_0 \, e^{-kt}. \tag{7.18}$$
- The **half-life** of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay:
$$\frac{1}{2}y_0 = y_0 \, e^{-kt}.$$
Thus the half-life reads
$$\text{Half-life} = \frac{\ln 2}{k}. \tag{7.19}$$

Example 7.10. Plutonium-239. The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?

Solution.

Ans: $24360 \cdot \frac{\ln 5}{\ln 2} \approx 56562.$

Heat Transfer: Newton's Law of Cooling

- **The rate of heat exchange** between an object and its surroundings is **proportional to the difference in temperature between the object and the surroundings**.
- This observation is called **Newton's Law of Cooling**, although it applies to warming as well.
- **Modeling**: If H is the temperature of the object at time t and H_s is the constant surrounding temperature, then the differential equation is

$$\frac{dH}{dt} = -k(H - H_s), \tag{7.20}$$

so that

$$\frac{dH}{H - H_s} = -k\,dt \quad \Rightarrow \quad \ln(H - H_s) = -kt + C$$
$$\Rightarrow H - H_s = Ae^{-kt}, \text{ for } A = H_0 - H_s.$$

where H_0 is the temperature at $t = 0$.

- Thus the solution reads

$$H = H_s + (H_0 - H_s)e^{-kt}. \tag{7.21}$$

Example 7.11. A hard-boiled egg at $98°$ C is put in a sink of $18°$ C water. After 5 min, the egg's temperature is $38°$ C. Assuming that the water has not warmed appreciably, how much longer will it take the egg to reach $20°$ C?

Solution. $H = 18 + (98 - 18)e^{-kt} = 18 + 80e^{-kt}$.

Ans: $t = \frac{\ln 40}{0.2 \ln 4} \approx 13\,\text{min} \Rightarrow$ It will take about 8 min more.

Exercises 7.2

1. Solve the differential equations.

(a) $\dfrac{dy}{dx} = e^{x-y}$

(b) $\dfrac{dy}{dx} = 2x\sqrt{1-y^2}, \quad -1 < y < 1$

(c) $\dfrac{1}{x}\dfrac{dy}{dx} = ye^{x^2} + 2\sqrt{y}\, e^{x^2}$

(d) $\dfrac{dy}{dx} = xy + 3x - 2y - 6$

Ans: (a) $e^y - e^x = C$. (b) $y = \sin(x^2 + C)$.

2. The population of **Starkville**, Mississippi, was 2,689 in the year 1900 ($t = 0$) and 25,495 in 2020 ($t = 120$). Assume that the population in Starkville has grown and will grow *exponentially*.

(a) Estimate the population in 1950 and 2000.

(b) Approximately when is the population going to reach 50,000?

Ans: (b) year 2056.

Note: Scientists who do Carbon-14 dating often use a figure of 5730 years for its half-life.

3. **Carbon-14**. The oldest known *frozen human mummy*, discovered in the *Schnalstal glacier* of the Italian Alps in 1991 and called **Otzi**, was found wearing straw shoes and a leather coat with goat fur, and holding a copper ax and stone dagger. It was estimated that Otzi died 5000 years before he was discovered in the melting glacier. How much of the original carbon-14 remained in Otzi at the time of his discovery?

Ans: $\approx 54.62\,\%$.

4. **Surrounding medium of unknown temperature**. A pan of warm water (46° C) was put in a refrigerator. Ten minutes later, the water's temperature was 39° C; 10 min after that, it was 33° C. Use Newton's Law of Cooling to estimate how cold the refrigerator was.

Ans: -3° C.

7.3. Hyperbolic Functions

Definition 7.12. Certain combinations of e^x and e^{-x} arise so frequently in mathematics and its applications that they deserve special names. These are called the **hyperbolic functions**.

$$\begin{aligned} \sinh x &= \frac{e^x - e^{-x}}{2} & \operatorname{csch} x &= \frac{1}{\sinh x} \\ \cosh x &= \frac{e^x + e^{-x}}{2} & \operatorname{sech} x &= \frac{1}{\cosh x} \\ \tanh x &= \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} & \coth x &= \frac{1}{\tanh x} \end{aligned} \tag{7.22}$$

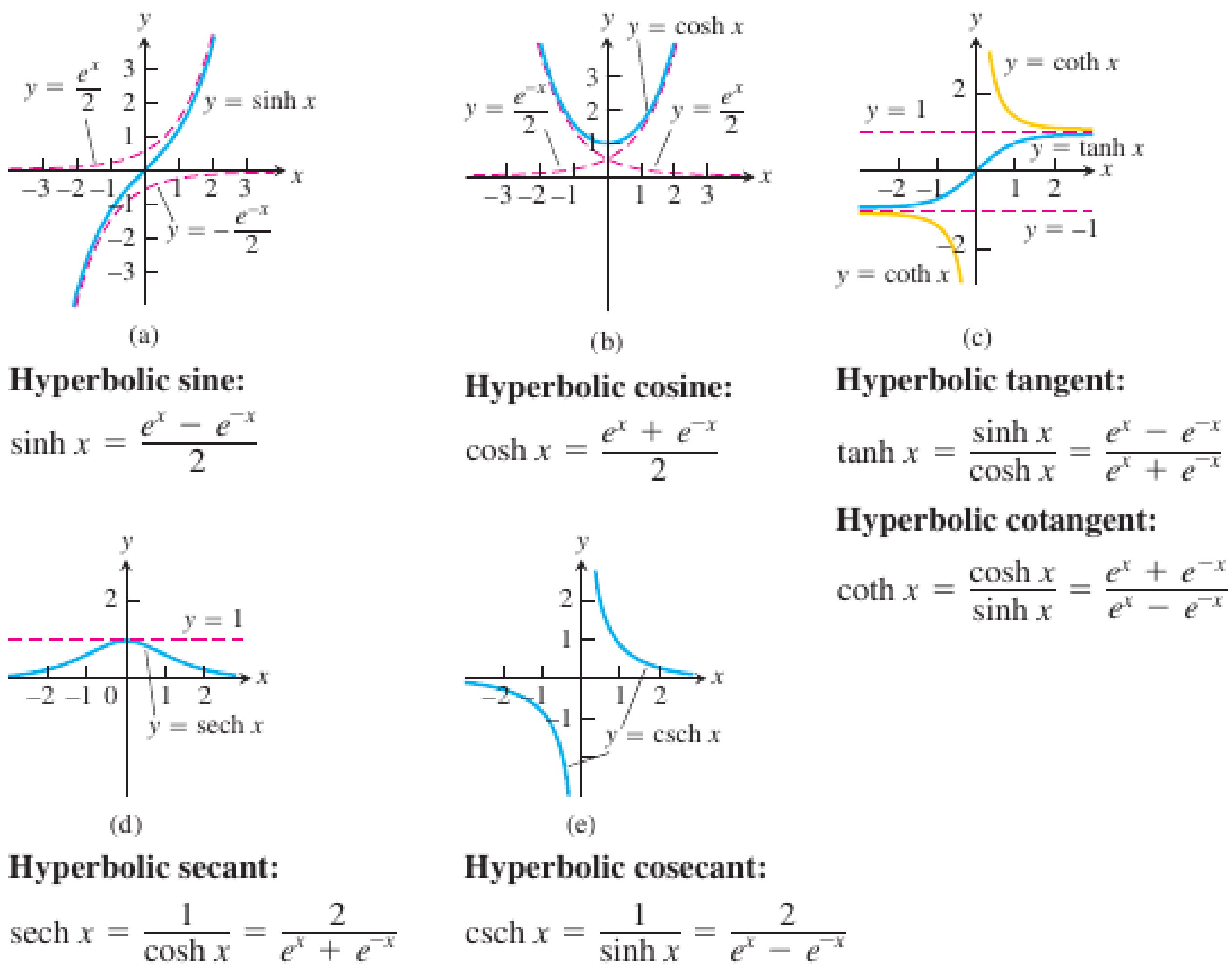


Figure 7.2: The six basic hyperbolic functions.

Formula 7.13. Helpful Hyperbolic Identities:

$$\begin{array}{ll} \sinh(-x) = -\sinh x & \cosh(-x) = \cosh x \\ \cosh^2 x - \sinh^2 x = 1 & 1 - \tanh^2 x = \operatorname{sech}^2 x \\ \sinh(2x) = 2\sinh x \cosh x & \cosh(2x) = \cosh^2 x + \sinh^2 x \end{array} \tag{7.23}$$

Note: Consider $\cosh^2 x - \sinh^2 x = 1$, in which the point $(\cosh x, \sinh x)$ lies on the right-hand branch of the hyperbola $x_1^2 - x_2^2 = 1$. This is where the **hyperbolic functions** get their names. The trigonometric functions are sometimes called the **circular functions**.

Example 7.14. Prove the following identity: $\cosh x - \sinh x = e^{-x}$

Solution.

Example 7.15. Find derivatives of the hyperbolic functions.

(a) $\sinh x$

(b) $\operatorname{csch} x = \dfrac{1}{\sinh x}$

Solution.

Ans: (b) $\frac{d}{dx}(\operatorname{csch} x) = -\operatorname{csch} x \coth x$.

7.3.1. Derivatives of Hyperbolic Functions

Formula 7.16. Derivatives of Hyperbolic Functions:

$$\begin{aligned} \frac{d}{dx}(\sinh x) &= \cosh x & \frac{d}{dx}(\operatorname{csch} x) &= -\operatorname{csch} x \coth x \\ \frac{d}{dx}(\cosh x) &= \sinh x & \frac{d}{dx}(\operatorname{sech} x) &= -\operatorname{sech} x \tanh x \\ \frac{d}{dx}(\tanh x) &= \operatorname{sech}^2 x & \frac{d}{dx}(\coth x) &= -\operatorname{csch}^2 x \end{aligned} \tag{7.24}$$

Example 7.17. Find the derivative.

(a) $f(x) = \dfrac{1}{2}\sinh(2x+1)$

(b) $g(x) = x^2 \tanh\left(\dfrac{1}{x}\right)$

(c) $y = \ln\cosh x - \dfrac{1}{2}\tanh^2 x$

Inverse Hyperbolic Functions

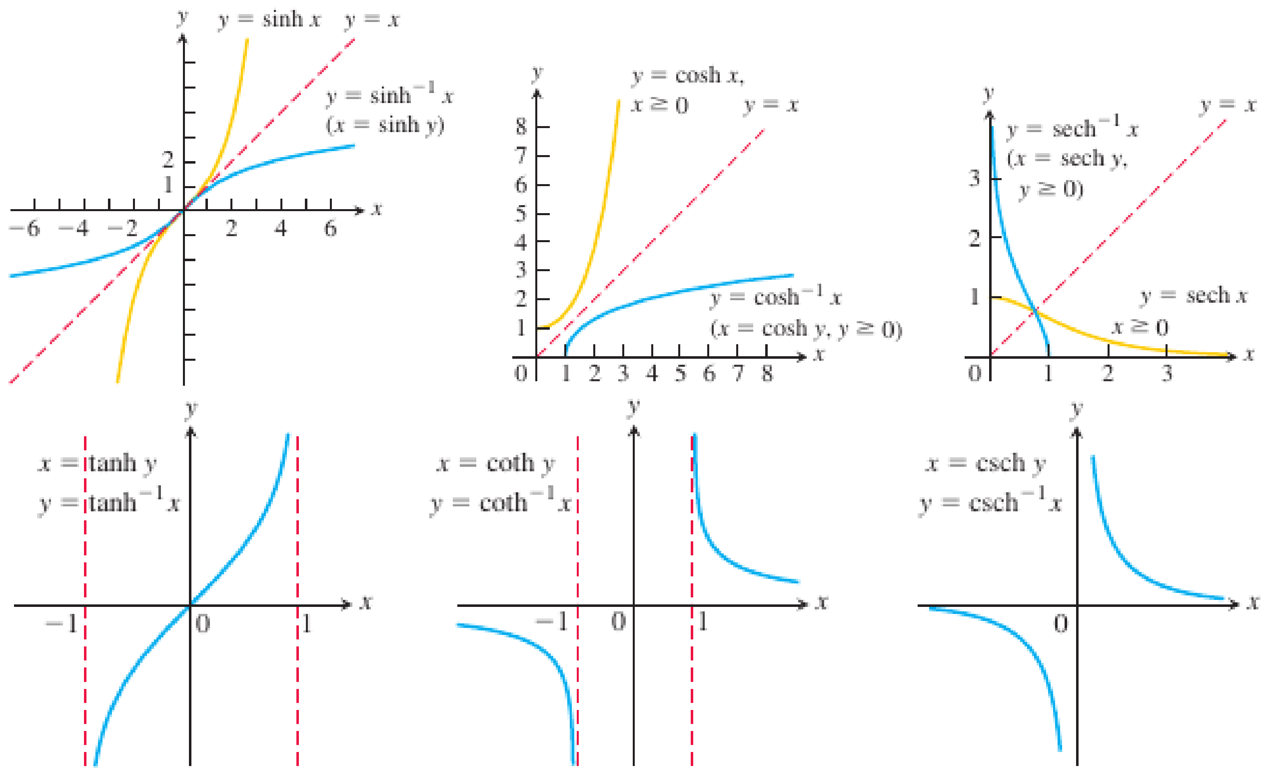


Figure 7.3: The graphs of the inverse hyperbolic functions.

Formula 7.18. Identities for inverse hyperbolic functions:

$$\begin{aligned} \operatorname{csch}^{-1} x &= \sinh^{-1}\frac{1}{x}, && x \neq 0 \\ \operatorname{sech}^{-1} x &= \cosh^{-1}\frac{1}{x}, && 0 < x \leq 1 \\ \coth^{-1} x &= \tanh^{-1}\frac{1}{x}, && |x| > 1 \end{aligned} \tag{7.25}$$

Example 7.19. Prove that $\operatorname{sech}^{-1} x = \cosh^{-1}\dfrac{1}{x}$.

Solution. For $0 < x \leq 1$,

$$\operatorname{sech}\left(\cosh^{-1}\frac{1}{x}\right) = \frac{1}{\cosh\left(\cosh^{-1}\frac{1}{x}\right)} = \frac{1}{\left(\frac{1}{x}\right)} = x,$$

which completes the proof. □

Formula 7.20. **Derivatives of Inverse Hyperbolic Functions**:

$$\begin{aligned}
&\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{1+x^2}} && \frac{d}{dx}(\operatorname{csch}^{-1} x) = -\frac{1}{|x|\sqrt{1+x^2}}, \quad x \neq 0 \\
&\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2-1}}, \quad x > 1 && \frac{d}{dx}(\operatorname{sech}^{-1} x) = -\frac{1}{x\sqrt{1-x^2}}, \quad 0 < x < 1 \\
&\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}, \quad |x| < 1 && \frac{d}{dx}(\coth^{-1} x) = \frac{1}{1-x^2}, \quad |x| > 1
\end{aligned} \tag{7.26}$$

Example 7.21. Show that if $x > 1$, then $\dfrac{d}{dx}(\cosh^{-1} x) = \dfrac{1}{\sqrt{x^2-1}}$.

Solution. $y = \cosh^{-1} x \Rightarrow \cosh y = x \Rightarrow (\sinh y) \cdot \dfrac{dy}{dx} = 1$. Thus

$$\begin{aligned}
\frac{dy}{dx} &= \frac{1}{\sinh y} = \frac{1}{\sinh(\cosh^{-1} x)} \\
&= \frac{1}{\big(\cosh^2(\cosh^{-1} x) - 1\big)^{1/2}} \qquad [\Leftarrow \cosh^2 \mathbf{x} - \sinh^2 \mathbf{x} = \mathbf{1}] \\
&= \frac{1}{(x^2-1)^{1/2}}.
\end{aligned}$$

Example 7.22. Find the derivative.

(a) $f(x) = \cosh^{-1}(2\sqrt{x+1})$

(b) $g(x) = (x^2 + 2x)\tanh^{-1}(x+1)$

Solution.

7.3.2. Integrals of Hyperbolic Functions

Formula 7.23. Integrals of Hyperbolic Functions:
It follows from **Formula 7.16** that

$$\begin{aligned}
&\int \cosh x\,dx = \sinh x + C && \int \operatorname{csch} x \coth x\,dx = -\operatorname{csch} x + C \\
&\int \sinh x\,dx = \cosh x + C && \int \operatorname{sech} x \tanh x\,dx = -\operatorname{sech} x + C \\
&\int \operatorname{sech}^2 x\,dx = \tanh x + C && \int \operatorname{csch}^2 x\,dx = -\coth x + C
\end{aligned} \tag{7.27}$$

Formula 7.24. Integrals of Inverse Hyperbolic Functions:
It follows from the **substitution rule** (**Theorem 5.48**) and **Formula 7.20** that for $a > 0$,

$$\begin{aligned}
&\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\left(\frac{x}{a}\right) + C && \int \frac{1}{x\sqrt{a^2+x^2}}\,dx = -\frac{1}{a}\operatorname{csch}^{-1}\left|\frac{x}{a}\right| + C \\
&\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\left(\frac{x}{a}\right) + C && \int \frac{1}{x\sqrt{a^2-x^2}}\,dx = -\frac{1}{a}\operatorname{sech}^{-1}\left(\frac{x}{a}\right) + C \\
&\int \frac{1}{a^2-x^2}\,dx = \begin{cases} \dfrac{1}{a}\tanh^{-1}\left(\dfrac{x}{a}\right) + C, & \textbf{if } x^2 < a^2 \\ \dfrac{1}{a}\coth^{-1}\left(\dfrac{x}{a}\right) + C, & \textbf{if } x^2 > a^2 \end{cases}
\end{aligned} \tag{7.28}$$

Example 7.25. Evaluate the integrals.

(a) $\displaystyle\int \sinh 2x\,dx$

(c) $\displaystyle\int_1^2 \frac{\cosh(\ln t)}{t}\,dt$

(b) $\displaystyle\int \operatorname{sech}^2(x - 1/2)\,dx$

Example 7.26. Evaluate the integrals.

(a) $\displaystyle\int_0^{2\sqrt{3}} \frac{1}{\sqrt{16+4x^2}}\,dx$

(c) $\displaystyle\int_1^{e} \frac{1}{x\sqrt{1+(\ln x)^2}}\,dx$

(b) $\displaystyle\int_0^{1/2} \frac{1}{1-x^2}\,dx$

Formula 7.27. Inverse Hyperbolic Functions in Logarithms: Since the hyperbolic functions can be written in terms of exponential functions, it is possible to express the inverse hyperbolic functions in terms of logarithms.

$$\begin{aligned}
\sinh^{-1} x &= \ln(x+\sqrt{x^2+1}), && -\infty < x < \infty \\
\cosh^{-1} x &= \ln(x+\sqrt{x^2-1}), && x \ge 1 \\
\tanh^{-1} x &= \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right), && |x| < 1 \\
\operatorname{csch}^{-1} x &= \ln\left(\frac{1}{x}+\frac{\sqrt{1+x^2}}{|x|}\right), && x \ne 0 \\
\operatorname{sech}^{-1} x &= \ln\left(\frac{1+\sqrt{1-x^2}}{x}\right), && 0 < x \le 1 \\
\coth^{-1} x &= \frac{1}{2}\ln\left(\frac{x+1}{x-1}\right), && |x| > 1
\end{aligned} \tag{7.29}$$

Exercises 7.3

1. Rewrite the expressions in terms of exponentials and simplify the results as much as you can.

   (a) $2\cosh(\ln x)$ (b) $(\sinh x + \cosh x)^4$

   Ans: (a) $x + \dfrac{1}{x}$.

2. Find the derivative of y with respect to the appropriate variable.

   (a) $y = \ln(\sinh t)$ (b) $y = (x^2+1)\,\text{sech}(\ln x)$ (c) $y = \sinh^{-1}(\tan x)$

   Hint: (b) You may simplify it first.

   Ans: (b) 2. (c) $|\sec x|$.

3. Evaluate the integrals.

   (a) $\displaystyle\int 6\cosh(x/2 - \ln 3)\,dx$ (b) $\displaystyle\int_{\ln 2}^{\ln 4} \coth x\,dx$

   (c) $\displaystyle\int_{-\ln 2}^{0} \cosh^2\left(\frac{x}{2}\right)dx$ (d) $\displaystyle\int_0^{\pi} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$

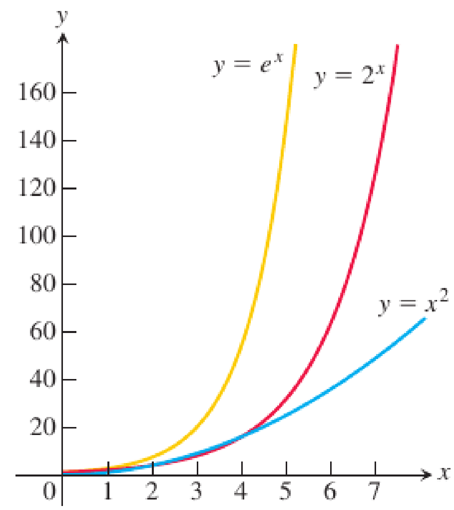
   Ans: (b) $\ln\dfrac{5}{2}$. (c) $\dfrac{3}{8} + \ln\sqrt{2}$.

4. **Challenge** Derive the formula $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ for all real x. Explain in your derivation why the plus sign is used with the square root instead of the minus sign.
   Hint: You will use the definition of $\sinh x$ and its inverse and the **quadratic formula**.

7.4. Relative Rates of Growth and Convergence: Big-oh and Little-oh

7.4.1. Relative Rates of Growth

You may have noticed that exponential functions like 2^x and e^x grow more rapidly (faster) than polynomials or logarithms, as x gets large.
Comparisons of exponential, polynomial, and logarithmic functions can be made precise by defining what it means for a function $f(x)$ to grow faster than another function $g(x)$ as $x \to \infty$.



Definition 7.28. Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

(a) f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

(b) f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L,$$

where L is finite and positive.

Example 7.29. Compare the growth rates of common functions, as $x \to \infty$. Use the **L'Hôpital's Rule**, if necessary.

(a) e^x and x^2

(b) 3^x and 2^x

(c) $\ln x$ and x^r

(d) $\log_a x$ and $\log_b x$, $\quad a, b > 1$

Example 7.30. You can show that $\sqrt{x^2+5}$ and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$, as follows.

(a) Prove that $\sqrt{x^2+5}$ and x grow at the same rate as $x \to \infty$.

(b) Prove that x and $(2\sqrt{x}-1)^2$ grow at the same rate as $x \to \infty$.

7.4.2. Big-oh and Little-oh

Definition 7.31. A function f is **of smaller order than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0. \tag{7.30}$$

We indicate this by writing $f = o(g)$ ("f is **little-oh** of g").

Definition 7.32. Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive number M such that

$$\frac{f(x)}{g(x)} \le M, \quad \text{for } x \text{ sufficiently large}. \tag{7.31}$$

We indicate this by writing $f = \mathcal{O}(g)$ ("f is **big-oh** of g").

Example 7.33. True, or false? As $x \to \infty$,

(a) $\dfrac{1}{x+3} = o\Big(\dfrac{1}{x}\Big)$

(b) $\sqrt{x^2+1} = \mathcal{O}(2x)$

(c) $x \ln x = o(x^2)$

(d) $\ln(\ln x) = \mathcal{O}(\ln x)$

7.4.3. Relative Rates of Convergence

Definition 7.34. Suppose $\lim_{h\to 0} G(h) = 0$.

(a) A quantity $F(h)$ is said to be **in little-oh** of $G(h)$ as $h \to 0$, if

$$\lim_{h\to 0} \frac{|F(h)|}{|G(h)|} = 0. \tag{7.32}$$

In this case, we denote $F(h) \in o(G(h))$ or $F(h) = o(G(h))$.

(b) A quantity $F(h)$ is said to be **in big-oh** of $G(h)$ as $h \to 0$, if there is a positive number K such that

$$\frac{|F(h)|}{|G(h)|} \le K, \text{ for } h \text{ sufficiently small}. \tag{7.33}$$

In this case, we denote $F(h) \in \mathcal{O}(G(h))$ or $F(h) = \mathcal{O}(G(h))$.

Example 7.35. Show that these assertions are not true.

(a) $e^x - 1 = \mathcal{O}(x^2)$, as $x \to 0$

(b) $x = o(\tan^{-1} x)$, as $x \to 0$

(c) $\sin x \cos x = o(x)$, as $x \to 0$

Solution.

Example 7.36. Prove that

$$\arctan(x) = x + \mathcal{O}(x^3), \quad \text{as } x \to 0.$$

Solution. Consider $\arctan(x) - x = \mathcal{O}(x^3)$.
Or, you may begin with $\dfrac{d}{dx}\arctan(x) = \dfrac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots,\ |x| < 1.$

Example 7.37. Let $f(h) = 1 + h - e^h$. What are the limit and the **rate of convergence** of $f(h)$ as $h \to 0$?

Clue: This problem asks you to find the largest k such that $f(h) = \mathcal{O}(h^k)$.

Solution.

Ans: $f(h) = \mathcal{O}(h^2)$.

Exercises 7.4

1. Which of the following functions grow faster than e^x as $x \to \infty$? Which grow at the same rate as e^x? Which grow slower?

   (a) $10x^4 + 30x + 1$ (b) $x \ln x$ (c) $\sqrt{1 + x^4}$
   (d) 2^x (e) 3^x (f) e^{-x}
   (g) xe^x (h) e^{x+1} (i) e^{2x}

2. Which of the following functions grow faster than $\ln x$ as $x \to \infty$? Which grow at the same rate as $\ln x$? Which grow slower?

   (a) $\log_3 x$ (b) $\ln 2x$ (c) $\ln \sqrt{x}$
   (d) $\sqrt{x}$ (e) $1/x$ (f) e^x

   Ans: Faster: (d) and (f).

3. **(Big-oh and Little-oh)** True, or false? As $x \to \infty$,

   (a) $x - 3x^2 = \mathcal{O}(x)$ (b) $\ln(x^2 + 1) = o(\ln(x + 1))$
   (c) $x + \ln x = \mathcal{O}(x)$ (d) $e^x = \mathcal{O}(e^{2x})$

   Ans: (b) False. (d) True.

4. **(Big-oh and Little-oh)** True, or false? As $x \to 0$,

   (a) $x - 3x^2 = \mathcal{O}(x)$ (b) $\ln(x^2 + 1) = o(\ln(x + 1))$

5. Prove the following.

   (a) As $x \to \infty$, e^x grows faster than x^n for any positive integer n, even $x^{1,000,000,000}$.
   (b) As $x \to \infty$, $\ln x$ grows slower than $x^{1/n}$ for any positive integer n, even $x^{1/1,000,000,000}$.

CHAPTER **8**

Techniques of Integration

In this chapter, ...

Contents of Chapter 8

8.1. Using Basic Integration Formulas

Formula 8.1. Basic Integration Formulas:
Here we summarize the indefinite integrals of many of the functions we have studied so far.

1. $\displaystyle\int k\,dx = kx + C$
2. $\displaystyle\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C \quad (n \neq -1)$
3. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$
4. $\displaystyle\int e^x\,dx = e^x + C$
5. $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C \quad (a > 0, a \neq 1)$
6. $\displaystyle\int \sin x\,dx = -\cos x + C$
7. $\displaystyle\int \cos x\,dx = \sin x + C$
8. $\displaystyle\int \sec^2 x\,dx = \tan x + C$
9. $\displaystyle\int \csc^2 x\,dx = -\cot x + C$
10. $\displaystyle\int \sec x \tan x\,dx = \sec x + C$
11. $\displaystyle\int \csc x \cot x\,dx = -\csc x + C$
12. $\displaystyle\int \tan x\,dx = \ln|\sec x| + C$
13. $\displaystyle\int \cot x\,dx = \ln|\sin x| + C$
14. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$
15. $\displaystyle\int \csc x\,dx = -\ln|\csc x + \cot x| + C$
16. $\displaystyle\int \sinh x\,dx = \cosh x + C$
17. $\displaystyle\int \cosh x\,dx = \sinh x + C$
18. $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + C$
19. $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$
20. $\displaystyle\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{x}{a}\right| + C$
21. $\displaystyle\int \frac{dx}{\sqrt{a^2 + x^2}} = \sinh^{-1}\left(\frac{x}{a}\right) + C \quad (a > 0)$
22. $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1}\left(\frac{x}{a}\right) + C \quad (x > a > 0)$

Example 8.2. Evaluate the integrals.

(a) $\displaystyle\int_0^1 \frac{16x}{8x^2 + 1}\,dx$

(b) $\displaystyle\int \frac{1 - x}{\sqrt{1 - x^2}}\,dx$

Key Idea 8.3. To transform the integrand into **a sum of elementary derivatives**, use algebraic manipulation, identities, and substitution. We will see ***very typical techniques***.

Example 8.4. Evaluate the integrals.

(a) $\displaystyle\int \frac{2}{x\sqrt{1-4(\ln x)^2}}\,dx$ (b) $\displaystyle\int_0^{\pi/4} \frac{1}{1-\sin\theta}\,d\theta$

Hint: (b) Multiply $\frac{1+\sin\theta}{1+\sin\theta}$ and then use $1-\sin^2\theta=\cos^2\theta$.

Example 8.5. Evaluate the integrals.

(a) $\displaystyle\int \frac{2x^2-5x}{2x+1}\,dx$ (b) $\displaystyle\int \frac{1}{\sqrt{4x-x^2}}\,dx$

Hint: (a) Perform long division first. (b) Complete the square: $4x-x^2=4-(x-2)^2$ and let $u=x-2$.

$\boxed{\textbf{Example}}$ **8.6.** Evaluate the integrals.

(a) $\displaystyle\int x^{11}\sqrt{x^6+2}\,dx$ (b) $\displaystyle\int \frac{\sqrt{x}}{1+x^3}\,dx$

Hint: (a) Use the substitution $u = x^6 + 2$. (b) Let $u = x^{3/2}$.

$\boxed{\textbf{Example}}$ **8.7.** Evaluate the integrals.

(a) $\displaystyle\int \frac{1}{1+e^x}\,dx$ (b) $\displaystyle\int_{-\pi/2}^{\pi/2} x^3\cos x\,dx$

Exercises 8.1

1. Evaluate each integral using any algebraic method, trigonometric identities, or a substitution.

   (a) $\displaystyle\int \frac{x^2}{x^2+1}\,dx$

   (b) $\displaystyle\int \frac{1}{x+\sqrt{x}}\,dx$

   (c) $\displaystyle\int \frac{1}{e^z+e^{-z}}\,dz$

   (d) $\displaystyle\int \frac{\ln x}{x+4x\ln^2 x}\,dx$

   (e) $\displaystyle\int_0^{\pi/2} \sqrt{1-\cos\theta}\,d\theta$

   (f) $\displaystyle\int e^{t+e^t}\,dt$

   Hint: (b) Rewrite $x+\sqrt{x}=\sqrt{x}(\sqrt{x}+1)$ and let $u=\sqrt{x}+1$. (c) Multiply both the numerator and the denominator by e^z and let $u=e^z$. (e) See the trigonometric formulas in (1.22), p. 33.

   Ans: (e) $2(\sqrt{2}-1)$. (f) $e^{e^t}+C$.

2. **Area**. Find the area of the region bounded above by $y=2\cos x$ and below by $y=\sec x$, $-\pi/4\le x\le \pi/4$.

   Ans: $2\sqrt{2}-\ln(3+2\sqrt{2})$.

3. **Volume**. Find the volume of the solid generated by revolving the region in the above problem (Exercise 2) about the x-axis.

4. **Arc length**. Find the length of the curve $y=\ln(\sec x)$, $0\le x\le \pi/4$.

   Ans: $\ln(\sqrt{2}+1)$.

8.2. Integration by Parts

8.2.1. Integration by Parts: Indefinite Integrals

Note: If we wished to evaluate integrals like $\int x\cos x\,dx$ or $\int x^2e^x\,dx$, the techniques we have studied would not work. ($\because$ It is not **a sum of elementary derivatives**.)

- **Integration by parts** is a technique for simplifying integrals of the form
$$\int u(x)v'(x)\,dx.$$
- **Strategy**:
  - For differentiable functions u and v the **Product Rule** says that
$$\frac{d}{dx}[u(x)v(x)] = u'(x)v(x) + u(x)v'(x). \tag{8.1}$$
  - Integrating both sides makes
$$u(x)v(x) = \int u'(x)v(x)\,dx + \int u(x)v'(x)\,dx. \tag{8.2}$$

Formula 8.8. **Integration by Parts**: It follows from (8.2) that
$$\int u(x)v'(x)\,dx \;=\; u(x)v(x) - \int u'(x)v(x)\,dx, \tag{8.3}$$
whose differential version reads
$$\int u\,dv = uv - \int v\,du. \tag{8.4}$$

Remark 8.9. **Integration by Parts: Alternative Form**
Let v_1 is the antiderivative of v with $C = 0$. Then
$$\int u(x)v(x)\,dx \;=\; u(x)v_1(x) - \int u'(x)v_1(x)\,dx. \tag{8.5}$$

Example 8.10. Evaluate the integrals.

(a) $\displaystyle\int x\cos 2x\,dx$ (b) $\displaystyle\int \tan^{-1} x\,dx$

Note: If a function is raised to a power, we might go through the process a few times.

Example 8.11. Evaluate the integrals.

(a) $\displaystyle\int (\ln x)^2\,dx$ (b) $\displaystyle\int x^2 e^{4x}\,dx$

$\boxed{\textbf{Example}}$ **8.12.** Evaluate the integrals.

(a) $\displaystyle\int x^2 \sin x \, dx$ (b) $\displaystyle\int y \sinh(2y) \, dy$

$\boxed{\textbf{Example}}$ **8.13.** Evaluate the integrals.

(a) $\displaystyle\int t^4 \ln t \, dt$ (b) $\displaystyle\int e^x \cos x \, dx$

Solution.

Ans: (b) $\displaystyle\int e^x \cos x \, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$

Remark 8.14. Tabular Integration by Parts:
While the aforementioned **recursive definition** is correct, it is often tedious to remember and implement. A much easier visual representation of this process is often taught to students and is called the **tabular method** or the **tic-tac-toe method**.

- **Let v_{k+1} be the antiderivative of v_k with $C = 0$**, where $v = v_0$. Then

$$\begin{aligned} \int uv &= \mathbf{uv_1} - \int \mathbf{u'v_1} = uv_1 - \left(u'v_2 - \int u''v_2\right) \\ &= \mathbf{uv_1} - \mathbf{u'v_2} + \int \mathbf{u''v_2} \\ &= \mathbf{uv_1} - \mathbf{u'v_2} + \mathbf{u''v_3} - \int \mathbf{u'''v_3} \\ &= \mathbf{uv_1} - \mathbf{u'v_2} + \mathbf{u''v_3} - \mathbf{u'''v_4} + \int \mathbf{u''''v_4} = \cdots \end{aligned} \tag{8.6}$$

  Note: u is involved in the form of $\boldsymbol{(-1)^k u^{(k)}}$.
- This method works best when one of the two functions in the product is a **polynomial**, so that its repeated derivatives go to a constant.
- It is also applicable for **functions that repeat themselves**.

Example 8.15. Use the tic-tac-toe method to evaluate the integrals.

(a) $\displaystyle\int x^3 e^{2x}\,dx$

(b) $\displaystyle\int e^x \sin x\,dx$

Ans: (b) $\dfrac{e^x \sin x - e^x \cos x}{2} + C$.

8.2.2. Integration by Parts: Definite Integrals

The **Integration by Parts Formula** in (8.3) can be combined with **Part 2 of the Fundamental Theorem** in order to evaluate definite integrals by parts.

Formula 8.16. Integration by Parts for Definite Integrals:

$$\int_a^b u(x)v'(x)\,dx = u(x)v(x)\Big]_a^b - \int_a^b u'(x)v(x)\,dx. \tag{8.7}$$

Example 8.17. Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution.

Ans: $1 - 5e^{-4} \approx 0.9084$

8.2.3. Integration by Parts: Special Functions

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int y f'(y)\,dy \qquad && y = f^{-1}(x),\ x = f(y),\\ & && dx = f'(y)dy\\ &= y f(y) - \int f(y)\,dy \\ &= x f^{-1}(x) - \int f(y)\,dy\end{aligned} \tag{8.8}$$

Note that $y = f^{-1}(x)$ (on the first line) can be viewed as a substitution.

Formula 8.18. **Integrating Inverses of Functions**:

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int f(y)\,dy. \qquad y = f^{-1}(x) \tag{8.9}$$

Example 8.19. Evaluate the integrals, using the formula in (8.9).

(a) $\displaystyle\int \arccos x\,dx$ (b) $\displaystyle\int_1^e \log_2 x\,dx$

Solution.

Ans: (a) $x \arccos x - \sin(\arccos x) + C$. (b) $1/\ln 2$.

Recursive Formulas

Example 8.20. Obtain a formula that expresses the integral

$$\int \cos^n x\,dx$$

in terms of an integral of a lower power of cos x.

Solution. Let $u = \cos^{n-1} x$ and $v' = \cos x$.
$\Rightarrow u' = -(n-1)\cos^{n-2} x \sin x$ and $v = \sin x$. Thus

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \cdot \sin^2 x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x \cdot (1-\cos^2 x)\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

Moving the last term to the left, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

Formula 8.21. For $n \ge 2$,

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx + C. \tag{8.10}$$

For example,

$$\int \cos^3 x\,dx = \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx + C = \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.$$

Note: Similarly you may develop a formula for $\int \sin^n x\,dx$. However, it is not easy to memorize. We will deal with details of **trigonometric integrals** in the next section, Section 8.3.

Exercises 8.2

1. Evaluate the integrals.

(a) $\displaystyle\int x^2 e^{-x}\,dx$

(b) $\displaystyle\int \sin^{-1} x\,dx$

(c) $\displaystyle\int e^{\theta}\sin\theta\,d\theta$

(d) $\displaystyle\int_1^e x^3 \ln x\,dx$

Ans: (b) $x\arcsin x + \sqrt{1-x^2} + C$. (d) $(3e^4+1)/16$.

2. Evaluate the integrals by using a substitution or algebraic manipulation prior to integration by parts.

(a) $\displaystyle\int e^{\sqrt{x}}\,dx$

(b) $\displaystyle\int \ln(x^2+x)\,dx$

Ans: (a) $2(\sqrt{x}-1)e^{\sqrt{x}} + C$.

3. Evaluate the integrals. Some integrals do not require integration by parts.

(a) $\displaystyle\int x^3 e^{x^4}\,dx$

(b) $\displaystyle\int \sqrt{x}\ln x\,dx$

(c) $\displaystyle\int \cos\sqrt{x}\,dx$

(d) $\displaystyle\int \frac{xe^x}{(x+1)^2}\,dx$

Hint: (d) Let $u = xe^x$ and $v' = 1/((x+1)^2$.

Ans: (c) $2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$.

4. Evaluate the integrals, using the formula in (8.9).

(a) $\displaystyle\int \arctan x\,dx$

(b) $\displaystyle\int \operatorname{arcsec} x\,dx$

(c) $\displaystyle\int \sinh^{-1} x\,dx$

Hint: (b) $\displaystyle\int \sec y\,dy = \ln|\sec y + \tan y| + C$; see Formula 5.54, p. 252.

Ans: (b) $x\sec^{-1}x - \ln|x + \sqrt{x^2-1}| + C$. (c) $x\sinh^{-1}x - \cosh(\sinh^{-1}x) + C$.

5. **Finding volume**. Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

(a) the y-axis.

(b) the line $x = \pi/2$.

Ans: (b) 2π.

8.3. Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions.

Formula 8.22. **Integrals of Powers of $\sin x$ and $\cos x$:**

Integrals of the Form $\int \sin^m x \cos^n x \, dx$

- **Case 1: m is odd.**

  (a) Write $m = 2k+1$ and use $\sin^2 x = 1 - \cos^2 x$ to obtain

  $$\sin^m x = \sin^{2k+1} x = (1-\cos^2 x)^k \sin x.$$

  (b) Substitute $u = \cos x \Rightarrow du = -\sin x \, dx$

- **Case 2: n is odd.**

  (a) Write $n = 2k+1$ and use $\cos^2 x = 1 - \sin^2 x$ to obtain

  $$\cos^n x = \cos^{2k+1} x = (1-\sin^2 x)^k \cos x.$$

  (b) Substitute $u = \sin x \Rightarrow du = \cos x \, dx$

- **Case 3: Otherwise (Both m and n are even).** Use the **Half-Angle Formulas**

  $$\sin^2 x = \frac{1-\cos 2x}{2} \qquad \cos^2 x = \frac{1+\cos 2x}{2} \tag{8.11}$$

  to **reduce the integrand to one in lower powers**.

Example 8.23. Evaluate the integrals.

(a) $\displaystyle\int \sin^5 x \cos^3 x \, dx$

(b) $\displaystyle\int_0^{\pi/6} 3\sin^5(3x) \, dx$

Example 8.24. Evaluate the integrals.

(a) $\displaystyle\int \cos^2 x\,dx$

(b) $\displaystyle\int \sin^2 x \cos^4 x\,dx$

(c) $\displaystyle\int_0^{\pi/4} \sqrt{1+\cos 4x}\,dx$

Ans: **(c)** $1/\sqrt{2}$.

Formula 8.25. Integrals of Powers of $\tan x$ and $\sec x$:
Integrals of the Form $\int \sec^m x \tan^n x \, dx$

- **Case 1: m is even.**

  (a) Write $m = 2k+2$ and use $\sec^2 x = 1 + \tan^2 x$ to obtain

  $$\sec^m x = (sec^2 x)^k \sec^2 x = (1+\tan^2 x)^k \sec^2 x.$$

  (b) Substitute $u = \tan x$ $\Rightarrow$ $du = \sec^2 x \, dx$

- **Case 2: n is odd.**

  (a) Write $n = 2k+1$, pull off one $\sec x$, and use $\tan^2 x = \sec^2 x - 1$ to obtain

  $$\begin{aligned}\sec^m x \tan^n x &= \sec^{m-1} x (\tan^2 x)^k \sec x \tan x \\ &= \sec^{m-1} x (\sec^2 x - 1)^k \sec x \tan x.\end{aligned}$$

  (b) Substitute $u = \sec x$ $\Rightarrow$ $du = \sec x \tan x \, dx$

- **Case 3: Otherwise (Neither m is even nor n is odd).** Use the identities

  $$\sec^2 x = 1 + \tan^2 x \qquad \tan^2 x = \sec^2 x - 1 \tag{8.12}$$

  and **use integration by parts**.

Example 8.26. Evaluate the integrals.

(a) $\int \tan^2 x \sec^4 x \, dx$

(b) $\int \tan^5 x \sec^3 x \, dx$

Example 8.27. Evaluate the integrals.

(a) $\displaystyle\int \sec^3 x\,dx$ (b) $\displaystyle\int \tan^2 x \sec x\,dx$

Formula: $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C.$

Solution. (a) Integration by parts:
$u = \sec x,\ \ v' = \sec^2 x$

Example 8.28. Evaluate the integral $\displaystyle\int \sin 3x \cos 5x\,dx.$

Formula: $\sin A \cos B\ =\ \dfrac{1}{2}[\sin(A-B) + \sin(A+B)]$; see (1.22), p.33, for details.

Solution.

Exercises 8.3

1. Evaluate the integrals.

   (a) $\displaystyle\int \cos^3 x \sin^3 x\, dx$ (b) $\displaystyle\int_0^{\pi} 8\sin^4\theta\, d\theta$

   Ans: (b) 3π.

2. Evaluate the integrals.

   (a) $\displaystyle\int_0^{2\pi} \sqrt{\frac{1-\cos x}{2}}\, dx$ (b) $\displaystyle\int_0^{\pi/6} \sqrt{1+\sin x}\, dx$

   Hint: (b) Multiply by $\dfrac{\sqrt{1-\sin x}}{\sqrt{1-\sin x}}$.

   Ans: (a) 4.

3. Evaluate the integrals.

   (a) $\displaystyle\int \sec^4 x\, dx$ (b) $\displaystyle\int_{-\pi/4}^{\pi/4} 6\tan^4 x\, dx$

   Hint: (b) $\tan^4 x = \tan^2 x(\sec^2 x - 1) = \tan^2 x \sec^2 x - \sec^2 x + 1$.

   Ans: (b) $8 + 3\pi$.

4. **Arc length**. Find the length of the curve

$$y = \ln(\sin x), \quad \frac{\pi}{6} \le x \le \frac{\pi}{2}.$$

   Formula: Arc length: $L = \displaystyle\int_a^b \sqrt{1+(f')^2}\, dx$; see (6.14) for details.

5. **Volume**. Find the volume of the solid formed by revolving the region bounded by the graphs of $y = \arctan x$, $x = 0$, and $y = \pi/4$ about the y-axis.

   Ans: $\pi(1 - \pi/4)$.

8.4. Trigonometric Substitutions

In some circumstances, the u-substitution can be in the form of a trig equation. This is particularly true with square roots.

Table 8.1: Trigonometric substitutions.

Expression	**Substitution**	**Identity**
$\sqrt{a^2+x^2}$	$x = a\tan\theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{a^2-x^2}$	$x = a\sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta, \quad 0 \le \theta < \frac{\pi}{2}$ **or** $\frac{\pi}{2} < \theta \le \pi$	$\sec^2\theta - 1 = \tan^2\theta$

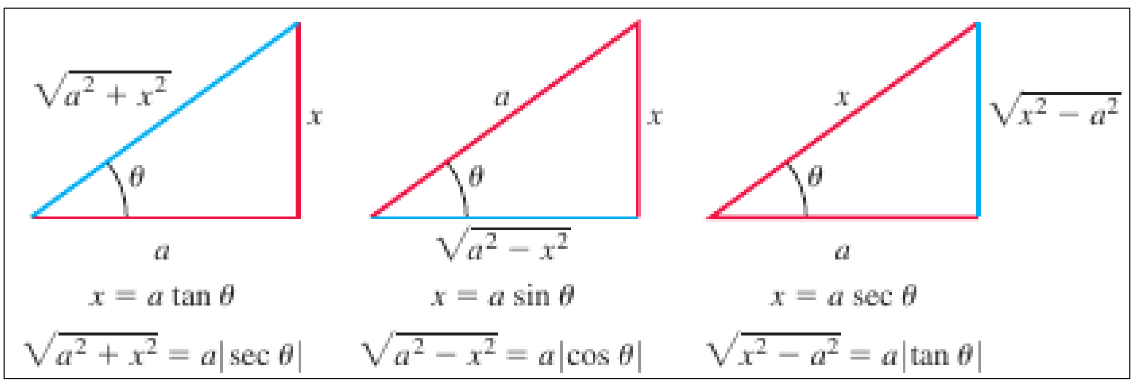


Figure 8.1: Reference triangles for the three basic trigonometric substitutions.

Remark 8.29. With the substitution $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta. \tag{8.13}$$

The substitution requires

$$\theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2}, & \text{if } \frac{x}{a} \ge 1, \\ \frac{\pi}{2} < \theta \le \pi, & \text{if } \frac{x}{a} \le -1. \end{cases} \tag{8.14}$$

For $\sec^{-1}$, see Figure 1.25, p. 54.

Example 8.30. Evaluate the integrals.

(a) $\displaystyle\int \frac{\sqrt{x^2-49}}{x}\,dx,\quad x \ge 7$

Solution. $x = 7\sec\theta,\ 0 \le \theta < \pi/2$

Ans: $7[\tan\theta - \theta] + C = 7\big[\frac{\sqrt{x^2-49}}{7} - \sec(x/7)\big] + C$

(b) $\displaystyle\int \sqrt{25-y^2}\,dy$

Solution. $y = 5\sin\theta$

Ans: $\frac{25}{2}(\theta + \sin\theta\cos\theta) + C = \frac{25}{2}\sin^{-1}(y/5) + \frac{y\sqrt{25-y^2}}{2} + C$

Note: For trigonometric substitutions, you may first consider the fact that ***the radicand must be nonnegative***; (a) $|x| \ge 7$, (b) $|y| \le 5$.

Example 8.31. Evaluate the integrals.

(a) $\displaystyle\int \frac{x^3}{\sqrt{x^2+4}}\,dx$

Ans: $8\left[\frac{1}{3}\sec^3\theta - \sec\theta\right] + C = \frac{1}{3}(x^2+4)^{3/2} - 4\sqrt{x^2+4} + C$

(b) $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^2}{(1-x^2)^{3/2}}\,dx$

Ans: $\sqrt{3} - \pi/3$

Summary 8.32. Procedure for a Trigonometric Substitution

1. Write down the substitution for x, calculate the differential dx, and specify the selected values of θ for the substitution.
2. Substitute the trigonometric expression and the calculated differential into the integrand, and then simplify the results algebraically.
3. Integrate the trigonometric integral, keeping in mind ***the restrictions on the angle θ*** for reversibility.
4. Draw an appropriate ***reference triangle*** to reverse the substitution in the integration result and convert it back to the original variable x.

Example 8.33. Evaluate the integrals. You may have to use the formula $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$.

(a) $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{25x^2-4}},\ \ x > \frac{2}{5}$

Ans: $\frac{1}{5}\left|\frac{5x}{2} + \frac{\sqrt{25x^2-4}}{2}\right| + C$

Exercises 8.4

1. Evaluate the integrals.

(a) $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}},\ x>1$ (b) $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

(c) $\displaystyle\int \frac{x^3\,dx}{x^2-1}$ (d) $\displaystyle\int_0^1 \frac{dx}{(4-x^2)^{3/2}}$

Ans: (b) $-\frac{1}{5}\left(\frac{\sqrt{1-x^2}}{x}\right)^5+C$. (c) $\frac{1}{2}x^2+\frac{1}{2}\ln|x^2-1|+C$.

2. Use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

(a) $\displaystyle\int_0^{\ln 4} \frac{e^y}{\sqrt{e^{2y}+9}}\,dy$ (b) $\displaystyle\int \sqrt{\frac{x}{1-x^3}}\,dx$ (*Hint*: Let $u=x^{3/2}$.)

Ans: (a) $\ln 9-\ln(1+\sqrt{10})$.

3. Complete the square before using an appropriate trigonometric substitution.

(a) $\displaystyle\int \sqrt{8-2x-x^2}\,dx$ (b) $\displaystyle\int \frac{\sqrt{x^2+4x+3}}{x+2}\,dx$

Ans: (b) $\sqrt{x^2+4x+3}-\operatorname{arcsec}(x+2)+C$.

4. **Challenge** Find the average value of $f(x)=\dfrac{\sqrt{x+1}}{\sqrt{x}}$ on the interval $[1,3]$.

Ans: $\sqrt{3}-\frac{\sqrt{2}}{2}+\frac{1}{2}\ln\left(\frac{2+\sqrt{3}}{1+\sqrt{2}}\right)$.

8.5. Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function as **a sum of simpler fractions** (called **partial fractions**) which are easily integrated.

Example 8.34. To illustrate the **method of partial fractions**, consider

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} = \frac{x+5}{x^2+x-2}.$$

If we now reverse this process, we see how to integrate the function on the right side of this equation:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx =$$

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- **The degree of $f(x)$ must be less than the degree of $g(x)$.**
  - That is, the fraction must be **proper**.
  - If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.

$$\frac{f(x)}{g(x)} = Q(x) + \frac{r(x)}{g(x)}. \tag{8.15}$$

- **We must know the factors of $g(x)$.**
  - In theory, **any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors.**

Method of Undetermined Coefficients

Example 8.35. Verify the following.

(a) $$\frac{5x-3}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3} = \frac{2}{x+1} + \frac{3}{x-3}$$

(b) $$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{Ax+B}{x^2+1} + \frac{C}{x-1} + \frac{D}{(x-1)^2} = \frac{2x+1}{x^2+1} + \frac{-2}{x-1} + \frac{1}{(x-1)^2}$$

(c) $$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2} = \frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}$$

Strategy 8.36. Method of Partial Fractions
When $f(x)/g(x)$ is Proper

1. **Linear factors of g**:
   Let $(x-r)$ be a factor of $g(x)$ with $(x-r)^m$ being its highest power. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x-r}+\frac{A_2}{(x-r)^2}+\cdots+\frac{A_m}{(x-r)^m}. \tag{8.16}$$

2. **Quadratic factors of g**:
   Let (x^2+px+q) be a factor of $g(x)$ with $(x^2+px+q)^n$ being its highest power. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x+C_1}{x^2+px+q}+\frac{B_2x+C_2}{(x^2+px+q)^2}+\cdots+\frac{B_nx+C_n}{(x^2+px+q)^n}. \tag{8.17}$$

3. **Combine all the partial fractions**:
   Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions.

4. **Determine the coefficients**:
   Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Example 8.37. Evaluate the integrals.

(a) $\displaystyle\int \frac{x^3+x}{x-1}\,dx$

(b) $\displaystyle\int \frac{2x^3-2x^2+1}{x^2-x}\,dx$

Example 8.38. Evaluate the integrals.

(a) $\displaystyle\int \frac{x-1}{(x+1)^3}\,dx$ (b) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

Example 8.39. Evaluate the integrals.

(a) $\displaystyle\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx$ (b) $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$ (*Hint*: Let $u=\sqrt{x+1}$)

Ans: (a) $\ln(x^2+1)-\frac{1}{x^2+1}+C$. (b) $2[\sqrt{x+1}-\frac{1}{2}\ln(\sqrt{x+1}+1)+\frac{1}{2}\ln|\sqrt{x+1}-1|+C$.

Exercises 8.5

1. Evaluate the integrals.

   (a) $\displaystyle\int_4^8 \frac{y\,dy}{y^2-2y-3}$ (b) $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2-2x+1}$

   Ans: (a) $(\ln 15)/2$.

2. Evaluate the integrals.

   (a) $\displaystyle\int \frac{2s+2}{(s^2+1)(s-1)^3}\,ds$ (b) $\displaystyle\int \frac{1}{(x^{1/3}-1)\sqrt{x}}\,dx$

   Hint: (b) Let $u = x^{1/6}$, equivalently, $x = u^6$.

   Ans: (a) $-(s-1)^{-2}+(s-1)^{-1}+\tan^{-1}s + C$.

3. Evaluate the integrals.

   (a) $\displaystyle\int \frac{1}{x(x^4+1)}\,dx$ (b) $\displaystyle\int \frac{\sqrt{1+\sqrt{x}}}{x}\,dx$ (c) $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y+\sin y-6}$

   Hint: (a) Multiply $\frac{x^3}{x^3}$.

   Ans: (b) $4\sqrt{1+\sqrt{x}}+2\ln\left|\frac{\sqrt{1+\sqrt{x}}-1}{\sqrt{1+\sqrt{x}}+1}\right|+C$.

4. Find the length of the curve $y=\ln(1-x^2),\ 0\le x\le \dfrac{1}{2}$.

   Ans: $\ln 3 - \frac{1}{2}$.

8.6. Integral Tables and Computer Algebra Systems

Note: One can find integrals by using **Integral Tables** or a **Computer Algebra System** (**CAS**). For example,

- **Integral Tables**: You can google various tables of integrals. Here is an example: CRC_integrals.pdf
- **Computer Algebra Systems**: Maple, Mathematica, Python

We will skip this section.

8.7. Numerical Integration

You may watch the project:

Numerical Integration: Trapezoid Rule and Simpson's Rule
YouTube click https://www.youtube.com/watch?v=5DsALI8s8g8

Approximation of Definite Integrals

- The antiderivatives of many functions have no elementary formulas. Examples: $\sin(x^2)$, $1/\ln x$, $\sqrt{1+x^4}$, $\cdots$.
- When we cannot find a workable antiderivative for a function f, we can partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the definite integral of f.
- This procedure is an example of **numerical integration**.
- Here we study two such methods, the **Trapezoid Rule** and **Simpson's Rule**.
- A key goal in our analysis is to **control the possible error** that is introduced when computing an approximation to an integral.

Approximating Integrals with the Midpoint Rule

Recall: In Section 5.1, we introduced the **Midpoint Rule** to approximate a definite integral over an interval $[a, b]$.

- Let the interval $[a, b]$ be partitioned into n equal subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b, \quad x_k = a + k \cdot \Delta x, \;\; \Delta x = \frac{b-a}{n}.$$

- We then approximate the integral using n rectangles:

$$\int_a^b f(x)\,dx \approx \sum_{k=1}^{n} f(c_k)\Delta x, \tag{8.18}$$

where $c_k = (x_{k-1} + x_k)/2$.

8.7.1. Trapezoid Rule

- The Trapezoid Rule is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles, as in Figure 8.2.

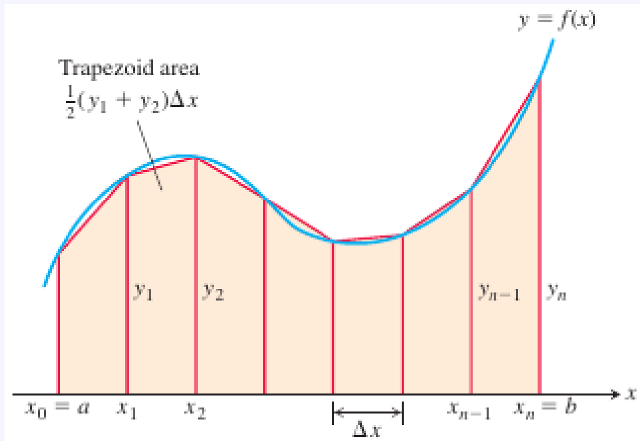


Figure 8.2: The Trapezoid Rule.

- It is not necessary for the subdivision points $x_0, x_1, x_2, \cdots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are.
- We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b-a}{n},$$

which is called the **step size** or **mesh size**.
- The area of the trapezoid that lies above the k-th subinterval is

$$\Delta x \cdot \frac{y_{k-1} + y_k}{2} = \frac{\Delta x}{2}(y_{k-1} + y_k), \tag{8.19}$$

where $y_{k-1} = f(x_{k-1})$ and $y_k = f(x_k)$.

Algorithm 8.40. The Trapezoid Rule

To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} T &= \sum_{k=1}^{n} \frac{\Delta x}{2}(y_{k-1}+y_k) \\ &= \frac{\Delta x}{2}\big[(y_0+y_1)+(y_1+y_2)+(y_2+y_3)+\cdots+(y_{n-1}+y_n)\big] \\ &= \frac{\Delta x}{2}\big[y_0+2y_1+2y_2+\cdots+2y_{n-1}+y_n\big] \\ &= \Delta x\Big(\frac{y_0}{2}+y_1+y_2+\cdots+y_{n-1}+\frac{y_n}{2}\Big). \end{aligned} \tag{8.20}$$

Example 8.41. Use the Trapezoid Rule with $n = 4,\, 8,\, 16$ to estimate $\displaystyle\int_0^{\pi/2} (\sin x + x^3)\,dx$. Compare the estimates with the exact value.

Solution.

trapezoid.m

```
f = @(x) sin(x)+x.^3;

a=0; b=pi/2; exact=1+(1/4)*b^4;
n=4;

Dx = (b-a)/n;
X = linspace(a,b,n+1);
Y = f(X);

T = Dx * ( sum(Y(2:n)) + (Y(1)+Y(n+1))/2 );
fprintf('T_%d = %g; Rel-Error = %g\n',n,T, abs(T-exact)/exact)
```

Output

```
T_4  = 2.60426; Rel-Error = 0.0326096
T_8  = 2.54258; Rel-Error = 0.00815486
T_16 = 2.52716; Rel-Error = 0.00203887
```

The Error = $\mathcal{O}(\Delta x^2)$

8.7.2. The Simpson's Rule

Another rule for approximating the definite integral of a continuous function results from **using parabolas**, instead of the straight-line segments that produced trapezoids.

- In the **Simpson's Rule**, **on each consecutive pair of intervals**, we approximate the curve $y = f(x)$ by **a parabola**.

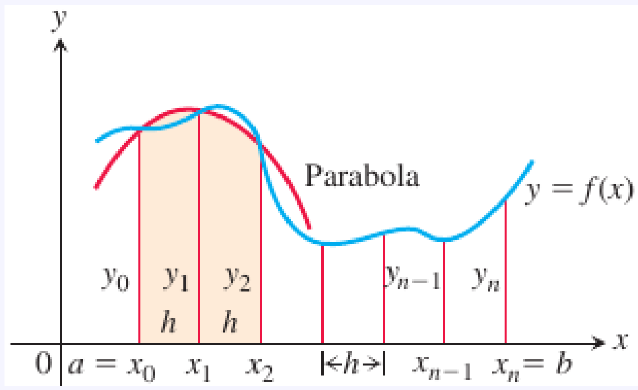


Figure 8.3: Simpson's Rule.

- The parabola which passes (x_0, y_0), (x_1, y_1), and (x_2, y_2) can be constructed in the form of **Lagrange polynomial**

$$P_{[x_0,x_2]}(x) = y_0 L_{2,0}(x) + y_1 L_{2,1}(x) + y_2 L_{2,2}(x), \tag{8.21}$$

where

$$\begin{aligned} L_{2,0}(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}, \\ L_{2,1}(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \\ L_{2,2}(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \end{aligned}$$

- Then

$$\int_{x_0}^{x_2} f(x)\,dx \approx \int_{x_0}^{x_2} P_{[x_0,x_2]}(x)\,dx = \frac{2h}{6}(y_0 + 4y_1 + y_2). \tag{8.22}$$

Algorithm 8.42. The Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$\begin{aligned} S &= \sum_{k=1}^{n/2} \frac{2\Delta x}{6}(y_{2k-2} + 4y_{2k-1} + y_{2k}) \\ &= \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned} \tag{8.23}$$

Example 8.43. Use the Simpson's Rule with $n = 4, 8, 16$ to estimate $\displaystyle\int_0^{\pi/2} (\sin x + x^3)\,dx$. Compare the estimates with the exact value.

Solution.

simpsons.m

```
f = @(x) sin(x)+x.^3;

a=0; b=pi/2; exact=1+(1/4)*b^4;
n=4;

Dx = (b-a)/n;
X = linspace(a,b,n+1);
Y = f(X);

S = 0;
for k=2:2:n
    S = S + (Y(k-1)+4*Y(k)+Y(k+1));
end
S = S* (Dx/3);

fprintf('S_%d = %g; Rel-Error = %g\n',n,S, abs(S-exact)/exact)
```

Output

```
S_4  = 2.52215; Rel-Error = 5.3364e-05
S_8  = 2.52203; Rel-Error = 3.2892e-06
S_16 = 2.52202; Rel-Error = 2.0487e-07
```

The Error $= \mathcal{O}(\Delta x^4)$

8.7.3. Gauss quadrature

Example 8.44. Find points $\{t_1, t_2\} \subset [-1, 1]$ and weights $\{w_1, w_2\}$ such that

$$\int_{-1}^{1} f(t)\, dt \approx w_1 f(t_1) + w_2 f(t_2) = \sum_{i=1}^{2} w_i f(t_i) \tag{8.24}$$

is as accurate as possible.

Solution.

Ans: $t_1 = -1/\sqrt{3}$, $t_2 = 1/\sqrt{3}$, $w_1 = w_2 = 1$.

Example 8.45. Use the Gauss quadrature to estimate $\int_{-1}^{1} \cos t\, dt$.

- $1 \cdot \cos(-1/\sqrt{3}) + 1 \cdot \cos(1/\sqrt{3}) = 1.67582366$.
- The exact value $\int_{-1}^{1} \cos t\, dt = \sin(1) - \sin(-1) = 2 \cdot \sin(1) = 1.68294197$. □

Remark 8.46. **Integration over General Intervals.**
Over an arbitrary interval $[a, b]$, we may introduce a transformation:

$$x = \frac{b-a}{2} t + \frac{a+b}{2} \ :\ [-1, 1] \to [a, b]. \tag{8.25}$$

Then, since $dx = \frac{b-a}{2}\, dt$, we have

$$\begin{aligned}\int_a^b f(x)\, dx &= \int_{-1}^{1} f\Big(\frac{b-a}{2} t + \frac{a+b}{2}\Big) \frac{b-a}{2}\, dt \\ &\approx \widehat{w}_1 f(\widehat{x}_1) + \widehat{w}_2 f(\widehat{x}_2),\end{aligned} \tag{8.26}$$

where

$$\boldsymbol{\widehat{w}_i = \frac{b-a}{2} w_i}, \quad \boldsymbol{\widehat{x}_i} = \frac{b-a}{2} t_i + \frac{a+b}{2} = \boldsymbol{(t_i + 1)\frac{b-a}{2} + a}. \tag{8.27}$$

Definition 8.47. The **Gauss quadrature** for q nodal points is defined in such a way that the approximation

$$\int_{-1}^{1} f(t)\,dt \approx \sum_{i=1}^{q} w_i f(t_i) \tag{8.28}$$

shows an optimal accuracy over the choices of nodal points $\{t_1, t_2, \cdots, t_q\}$ and weights $\{w_1, w_2, \cdots, w_q\}$.

Example 8.48. The Gauss quadrature can be applied for each subinterval of a partition of interval $[a, b]$. Consider

$$\int_{0}^{\pi/2} (\sin x + x^3)\,dx.$$

Use the Gauss quadrature to estimate the integral, with $n = 4,\ 8,\ 16$ and $q = 1,\ 2,\ 3$.

- Compare the estimates with the exact value.
- Compare accuracy with that of the Simpson's Rule.

Note: When $q = 1$, the Gauss quadrature becomes the **midpoint rule**.

Solution.

call_gauss_quadrature.m

```
f = @(x) sin(x)+x.^3;

a=0; b=pi/2; exact=1+(1/4)*b^4;
n=4;

for q=1:3
    GQ(q) = gauss_quadrature(f,a,b,n,q);
end

format shorte
REL_ERROR =abs((GQ-exact)/exact)
```

gauss_quadrature.m

```
function v = gauss_quadrature(f,a,b,n,q)
% n = the number of subintervals
% q = the number of quadrature points in each subinterval

%--- Initial setting -------------------------
h = (b-a)/n; partition = linspace(a,b,n+1);

%--- Standard T and W, on [-1,1] -------------
T = zeros(1,q); W = zeros(1,q);
if     q==1, T = [0];                          W = [2];
elseif q==2, T = [-1/sqrt(3),1/sqrt(3)];    W = [1,1];
elseif q==3, T = [-sqrt(3/5),0,sqrt(3/5)]; W = [5/9,8/9,5/9];
end

%--- Transform P to X & Scale W --------------
X = (T+1)*(h/2); W = W*(h/2);

%--- Now, Gauss Quadrature -------------------
v=0;
for i=1:n
    fX = f( partition(i)+X );
    v = v + sum(fX.*W);
end
```

Output

```
 Output:                  q=1          q=2          q=3
----------------------------------------------------------
 n=4:  REL_ERROR = 1.6300e-02   2.1935e-06   7.2511e-10
 n=8:  REL_ERROR = 4.0771e-03   1.3659e-07   1.1285e-11
 n=16: REL_ERROR = 1.0194e-03   8.5291e-09   1.7644e-13
```

Simpson's Output

```
S_4  = 2.52215; Rel-Error = 5.3364e-05
S_8  = 2.52203; Rel-Error = 3.2892e-06
S_16 = 2.52202; Rel-Error = 2.0487e-07
```

Exercises 8.7

1. CAS Use the Trapezoid Rule and the Simpson's Rule with $n = 4, 8, 16$ to approximate $\int_0^1 \sqrt{1+x^4}\, dx$. How accurate is the Simpson's Rule with $n = 16$?

2. CAS A car laps a race track in 60 seconds. The speed of the car at each 5 second interval is determined by using a radar gun and is given from the beginning of the lap, in feet/second, by the entries in the following table:

Time	0	5	10	15	20	25	30	35	40	25	50	55	60
Speed	0	90	124	142	156	147	133	121	109	99	95	78	89

   How long is the track?

   (a) Use the Trapezoid Rule
   (b) Use the Simpson's Rule

3. CAS The length of one arch of the curve $y = \sin x$ is given by $L = \int_0^\pi \sqrt{1+\cos^2 x}\, dx$. Estimate L by Simpson's Rule with $n = 8$.

4. CAS Estimate the value of
$$\pi = \int_0^1 \frac{4}{1+x^2}\, dx, \tag{8.29}$$
   to 12 decimal places.

   (a) Use the Simpson's Rule to find the number of subintervals.
   (b) Use the Gauss quadrature with $q = 2$ to find the number of subintervals.

8.8. Improper Integrals

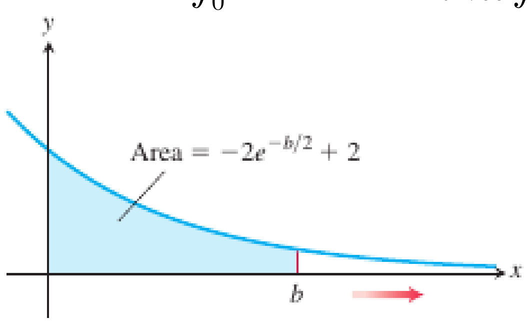
Expansion 8.49. Definite Integrals

- Up to now, we have required **definite integrals** to satisfy two properties:
  1. The **domain of integration** $[a, b]$ must be **finite**.
  2. The **range of the integrand** must be **finite** on this domain.
- In practice, we may encounter problems that fail to meet one or both of these conditions.
- In either case, the integrals are said to be **improper** and are calculated as limits.
  - We will see in Section 8.9 that improper integrals play an important role in probability.

8.8.1. Type I: Infinite Intervals

Example 8.50. Evaluate $\displaystyle\int_0^\infty e^{-x/2}\,dx$.

Solution. $\displaystyle\int_0^\infty e^{-x/2}\,dx = \lim_{b\to\infty}\int_0^b e^{-x/2}\,dx.$



Definition 8.51. Integrals with infinite intervals are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a,\infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty}\int_a^b f(x)\,dx. \tag{8.30}$$

2. If $f(x)$ is continuous on $(-\infty,b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty}\int_a^b f(x)\,dx. \tag{8.31}$$

3. If $f(x)$ is continuous on $(-\infty,\infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \tag{8.32}$$
   where c is any real number.

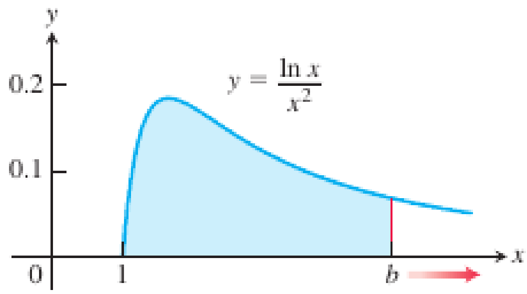
In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral **diverges**.

Example 8.52. Determine whether the integral $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is convergent or divergent.

Solution.

Example 8.53. Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is its value?

Solution.



The Integral $\int_1^{\infty} \frac{dx}{x^p}$

Example 8.54. For what values of p does the integral $\int_1^{\infty} \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

Solution. Consider cases: ① $p = 1$, ② $p \neq 1$

Ans: **When $p > 1$, the improper integral converges to $1/(p-1)$.**

$\boxed{\textbf{Example}}$ **8.55.** Determine whether the integral converges or diverges. Evaluate if it converges.

$$\int_1^{\infty} 10e^{-5x}\,dx$$

Solution.

$\boxed{\textbf{Example}}$ **8.56.** Determine whether the integral converges or diverges. Evaluate if it converges.

$$\int_1^{\infty} \frac{1}{x^2+3x}\,dx$$

Solution.

Example 8.57. Determine whether the integral converges or diverges. Evaluate if it converges.
$$\int_{-\infty}^{2} \frac{2}{x^2+4}\,dx$$
Solution.

Example 8.58. Determine whether the integral converges or diverges. Evaluate if it converges.
$$\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx$$
Solution.

8.8.2. Type II: Discontinuous Integrands

Definition 8.59. Integrals of functions that **become infinite at a point** within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx. \tag{8.33}$$
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx. \tag{8.34}$$
3. If $f(x)$ is continuous on $[a, b]$ except at c, $a < c < b$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{8.35}$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the value of the improper integral. If the limit does not exist, the integral **diverges**.

Example 8.60. Determine whether the integral converges or diverges. Evaluate if it converges.
$$\int_0^{32} \frac{1}{\sqrt[5]{x}}\,dx$$
Solution.

Example 8.61. Determine whether the integral converges or diverges. Evaluate if it converges.

$$\int_0^{\pi/2} \tan x \, dx$$

Solution.

Example 8.62. Determine whether the integral converges or diverges. Evaluate if it converges.

$$\int_{1/2}^{2} \frac{1}{x \ln x} \, dx$$

Solution.

8.8.3. Tests for Convergence and Divergence

Remark 8.63. When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges.

- If the integral diverges, that's the end of the story.
- If it converges, we can use **numerical methods** to approximate its value.
- The principal tests for convergence or divergence are the **Direct Comparison Test** and the **Limit Comparison Test**.

Theorem 8.64. Direct Comparison Test
Let f and g be continuous on $[a,\infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. *If $\int_a^\infty g(x)\,dx$ converges, then $\int_a^\infty f(x)\,dx$ converges.*
2. *If $\int_a^\infty f(x)\,dx$ diverges, then $\int_a^\infty g(x)\,dx$ diverges.*

Example 8.65. These examples illustrate how we use Theorem 8.64.

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges, because $0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$ and

$$\int_1^\infty \frac{1}{x^2}\,dx \quad \text{converges.}$$

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2-0.1}}\,dx$ diverges, because $\dfrac{1}{\sqrt{x^2-0.1}} \ge \dfrac{1}{x}$ and

$$\int_1^\infty \frac{1}{x}\,dx \quad \text{diverges.}$$

(c) $\displaystyle\int_0^{\pi/2} \frac{\cos x}{\sqrt{x}}\,dx$ converges, because $0 \le \dfrac{\cos x}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$ and

$$\int_0^{\pi/2} \frac{1}{\sqrt{x}}\,dx = \lim_{a\to 0^+} 2\sqrt{x}\Big|_a^{\pi/2} = \lim_{a\to 0^+} 2(\sqrt{\pi/2}-\sqrt{a}) = \sqrt{2\pi} \quad \text{converges.}$$

Theorem 8.66. **Limit Comparison Test**
If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty, \tag{8.36}$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either both converge *or* both diverge.

Example 8.67. Consider $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$.

(a) Show that the integral converges by comparison with $\int_1^\infty 1/x^2\,dx$.

(b) Find and compare the two integral values.

Solution.

$\boxed{\textbf{Example}}$ **8.68.** Investigate the convergence of $\displaystyle\int_1^\infty \frac{1-e^{-x}}{x}\,dx$.

Solution.

$\boxed{\textbf{Example}}$ **8.69.** Investigate the convergence of $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$.

Solution.

Exercises 8.8

1. Evaluate the integrals.

(a) $\displaystyle\int_{-\infty}^{-2} \frac{2}{x^2-1}\,dx$ (b) $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{(x^2+1)^2}\,dx$

(c) $\displaystyle\int_{0}^{1} x\ln x\,dx$ (d) $\displaystyle\int_{0}^{2} \frac{ds}{\sqrt{4-s^2}}$

Ans: (b) 0. (c) $-1/4$. (d) $\pi/2$.

2. Use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. (If more than one method applies, use whatever method you prefer.)

(a) $\displaystyle\int_{0}^{1} \frac{\ln x}{x^2}\,dx$ (b) $\displaystyle\int_{-1}^{1} \ln|x|\,dx$

(c) $\displaystyle\int_{1}^{\infty} \frac{1}{x^3+1}\,dx$ (d) $\displaystyle\int_{1}^{\infty} \frac{1}{\sqrt{e^x-x}}\,dx$

Ans: (a) Diverges.

3. Find the values of p for which each integral converges.

(a) $\displaystyle\int_{1}^{2} \frac{dx}{x(\ln x)^p}$ (b) $\displaystyle\int_{2}^{\infty} \frac{dx}{x(\ln x)^p}$

Ans: (a) Converges when $p<1$.

4. Evaluate the integrals.

(a) $\displaystyle\int_{0}^{1} \frac{dt}{\sqrt{t}(1+t)}$ (b) $\displaystyle\int_{0}^{\infty} \frac{dt}{\sqrt{t}(1+t)}$

Ans: (b) π.

5. **$\displaystyle\int_{-\infty}^{\infty} f(x)\,dx$ may not equal $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$.**

(a) Show that $\displaystyle\int_{0}^{\infty} \frac{2x}{x^2+1}\,dx$ diverges.

(b) What can you say about the convergence of $\displaystyle\int_{-\infty}^{\infty} \frac{2x}{x^2+1}\,dx$?

(c) Show that $\displaystyle\lim_{b\to\infty}\int_{-b}^{b} \frac{2x}{x^2+1}\,dx = 0.$

8.9. Probability

8.9.1. Probability Density Functions

Definition 8.70. A **random variable** is a function X that assigns a numerical value to each outcome in a sample space.

- Random variables that have only finitely many values are called **discrete random variables**.
- A **continuous random variable** can take on values in an entire interval, and it is associated with a **distribution function**.

Definition 8.71. A **probability density function (PDF)** for a continuous random variable is a function f defined over $(-\infty, \infty)$ and having the following properties:

1. f is continuous, except possibly at a finite number of points.
2. f is nonnegative, so $f \geq 0$.
3. $\displaystyle\int_{-\infty}^{\infty} f(x)\, dx = 1$.

If X is a continuous random variable with the PDF f, the **probability** that X assumes a value in the interval between $X = c$ and $X = d$ is given by the integral

$$P(c \leq X \leq d) = \int_c^d f(x)\, dx. \tag{8.37}$$

Example 8.72. Let $f(t) = \begin{cases} 0, & \text{if } t < 0, \\ 2e^{-2t}, & \text{if } t \geq 0. \end{cases}$

(a) Verify that f is a PDF.

(b) The time T in hours until a car passes a spot on a remote road is described by the PDF f. Find the probability $P(T \leq 1)$ that a hitchhiker at that spot will see a car within one hour.

Solution.

Exponentially Decreasing Distributions

Definition 8.73. An **exponentially decreasing probability density function** is a PDF of the form

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ ce^{-cx}, & \text{if } x \geq 0, \end{cases} \tag{8.38}$$

for arbitrary $c > 0$.

Expected Values, Means, and Medians

Definition 8.74. The **expected value** or **mean** of a continuous random variable X with PDF f is the number

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx. \tag{8.39}$$

Example 8.75. Find the mean of the random variable X with the exponential PDF

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ ce^{-cx}, & \text{if } x \geq 0. \end{cases}$$

Solution.

Ans: $\mu = 1/c$.

Exponential Density Function for a Random Variable X with Mean μ

$$f(x) = \begin{cases} 0, & \text{if } x < 0, \\ \mu^{-1} e^{-x/\mu}, & \text{if } x \geq 0. \end{cases} \tag{8.40}$$

There are other ways to measure the **centrality** of a random variable with a given PDF.

Definition 8.76. The **median** of a continuous random variable X with PDF f is the number m for which

$$\int_{-\infty}^{m} f(x)\,dx = \frac{1}{2} \quad \text{and} \quad \int_{m}^{\infty} f(x)\,dx = \frac{1}{2}. \tag{8.41}$$

Example 8.77. Find the median of a random variable X with the exponential PDF

$$f(x) = \begin{cases} 0, & \textbf{if } x < 0, \\ ce^{-cx}, & \textbf{if } x \geq 0. \end{cases}$$

Solution.

$$\frac{1}{2} = \int_0^m ce^{-cx}\,dx = -e^{-cx}\Big|_0^m = 1 - e^{-cm}.$$

$$\frac{1}{2} = \int_m^\infty ce^{-cx}\,dx =$$

Ans: $m = \frac{1}{c}\ln 2 = \mu \ln 2.$

Example 8.78. A manufacturer of light bulbs finds that the mean lifetime of a bulb is 1200 hours. Assume the life of a bulb is exponentially distributed.

(a) Find the probability that a bulb will last less than its guaranteed lifetime of 1000 hours.

(b) In a batch of light bulbs, what is the expected time until half the light bulbs in the batch fail?

Solution.

Ans: (a) $1 - e^{-5/6} \approx 0.5654$. (b) median $= \mu \ln 2 = 1200 * \ln 2 \approx 831.7766$ hours.

Example 8.79. The mean waiting time to get served after walking into a bakery is 30 seconds. Assume that an exponential density function describes the waiting times.

(a) What is the probability a customer waits 15 seconds or less?

Ans: $1 - e^{-1/2} \approx 0.393$

(b) What is the probability a customer waits longer than one minute?

Ans: $1 - (1 - e^{-2}) \approx 0.135$

(c) What is the probability a customer waits exactly 5 minutes?

Ans: 0

(d) If 200 customers come to the bakery in a day, how many are likely to be served within three minutes?

Solution.
The probability that each customer is served within three minutes:

$$P(T \le 180) = \int_0^{180} \frac{1}{30} e^{-x/30}\,dx = 1 - e^{-6} \approx 0.997521.$$

The probability that a single customer waits longer than three minutes:

$$1 - (1 - e^{-6})^{200} \approx 0.3912.$$

$\Rightarrow$ ***Most likely, all 200 would be served within three minutes.***

8.9.2. Variance and Standard Deviation

Definition 8.80. The **variance** of a random variable X with probability density function f is the expected value of $(X-\mu)^2$:

$$\mathrm{Var}(X) = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx. \tag{8.42}$$

The **standard deviation** of X is

$$\sigma_X = \sqrt{\mathrm{Var}(X)}. \tag{8.43}$$

Example 8.81. Consider the PDF

$$f(t) = \begin{cases} 0, & \textbf{if } t < 0, \\ 0.1e^{-0.1t}, & \textbf{if } t \ge 0. \end{cases}$$

(a) Find the standard deviation of the random variable T.

(b) Find the probability that T lies within one standard deviation of the mean, $P(\mu-\sigma < T < \mu+\sigma)$.

Solution. $\mu = 10$.

Ans: (a) $\sigma = 10$. (b) $1 - e^{-2} \approx 0.865$.

Remark 8.82. Other Common Distributions

1. The **uniform distribution** is very simple, but it occurs commonly in applications:
$$f(x) = \frac{1}{b-a}, \quad a \le x \le b. \tag{8.44}$$

2. Numerous applications use the **normal distribution**, which is defined by the PDF
$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/2\sigma^2}, \quad x \in \mathbb{R}. \tag{8.45}$$

   In applications the values of the mean μ and the standard deviation σ are often estimated using large sets of data.

Example 8.83. Nearly 1.5 million high school students took the **ACT test** in 2009, and the composite mean score across the academic areas was $\mu = 21.1$ with standard deviation $\sigma = 5.1$.

(a) What percentage of the population had an ACT score between 18 and 24?

(b) What is the ranking of a student who scored 27 on the test?

(c) What is the minimal integer score a student needed to get in order to be in the top 8% of the scoring population?

act_test_2009.m

```
mu = 21.1; sigma=5.1;
f = @(x) (1/(sigma*sqrt(2*pi))) * exp(-(x-mu).^2/(2*sigma^2));

%% (a)
integral(f,18,24)          %output= 0.44355  ---> Ans: about 44%

%% (b)
ACT27 = integral(f,27,36)  %output= 0.12192  ---> Ans: 12%

%% (c)
% We look at how many students had a mark >=28
ACT28 = integral(f,28,36)  %output= 0.086296
% We look at the next higher integer score
ACT29 = integral(f,29,36)  %output= 0.058947 ---> Ans: 29
```

Exercises 8.9

1. Verify that the functions are PDF for a continuous random variable X over the given interval. Determine the specified probability.

   (a) $f(x) = xe^{-x}$ over $[0, \infty)$, $P(1 \le X \le 3)$ (b) $f(x) = \dfrac{3}{2}x(2-x)$ over $[0, 1]$, $P(X < 0.5)$

   Ans: (a) ≈ 0.537.

2. Find the value of the constant c so that the given function is a PDF for a random variable over the specified interval.

   (a) $f(x) = 4e^{-2x}$ over $[0, c]$ (b) $f(x) = cx\sqrt{25 - x^2}$ over $[0, 5]$

   Ans: (a) $\frac{\ln 2}{2}$.

3. Suppose f is a PDF for the random variable X with mean μ. Show that the variance defined in (8.42) satisfies
$$\text{Var}(X) = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2. \tag{8.46}$$

4. **Airport Waiting Time**. According to the U.S. Customs and Border Protection Agency, the average airport wait time at Chicago's *O'Hare International Airport* is 16 minutes for a traveler arriving during the hours 7-8 a.m., and 32 minutes for arrival during the hours 4-5 p.m. The wait time is defined as the total processing time from arrival at the airport until the completion of a passenger's security screening. Assume the wait time is *exponentially distributed*.

   (a) What is the probability of waiting between 10 and 30 minutes for a traveler arriving during the 7-8 a.m. hour?
   (b) What is the probability of waiting less than 20 minutes for a traveler arriving during the 4-5 p.m. hour?

   Formula: For 4-5 p.m arrivals, for example, the PDF is $f(t) = \begin{cases} 0, & \text{if } t < 0, \\ \dfrac{1}{32}e^{-t/32}, & \text{if } t \ge 0. \end{cases}$

5. CAS **Germination of Sunflower Seeds**. The germination rate of a particular seed is the percentage of seeds in the batch which successfully emerge as plants. Assume that the germination rate for a batch of sunflower seeds is 80%, and that among a large population of n seeds the number of successful germinations is *normally distributed* with mean $\mu = 0.8n$ and standard deviation $\sigma = 0.4\sqrt{n}$.

   (a) In a batch of $n = 2500$ seeds, what is the probability that at least 1960 will successfully germinate?
   (b) In a batch of $n = 2500$ seeds, what is the probability that at most 1980 will successfully germinate?
   (c) In a batch of $n = 2500$ seeds, what is the probability that between 1940 and 2020 will successfully germinate? *Ans*: 0.83999.

CHAPTER **9**

First-Order Differential Equations

In this chapter, ...

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TBA

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Sequences, Series, and Vector-Valued Functions

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CHAPTER **11**

Parametric Equations and Polar Coordinates

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Vectors and the Geometry of Space

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13.1. Curves in Space and Their Tangents

CALCULUS IV

Vector Calculus

What is Vector Calculus?

- A type of advanced mathematics, which has practical applications in physics and engineering (and other sciences)
- Concerned with differentiation and integration of vector fields.

Covers

Lecture Note: https://skim.math.msstate.edu/LectureNotes/Calculus-Lectures.pdf

- **Appendix A**. Vectors and the Geometry of Space: Preliminaries
  - A brief review of Chapter 12.
- **Chapter 14**. Partial Derivatives
  - Expansion of Differentiation (Calculus I)
- **Chapter 15**. Multiple Integrals
  - Expansion of Integration (Calculus II)
- **Chapter 16**. Integrals and Vector Fields
  - Applications to real-world problems in 2D/3D
  - Focusing on problems in physics and engineering

Appendix A

Vectors and the Geometry of Space: Preliminaries

In **vector calculus**, you will be frequently required to deal with

- **vectors** and various **geometric objects**
- in 2-dimensional (2D) and 3-dimensional (3D) spaces.

This appendix reviews vectors and equations of 3D geometric objects.
In particular, you will learn

- vectors
- dot product
- cross product
- equations of lines and planes, and
- cylinders and quadric surfaces

Contents of Chapter A

A.1. Vector Operations

There exists a lot to cover in the class of **vector calculus**; however, it is important to have a good foundation before we trudge forward. In that vein, let's review vectors and their geometry in space ($\mathbb{R}^3$) briefly.

A.1.1. 3D coordinate systems

Recall: Let $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ be points in $\mathbb{R}^2$. Then the **distance** from P to Q is

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}. \tag{A.1}$$

Definition A.1. *Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$ be points in $\mathbb{R}^3$. Then the **distance** from P to Q is*

$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{A.2}$$

Self-study A.2. Find the distance between $P(-3, 2, 7)$ and $Q(-1, 0, 6)$.

Solution.

Ans: 3

Recall: A **circle** in $\mathbb{R}^2$ is defined to be all of the points in the plane ($\mathbb{R}^2$) that are **equidistant** from **a central point** $C(a, b)$.

$$(x-a)^2 + (y-b)^2 = r^2. \tag{A.3}$$

A natural generalization of this to 3-D space would be to say that a sphere is defined to be all of the points in $\mathbb{R}^3$ that are equidistant from a central point C. This is exactly what the following definition does!

Definition A.3. *Let $C(h, k, l)$ be a point in $\mathbb{R}^3$. Then the* ***sphere*** *centered at C with radius r is defined by the equation*

$$(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2. \tag{A.4}$$

That is to say that this defines all points $(x, y, z) \in \mathbb{R}^3$ that are at the same distance r from the center $C(h, k, l)$.

Problem A.4. Show that $x^2 + y^2 + z^2 - 4x + 2y - 6z + 10 = 0$ is the equation of a sphere, and find its **center and radius**.

Solution.

Ans: $C(2, -1, 3)$ and $r = 2$

A.1.2. Vectors and vector operations

Definition A.5. *A **vector** is a mathematical object that stores both **length** (magnitude) and **direction**.*

Let $P = (x_1, y_1, z_1)$ and $Q = (x_2, y_2, z_2)$. Then the *vector* with initial point P and terminal point Q (denoted $\overrightarrow{PQ}$) is defined by

$$\overrightarrow{PQ} = \langle x_2 - x_1, y_2 - y_1, z_2 - z_1 \rangle = \overrightarrow{OQ} - \overrightarrow{OP},$$

where O is the origin, $O = (0, 0, 0)$. The vector $\overrightarrow{OP}$ is called the **position vector** of the point P. For convenience, we use bold-faced lower-case letters to denote vectors. For example, $\mathbf{v} =< v_1, v_2, v_3 >$ is a (position) vector in $\mathbb{R}^3$ associated with the point (v_1, v_2, v_3).

Definition A.6. *Two vectors are said to be **equal** if and only if they have the same length and direction, regardless of their position in $\mathbb{R}^3$. That is to say that a vector can be moved (with no change) anywhere in space as long as the magnitude and direction are preserved.*

Definition A.7. *Let $\mathbf{v} =< v_1, v_2, v_3 >$. Then the **magnitude** (a.k.a. **length** or **norm**) of $\mathbf{v}$ (denoted $|\mathbf{v}|$ or sometimes $||\mathbf{v}||$) is defined by*

$$|\mathbf{v}| = \sqrt{v_1^2 + v_2^2 + v_3^2}. \tag{A.5}$$

Definition A.8. *(Vector addition) Let $\mathbf{u} =< u_1, u_2, u_3 >$ and $\mathbf{v} =< v_1, v_2, v_3 >$. Then*

$$\mathbf{u} + \mathbf{v} =< u_1 + v_1, u_2 + v_2, u_3 + v_3 > .$$

Definition A.9. *(Scalar multiplication) Let* $\mathbf{v} =< v_1, v_2, v_3 >$ *and* $k \in \mathbb{R}$. *Then*

$$k\,\mathbf{v} =< kv_1, kv_2, kv_3 > .$$

Problem A.10. If $\mathbf{a} =< 0, 3, 4 >$ and $\mathbf{b} =< 1, 5, 2 >$, find $|\mathbf{a}|$, $2\mathbf{a} - 3\mathbf{b}$, and $|2\mathbf{a} - 3\mathbf{b}|$.

Solution.

Ans: $|\mathbf{a}| = 5$; $2\mathbf{a} - 3\mathbf{b} =< -3, -9, 2 >$; $|2\mathbf{a} - 3\mathbf{b}| = \sqrt{94}$

Definition A.11. *A* ***unit vector*** *is a vector whose magnitude is 1. Note that given a vector* $\mathbf{v}$, *we can form a unit vector (of the same direction) by dividing by its magnitude. That is, let* $\mathbf{v} =< v_1, v_2, v_3 >$. *Then*

$$\mathbf{u} = \frac{\mathbf{v}}{|\mathbf{v}|} \tag{A.6}$$

is a unit vector in the direction of $\mathbf{v}$.

Definition A.12. *Any vector can be denoted as the linear combination of the* ***standard unit vectors***

$$\mathbf{i} =< 1, 0, 0 >, \quad \mathbf{j} =< 0, 1, 0 >, \quad \mathbf{k} =< 0, 0, 1 > .$$

So given a vector $\mathbf{v} =< v_1, v_2, v_3 >$, one can express it with respect to the standard unit vectors as

$$\mathbf{v} =< v_1, v_2, v_3 >= v_1\,\mathbf{i} + v_2\,\mathbf{j} + v_3\,\mathbf{k}. \tag{A.7}$$

This text, however, will more often than not use the angle brace notation.

Definition A.13. *Let* $\mathbf{u} =< u_1, u_2, u_3 >$ *and* $\mathbf{v} =< v_1, v_2, v_3 >$*. Then the* **dot product** *is*

$$\mathbf{u} \cdot \mathbf{v} = u_1\, v_1 + u_2\, v_2 + u_3\, v_3, \tag{A.8}$$

which is sometimes referred as the ***Euclidean inner product****. Note that* $\mathbf{v} \cdot \mathbf{v} = |\mathbf{v}|^2$*.*

Theorem A.14. *Let* θ *be the angle between* $\mathbf{u}$ *and* $\mathbf{v}$ *(so* $0 \le \theta \le \pi$*). Then*

$$\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\, |\mathbf{v}|\, \cos(\theta). \tag{A.9}$$

Corollary A.15. *Two vectors* $\mathbf{u}$ *and* $\mathbf{v}$ *are orthogonal if and only if* $\mathbf{u} \cdot \mathbf{v} = 0$*.*

Problem A.16. Find the angle between the vectors $\mathbf{a} =< 2, 2, 1 >$ and $\mathbf{b} =< 3, 0, 3 >$.

Solution.

Ans: $\pi/4\,(= 45°)$

Definition A.17. *Let* $\mathbf{u} =< u_1, u_2, u_3 >$ *and* $\mathbf{v} =< v_1, v_2, v_3 >$. *Then the* ***cross product*** *is the determinant of the following matrix:*

$$\begin{aligned} \mathbf{u} \times \mathbf{v} &= \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} \\ &= \det \begin{bmatrix} u_2 & u_3 \\ v_2 & v_3 \end{bmatrix} \mathbf{i} - \det \begin{bmatrix} u_1 & u_3 \\ v_1 & v_3 \end{bmatrix} \mathbf{j} + \det \begin{bmatrix} u_1 & u_2 \\ v_1 & v_2 \end{bmatrix} \mathbf{k} \\ &= < u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 > . \end{aligned} \tag{A.10}$$

Problem A.18. Find the cross product $\mathrm{a} \times \mathrm{b}$, when $\mathrm{a} =< 1, 3, 4 >$ and $\mathrm{b} =< 3, -1, -2 >$.

Solution.

Ans: $< -2, 14, -10 >$

Theorem A.19. *The vector* $\mathrm{a} \times \mathrm{b}$ *is orthogonal to both* a *and* b.

Theorem A.20. *Let* θ *be the angle between* a *and* b *(so* $0 \le \theta \le \pi$*). Then*

$$|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}|\, |\mathbf{b}| \sin \theta. \tag{A.11}$$

Claim A.21. *The length of the cross product* $\mathrm{a} \times \mathrm{b}$ *is equal to the area of the parallelogram determined by* a *and* b.

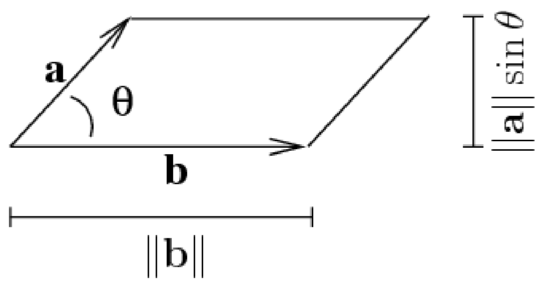


Figure A.1

Problem A.22. Prove that two nonzero vectors a and b are parallel if and only if $\mathrm{a} \times \mathrm{b} = 0$.

Solution.

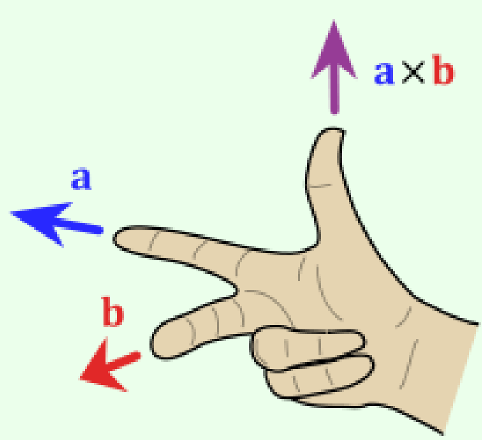


Figure A.2: Finding the direction of the cross product by the right-hand rule.

The **cross product** $\mathrm{a} \times \mathrm{b}$ is defined as a vector that is perpendicular (orthogonal) to both a and b, with a direction given by the **right-hand rule** and a magnitude equal to the area of the parallelogram that the vectors span.
If the fingers of your right hand curl in the direction of a rotation (through an angle less than $180°$) from a to b, then the thumb points in the direction of $\mathrm{a} \times \mathrm{b}$.

Exercises A.1

1. Find the cross product $\mathbf{a} \times \mathbf{b}$ and verify that it is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

   (a) $\mathbf{a} =< 1, 2, -1 >$, $\mathbf{b} =< 2, 0, -3 >$
   (b) $\mathbf{a} =< 1, t, 1/t >$, $\mathbf{b} =< t^2, t, 1 >$

2. Find $|\mathbf{u} \times \mathbf{v}|$ and determine whether $\mathbf{u} \times \mathbf{v}$ is directed into the page or out of the page.

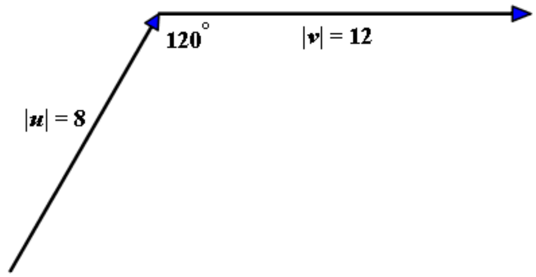


Figure A.3

3. (i) Find a nonzero vector orthogonal to the plane through the points P, Q, and R, and (ii) find the area of the triangle PQR.

   (a) $P(1, 0, 1)$, $Q(2, 1, 3)$, $R(-3, 2, 5)$

   Ans: $< 0, -12, 6 >$, $3\sqrt{5}$

   (b) $P(1, -1, 0)$, $Q(-3, 1, 2)$, $R(0, 3, -1)$

   Ans: $< -10, -6, -14 >$, $\sqrt{83}$

4. Find the angle between $\mathbf{a}$ and $\mathbf{b}$, when $\mathbf{a} \cdot \mathbf{b} = -\sqrt{3}$ and $\mathbf{a} \times \mathbf{b} =< 2, 2, 1 >$.

   Ans: $120°$

Note: Exercise problems are added for your homework; answers would be provided for some of them. However, you have to verify them, by showing solutions in detail.

A.2. Equations in the 3D Space

Objective: To build equations of lines, line segments, and planes.

Parametrization of a Line. Let $P_0 = (x_0, y_0, z_0)$ be a point in $\mathbb{R}^3$, and $\mathbf{v} = \langle a, b, c\rangle$ be a vector in $\mathbb{R}^3$. Then the line through P_0 parallel to $\mathbf{v}$ is

$$\mathbf{r} = P_0 + t\,\mathbf{v}, \quad t \in \mathbb{R}. \tag{A.12}$$

Since $\mathbf{r} = [x, y, z]^T$, this can also be written as (**parametric equation**)

$$x = x_0 + at, \quad y = y_0 + bt, \quad z = z_0 + ct; \quad t \in \mathbb{R}. \tag{A.13}$$

or as the **symmetric equation**

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}. \tag{A.14}$$

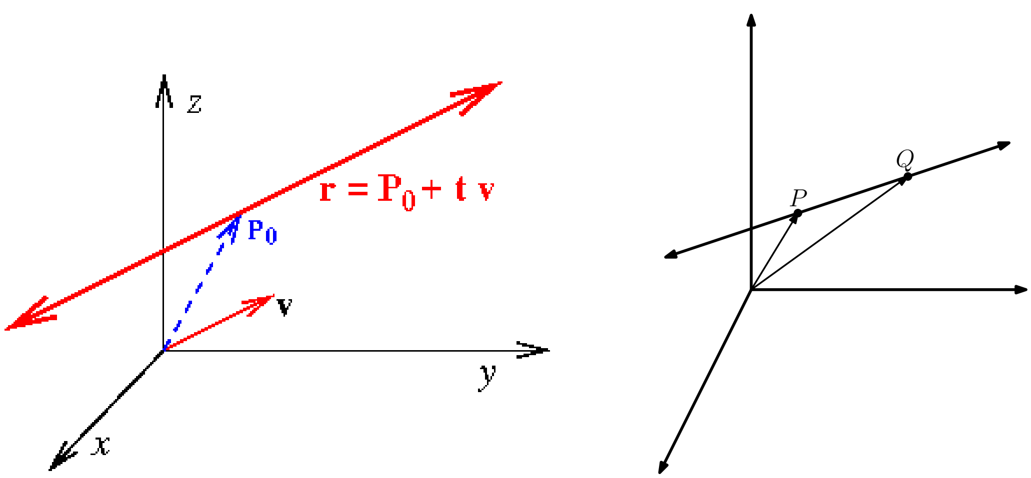


Figure A.4: Parametrization: (left) line and (right) line segment.

Parametrization of a Line Segment. Let P and Q be respectively the initial and terminal points of a line segment. Then the line segment $\overline{PQ}$ can be parametrized as

$$\mathbf{r}(t) = (1 - t)\,\overrightarrow{OP} + t\,\overrightarrow{OQ}, \quad 0 \le t \le 1. \tag{A.15}$$

Problem A.23. Find a vector equation and parametric equation for the line that passes through the point $(5, 1, 3)$ and is parallel to $\langle 1, 4, -2\rangle$.

Solution.

Ans: $x = 5 + t$, $y = 1 + 4t$, $z = 3 - 2t$

Problem A.24. Find the parametric equation of the line segment from $(2, 4, -3)$ to $(3, -1, 1)$.

Solution.

Ans: $\mathbf{r}(t) = \langle 2 + t, 4 - 5t, -3 + 4t\rangle$, $0 \le t \le 1$.

Planes. Let $\mathbf{x}_0 = (x_0, y_0, z_0)$ be a point in the plane and $\mathbf{n} = \langle a, b, c\rangle$ be a vector normal to the plane. Then the equation of the plane is

$$\mathbf{n} \cdot (\mathbf{x} - \mathbf{x}_0) = a\,(x - x_0) + b\,(y - y_0) + c\,(z - z_0) = 0. \tag{A.16}$$

Problem A.25. Find an equation of the plane that passes through the points $P(1, 2, 3)$, $Q(3, 2, 4)$, and $R(1, 5, 2)$.

Solution.

Ans: $-3(x - 1) + 2(y - 2) + 6(z - 3) = 0$.

Exercises A.2

1. Find an equation of the line which passes through $(1, 0, 3)$ and *perpendicular* to the plane $x - 3y + 2z = 4$.

2. Find the line of the intersection of planes $x + 2y + 3z = 6$ and $x - y + z = 1$. (***Hint***: The intersection is a line; consider how the direction of the line is related to the normal vectors of the planes.)

   Ans: $\mathbf{r} = P_0 + t\mathbf{v} =< 1, 1, 1 > +t < 5, 2, -3 >$

3. Find the vector equation for the line segment from $P(1, 2, -4)$ to $Q(5, 6, 0)$.

4. Find an equation of the plane.

   (a) The plane through the point $(0, 1, 2)$ and *parallel* to the plane $x - y + 2z = 4$.

   (b) The plane through the points $P(1, -2, 2)$, $Q(3, -4, 0)$, and $R(-3, -2, -1)$.

   Ans: $3(x - 1) + 7(y + 2) - 4(z - 2) = 0$.

5. Use intercepts to help sketch the plane $2x + y + 5z = 10$.

A.3. Cylinders and Quadric Surfaces

Objective: To sketch and visualize surfaces, given their equations.

Definition A.26. *A **cylinder** is a surface that consists of* **all lines** *that are parallel to a given line and* **pass through a given plane curve***.*

Problem A.27. Sketch $z = x^2$ in $\mathbb{R}^3$.

Problem A.28. Sketch $x^2 + y^2 = 1$ in $\mathbb{R}^3$.

Problem A.29. Sketch $y^2 + z^2 = 1$ in $\mathbb{R}^3$.

Definition A.30. *A* ***quadric surface*** *is the graph of a second-degree equation in three variables $x, y,$ and z. By translation and rotation, we can write the standard form of a quadric surface as*

$$Ax^2 + By^2 + Cz^2 + J = 0 \quad \textit{or} \quad Ax^2 + By^2 + Iz = 0. \tag{A.17}$$

Definition A.31. *The* ***trace*** *of a surface in $\mathbb{R}^3$ is the graph in $\mathbb{R}^2$ obtained by allowing one of the variables to be a specific real number. For example, $x = a$.*

Problem A.32. Use the traces to sketch $x^2 + \dfrac{y^2}{9} + \dfrac{z^2}{4} = 1$.

Problem A.33. Use the traces to sketch $z = 4x^2 + y^2$.

Exercises A.3

1. Sketch the surface.

   (a) $x^2 + y^2 = 1$
   (b) $x^2 + y^2 - 2y = 0$
   (c) $z = \sin x$

2. Use traces to sketch and identify the surface.

   (a) $z = y^2 - x^2$
   (b) $4y^2 + 9z^2 = x^2 + 36$

3. Sketch the region bounded by the surfaces $z = \sqrt{x^2 + y^2}$ and $z = 2 - x^2 - y^2$.

4. Sketch the surface obtained by rotating the line $\mathbf{r}(t) = \langle 0, 1, 3 \rangle t$ about the z-axis; find an equation of it. (***Hint***: The line can be expressed as $\{z = 3y,\ \ x = 0\}$.)

   Ans: $|z| = 3\sqrt{x^2 + y^2}$ or $z^2 = 9(x^2 + y^2)$

CHAPTER **14**

Partial Derivatives

In mathematics, a **partial derivative** of a function of several variables is its derivative with respect to one of those variables, with the others held constant. In this chapter, you will learn about the partial derivatives and their applications.

Subjects	Applications
Limits and continuity Partial derivatives	
	Tangent planes & linear approximations
Chain rule Directional derivatives and the Gradient Vector	
	Maximum and minimum values Method of Lagrange multipliers

Contents of Chapter 14

14.1. Functions of Several Variables

14.1.1. Domain and range

Definition 14.1. *A **function of two variables**, f, is a rule that assigns each ordered pair of real numbers (x, y) in a set $D \subset \mathbb{R}^2$ a unique real number denoted by $f(x, y)$. The set D is called the **domain** of f and its **range** is the set of values that f takes on, that is, $\{f(x, y) : (x, y) \in D\}$.*

Definition 14.2. *Let f be a function of two variables, and $z = f(x, y)$. Then x and y are called **independent variables** and z is called a **dependent variable**.*

Problem 14.3. Let $f(x, y) = \dfrac{\sqrt{x+y+1}}{x-1}$. Evaluate $f(3, 2)$ and give its domain.

Ans: $f(3, 2) = \sqrt{6}/2$; $D = \{(x, y) : x + y + 1 \geq 0,\ \ x \neq 1\}$

Problem 14.4. Find the domain of $f(x, y) = x \ln\left(y^2 - x\right)$.

Problem 14.5. Find the domain and the range of $f(x, y) = \sqrt{9 - x^2 - y^2}$.

14.1.2. Graphs

Definition 14.6. *If f is a function of two variables with domain D, then the **graph** of f is the set of all points $(x, y, z) \in \mathbb{R}^3$ such that $z = f(x, y)$ for all $(x, y) \in D$.*

Problem 14.7. Sketch the graph of $f(x, y) = 6 - 3x - 2y$.

Solution. The graph of f has the equation $z = 6-3x-2y$, or $3x+2y+z = 6$, which is a plane. Now, we can find intercepts to graph the plane.

Problem 14.8. Sketch the graph of $g(x, y) = \sqrt{9 - x^2 - y^2}$.

Solution. The graph of g has the equation $z = \sqrt{9 - x^2 - y^2}$, or $x^2+y^2+z^2 = 9$, $z \geq 0$, which is a upper hemi-sphere.

14.1.3. Level curves

Definition 14.9. *The **level curves** of a function of two variables, f, are the curves with equations $f(x,y)=k$, for $k \in K \subset$ Range(f).*

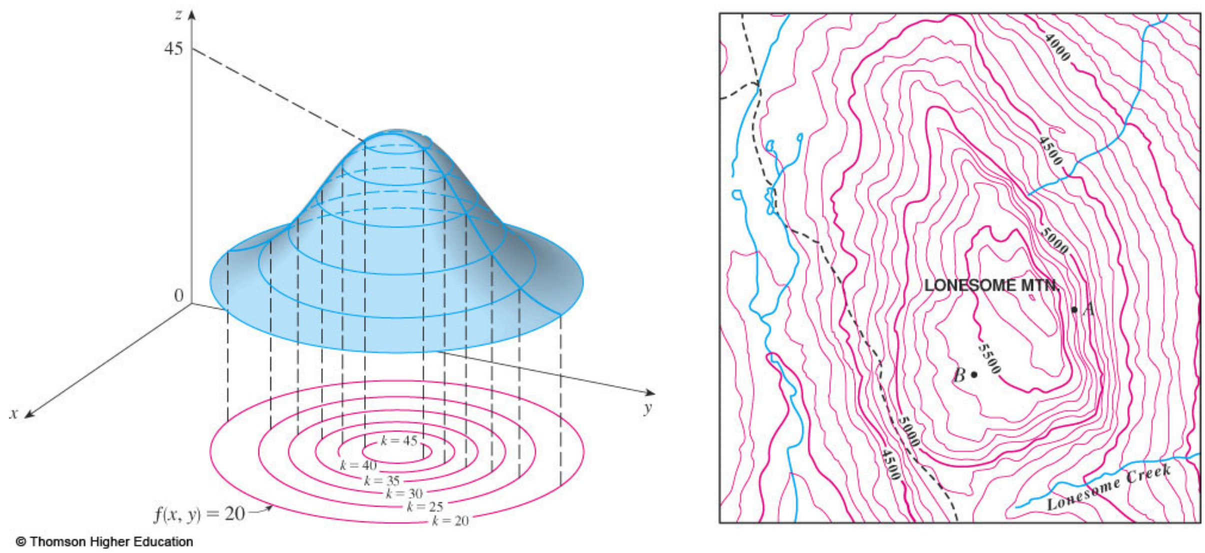


Figure 14.1: Level curves: (left) the graph of a function vs. level curves and (right) a topographic map of a mountainous region. Level curves are often considered for an effective visualization.

Problem 14.10. Sketch the level curves of $f(x,y) = 6 - 3x - 2y$ for $k \in \{-6, 0, 6, 12\}$.

Problem 14.11. Sketch the level curves of $g(x,y) = \sqrt{9 - x^2 - y^2}$ for $k \in \{0, 1, 2, 3\}$

Problem 14.12. Sketch the level curves of $h(x,y) = 4x^2 + y^2 + 1$.

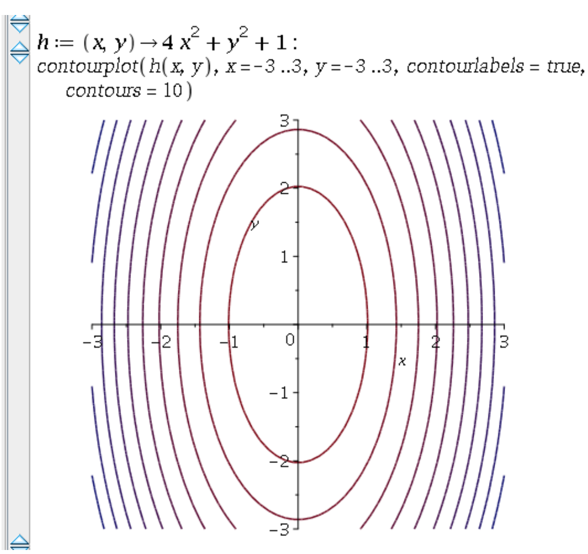


Figure 14.2

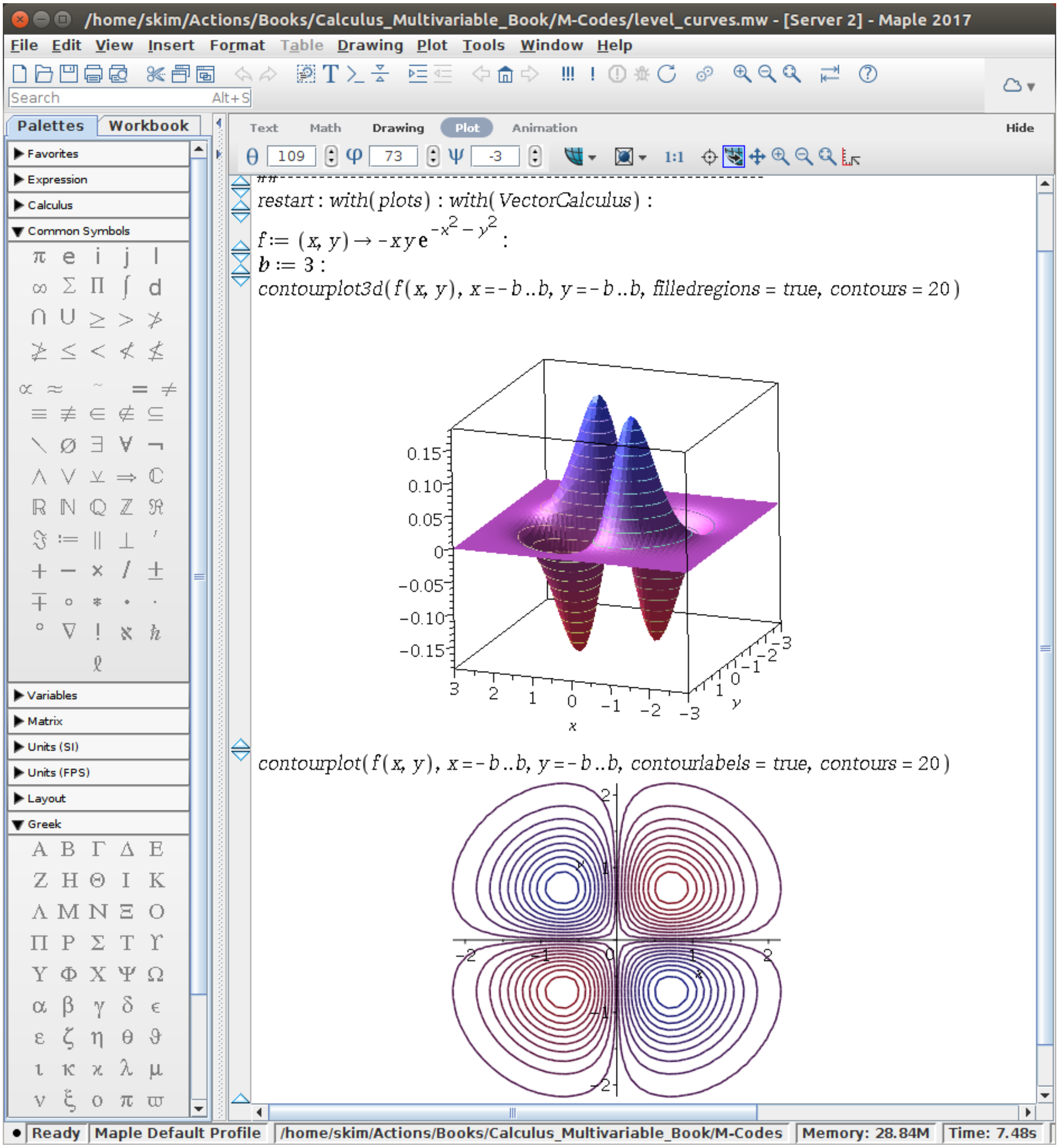


Figure 14.3: Computer-generated level curves.

Function visualization is now easy with e.g., Mathematica, Maple, and Matlab, as shown in Figure 14.3.[1]

[1]For plotting with Maple, you may exploit `plot`, `plot3d`, `contourplot3d`, and `contourplot`, which are available from the `plots` package. Maple can include packages with the `with` command, as in Figure 14.3.

14.1.4. Functions of three or more variables

Definition 14.13. *A* ***function of three variables****, f, is a rule that assigns each ordered pair of real numbers (x, y, z) in a set $D \subset \mathbb{R}^3$ a unique real number denoted by $f(x, y, z)$.*

Problem 14.14. Find the domain of f if $f(x, y, z) = \ln(z - y) + xy \sin z$.

Level Surfaces

Problem 14.15. Find the **level surfaces** (:=$f(x, y, z) = k$) of

$$f(x, y, z) = x^2 + y^2 + z^2.$$

Solution.

A level surface is the surface where the function values are all the same. Thus the **outer normal** is the **fastest increasing direction** of f.

Exercises 14.1

1. Find and sketch the domain of the function

   (a) $f(x, y) = \ln(9 - 9x^2 - y^2)$

   (b) $g(x, y) = \dfrac{\sqrt{x - y^2}}{1 - y^2}$

2. Let $f(x, y) = \sqrt{4 - x^2 - 4y^2}$.

   (a) Find the *domain* of f.

   (b) Find the *range* of f.

   (c) Sketch the *graph* of the function.

3. Match the function with its contour plot (labeled I–VI). Give reasons for your choices.

   (a) $f(x, y) = x^2 - y^2$

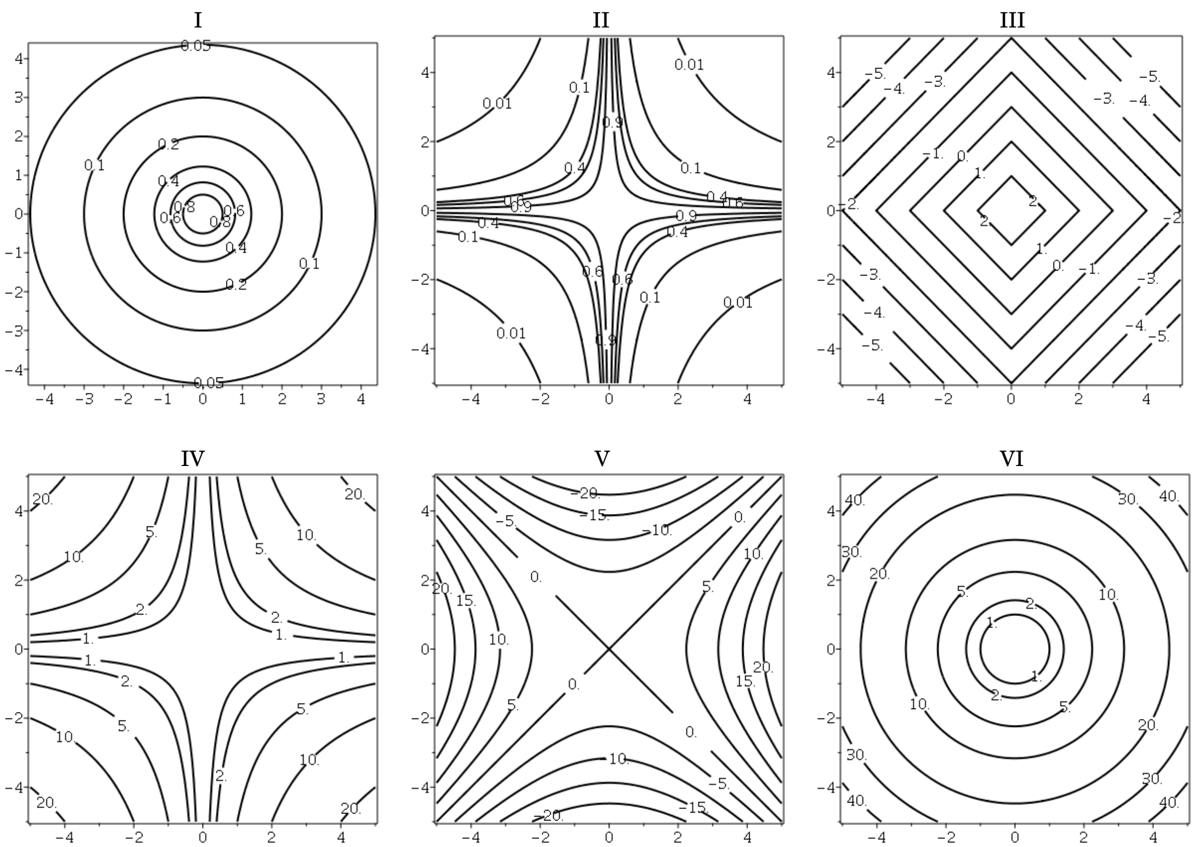
   (b) $f(x, y) = x^2 + y^2$

   (c) $f(x, y) = 3 - |x| - |y|$

   (d) $f(x, y) = |xy|$

   (e) $f(x, y) = \frac{1}{1+x^2+y^2}$

   (f) $f(x, y) = \frac{1}{1+x^2y^2}$



4. Describe the level surfaces of the function $f(x, y, z) = x^2 + y^2 - z^2$.

14.2. Limits and Continuity

Limits

Recall: **For $y = f(x)$**, we say that the **limit** of $f(x)$, as $x \to a$, is L, if

$$\lim_{x\to a^-} f(x) = L = \lim_{x\to a^+} f(x),$$

or, equivalently, if $\forall\, \varepsilon > 0$, **there exists $\delta = \delta(\varepsilon) > 0$ such that**

$$\textbf{if } 0 < |x-a| < \delta \textbf{ then } |f(x)-L| < \varepsilon. \tag{14.1}$$

In this case, we write

$$\lim_{x\to a} f(x) = L. \tag{14.2}$$

The above argument involving ε and δ is called the **ε-δ argument**.

Definition 14.16. *Let f be a function of two variables whose domain D includes points arbitrarily close to (a,b). Then we say that the **limit** of $f(x,y)$, as (x,y) approaches (a,b), is L:*

$$\lim_{(x,y)\to(a,b)} f(x,y) = L, \tag{14.3}$$

if $\forall\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\textit{if } (x,y) \in D \textit{ and } 0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta \textit{ then } |f(x,y) - L| < \varepsilon.$$

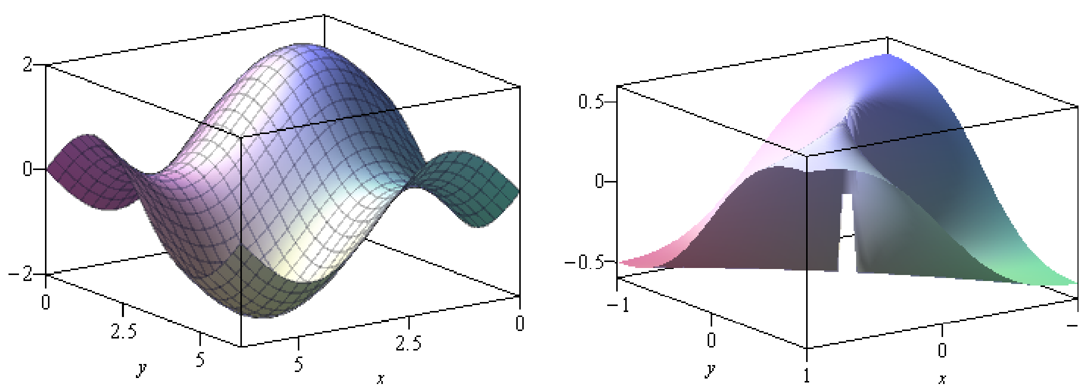


Figure 14.4: Plots of $z = \sin x + \sin y$ (left) and $z = \dfrac{xy}{x^2+y^2}$ (right).

When the Limit Does Not Exist

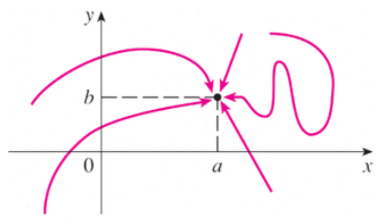


Figure 14.5

Claim 14.17. If $f(x,y) \to L_1$ as $(x,y) \to (a,b)$ along a path C_1 and $f(x,y) \to L_2$ as $(x,y) \to (a,b)$ along a path C_2, where $L_1 \neq L_2$, then $\lim_{(x,y)\to(a,b)} f(x,y)$ does not exist.

Problem 14.18. Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution. Consider two paths: e.g., $C_1 : \{y = 0\}$ and $C_2 : \{x = 0\}$.

Ans: no

Problem 14.19. Does $\displaystyle\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2}$ exist?

Solution. Consider a path $C : \{x = y\}$ with another.

Ans: no

Problem 14.20. Does $\lim_{(x,y)\to(1,1)} \dfrac{2xy}{x^2+y^2}$ exist?

Solution.

Ans: yes: $L = 1$

Problem 14.21. Does $\lim_{(x,y)\to(0,0)} \dfrac{xy^2}{x^2+y^4}$ exist?

Solution. Consider a path $C : \{x = y^2\}$ with another.

Ans: no. See Figure 14.6 on p. 422 below.

The Existence of a Limit

Problem 14.22. Use the **squeeze theorem** to show $\displaystyle\lim_{(x,y)\to(0,0)} \frac{3x^2y}{x^2+y^2} = 0.$

Solution.

Problem 14.23. Find the limit: $\displaystyle\lim_{(x,y)\to(0,0)} (x^2+y^2)\,\ln(x^2+y^2).$

Solution. Consider $\lim_{x\to 0} x\ln x$ and introduce a new variable $s = x^2+y^2$.

Ans: $L = 0$

Continuity

Recall: A function (of a single variable) f is **continuous** at $x = a$ if

$$\lim_{x\to a} f(x) = f(a).$$

The above means that

1. the limit on the left side exists,
2. $f(a)$ is defined, and
3. they are the same.

Definition 14.24. *A function of two variables f is called* ***continuous at point*** $(a, b) \in \mathbb{R}^2$ *if*

$$\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b). \tag{14.4}$$

If f is continuous at every point (x, y) in a region $D \subset \mathbb{R}^2$, then we say that f is ***continuous on*** D.

Problem 14.25. Is $f(x, y) = \dfrac{2xy}{x^2 + y^2}$ continuous at $(0, 0)$? What about at $(1, 1)$? Why?

Solution. See Problems 14.19 and 14.20.

Ans: no; yes

Problem 14.26. Is the following function continuous at $(0, 0)$? What about at elsewhere?

$$g(x, y) = \begin{cases} \frac{3x^2y}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \tag{14.5}$$

Solution. See Problem 14.22.

Ans: It is continuous everywhere.

Computer algebra

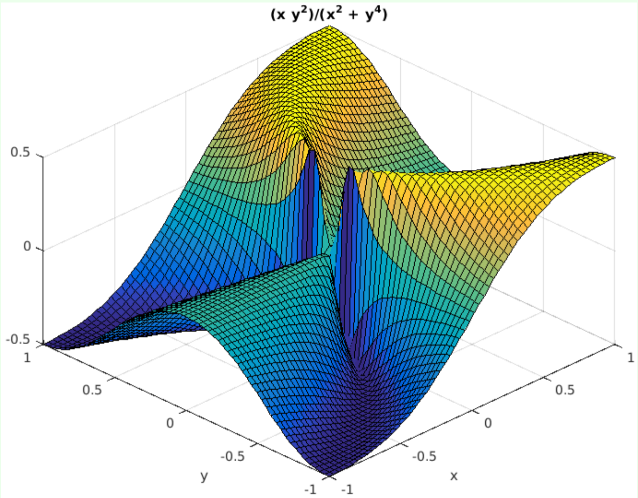


Figure 14.6: Matlab plot: using `ezsurf` for Problem 14.21, p. 419.

In computational mathematics, **computer algebra** (a.k.a. **symbolic computation**) refers to the study and development of algorithms and software for ***manipulating mathematical expressions***. It emphasizes **exact computation** with expressions containing variables that are manipulated as symbols.
There have been more than 40 computer algebra systems available; popular ones are **Maple, Mathematica, Matlab, and Python.**

Matlab script

```
1 syms x y
2
3 f = x*y^2/(x^2+y^4);
4 ezsurf(f,[-1,1,-1,1])
5 view(-45,45)
6 print('-r100','-dpng','matlab_ezsurf.png');
```

The above Matlab script results in Figure 14.6. Line 1 declares symbolic variables `x y`; line 3 defines the function f; line 4 plots a figure over the rectangular domain $[-1,1]\times[-1,1]$; line 5 changes the view angle to $(-45°, 45°)$ in the horizontal and vertical directions, respectively; and the final line saves the figure to `matlab_ezsurf.png` with the resolution level of 100.

Exercises 14.2

1. Find the limit, if it exists, or show that the limit does not exist.

   (a) $\displaystyle\lim_{(x,y)\to(\pi,\pi/2)} x\cos(x-y)$

   Ans: 0

   (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x}{\sqrt{x^2+y^2}}$

   (c) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}}$

   Ans: 0

2. Use polar coordinates to find the limit.

   (a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin(x^2+y^2)}{x^2+y^2}$ (b) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{4+x^2+y^2}-2}$

   Ans: (a) 1; (b) 4

3. CAS Use a computer graph of the function to explain why the limit does not exist.[2]

   $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2+2xy+4y^2}{3x^2+y^2}$

4. Determine and ***verify*** whether the following functions are continuous at $(0,0)$ or not.

   (a) $f(x,y) = \begin{cases} \dfrac{x^4\sin y}{x^4+y^4}, & \text{if } (x,y)\neq(0,0), \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

   Ans: continuous

   (b) $g(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2}, & \text{if } (x,y)\neq(0,0), \\ 0, & \text{if } (x,y)=(0,0). \end{cases}$

   Ans: discontinuous

[2]You have to perform a computer implementation for problems indicated by the *Computer Algebra System* sign CAS. Of course, you must copy-and-paste your implementation and results to the report.

14.3. Partial Derivatives

14.3.1. First-order partial derivatives

Recall: A function $y = f(x)$ is **differentiable** at a if

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h} \quad \text{exists.}$$

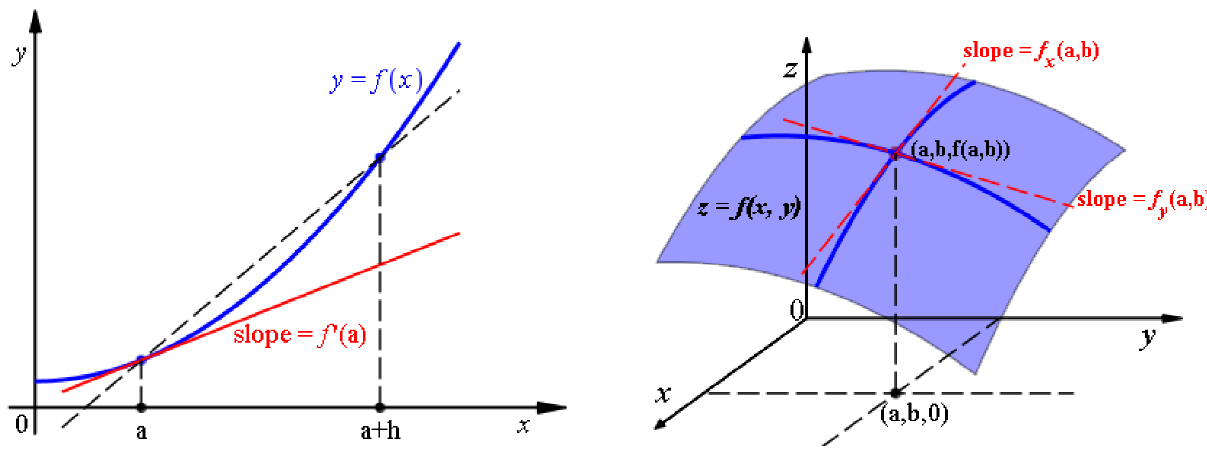


Figure 14.7: Ordinary derivative $f'(a)$ and partial derivatives $f_x(a,b)$ and $f_y(a,b)$.

Let f be a function of two variables (x, y). Suppose we let only x vary while keeping y fixed, say $\boldsymbol{y = b}$. Then $\boldsymbol{g(x) := f(x, b)}$ is a function of a single variable. If g is differentiable at a, then we call it the **partial derivative of f with respect to x at (a, b)** and denoted by $f_x(a, b)$.

$$\begin{aligned} g'(a) &= \lim_{h\to 0} \frac{g(a+h) - g(a)}{h} \\ &= \lim_{h\to 0} \frac{f(a+h, b) - f(a, b)}{h} =: f_x(a, b). \end{aligned} \tag{14.6}$$

Similarly, the **partial derivative of f with respect to y at (a, b)**, denoted by $f_y(a, b)$, is obtained keeping x fixed, say $\boxed{\boldsymbol{x = a}}$, and finding the ordinary derivative at b of $\boxed{\boldsymbol{G(y) := f(a, y)}}$:

$$\begin{aligned} G'(b) &= \lim_{h\to 0} \frac{G(b+h) - G(b)}{h} \\ &= \lim_{h\to 0} \frac{f(a, b+h) - f(a, b)}{h} =: f_y(a, b). \end{aligned} \tag{14.7}$$

Problem 14.27. Find $f_x(0, 0)$, when $f(x, y) = \sqrt[3]{x^3 + y^3}$.

Solution. Using the definition,

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h, 0) - f(0, 0)}{h}$$

Ans: 1

Definition 14.28. *If f is a function of two variables, its* ***partial derivatives*** *are the functions $f_x = \frac{\partial f}{\partial x}$ and $f_y = \frac{\partial f}{\partial y}$ defined by:*

$$\begin{aligned} f_x(x, y) &= \lim_{h\to 0} \frac{f(x+h, y) - f(x, y)}{h} \quad \textit{and} \\ f_y(x, y) &= \lim_{h\to 0} \frac{f(x, y+h) - f(x, y)}{h}. \end{aligned} \tag{14.8}$$

Observation 14.29. **The partial derivative with respect to x** represents **the slope of the tangent line to the curve** that are parallel to the xz-plane (i.e. in the direction of $\langle 1, 0, \cdot\rangle$).
Similarly, the partial derivative with respect to y represents the slope of the tangent line to the curve that are parallel to the yz-plane (i.e. in the direction of $\langle 0, 1, \cdot\rangle$).

Rule for finding Partial Derivatives of $\mathrm{z} = \mathrm{f}(\mathrm{x}, \mathrm{y})$

- To find f_x, regard y as a constant and differentiate f w.r.t. x.
- To find f_y, regard x as a constant and differentiate f w.r.t. y.

Problem 14.30. If $f(x, y) = x^3 + x^2y^3 - 2y^2$, find $f_x(2, 1)$ and $f_y(2, 1)$.
Solution.

Ans: $f_x(2, 1) = 16$; $f_y(2, 1) = 8$

Problem 14.31. Let $f(x, y) = \sin\left(\dfrac{x}{1+y}\right)$. Find the first partial derivatives of $f(x, y)$.
Solution.

Problem 14.32. Find the first partial derivatives of $f(x, y) = x^y$.

Solution. Use $\frac{d}{dx}a^x = a^x \ln a$.

> **Recall**: **(Implicit differentiation)**. When $y = y(x)$ and $x^2 + y^3 = 3$, you have $2x + 3y^2y' = 0$ so that $y' = -2x/(3y^2)$.

Problem 14.33. Find $\partial z/\partial x$ and $\partial z/\partial y$ if z is defined implicitly as a function of x and y by

$$x^3 + y^3 + z^3 + 6xyz = 1.$$

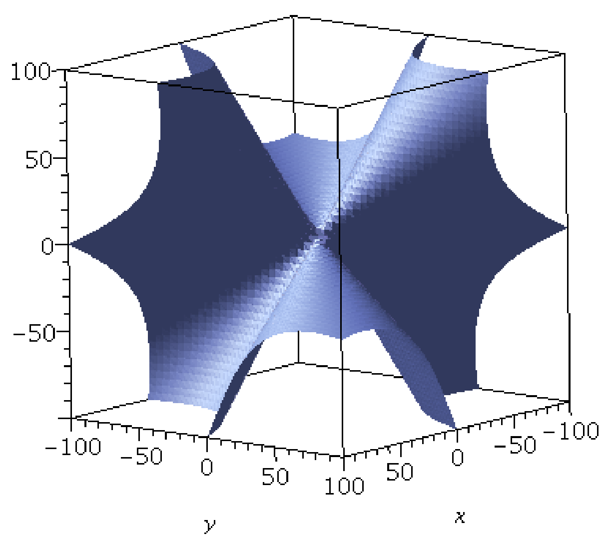


Figure 14.8: `implicitplot3d` in Maple: a plot of surface defined in Problem 14.33.

Problem 14.34. (Revisit of Problem 14.27). Find $f_x(x, y)$, when $f(x, y) = \sqrt[3]{x^3 + y^3}$. Can you evaluate $f_x(0, 0)$ easily?

Solution.

Functions of more than two variables

Problem 14.35. Let $f(x, y, z) = e^{xy} \ln z$. Find f_x, f_y, and f_z.

Solution.

14.3.2. Higher-order partial derivatives

Second partial derivatives of $z = f(x, y)$

$$
\begin{aligned}
(f_x)_x &= f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{11} \\
(f_x)_y &= f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{12} \\
(f_y)_x &= f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{21} \\
(f_y)_y &= f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{22}
\end{aligned}
$$

Problem 14.36. Find the second partial derivatives of $f(x, y) = x^3 + x^2y^3 - 2y^2$.

Solution.

Theorem 14.37. ***(Clairaut's theorem)*** *Suppose f is defined on a disk $D \subset \mathbb{R}^2$ that contains the point (a, b). If **both** f_{xy} **and** f_{yx} **are continuous on** D, then*

$$f_{xy}(a, b) = f_{yx}(a, b). \tag{14.9}$$

Claim 14.38. It can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$, when these partial derivatives are continuous.

$\boxed{\textbf{Problem}}$ **14.39.** Verify Clairaut's theorem for $f(x, y) = xye^y$.

$\boxed{\textbf{Problem}}$ **14.40.** Calculate $f_{xyzx}(x, y, z)$, given $f(x, y, z) = \sin(3x + yz)$.
Solution.

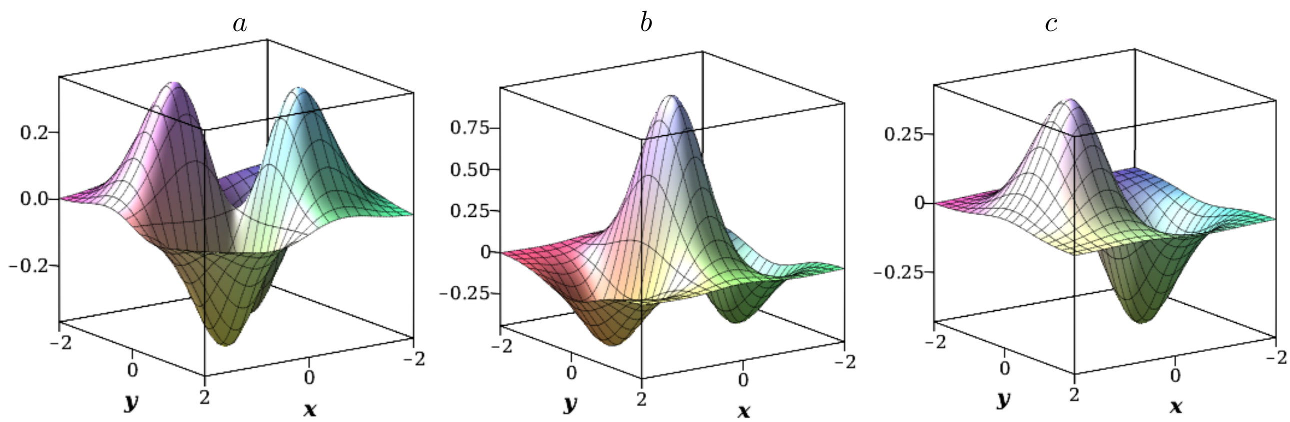
Exercises 14.3

1. The temperature T (in $^\circ$F) at a location in the Northern Hemi-sphere depends on the longitude x, latitude y, and time t; so we can write $T = f(x, y, t)$. Let's measure time in hours from the beginning of January.

   (a) What do the partial derivatives $\partial T/\partial x$, $\partial T/\partial y$, and $\partial T/\partial t$ mean?

   (b) **Mississippi State University** (MSU)[3] has longitude 88.8° W and latitude 33.5° N. Suppose that at noon on January first, the wind is blowing warm air to northeast, so the air to the west and south is warmer than that in the north and east. Would you expect $f_x(88.8, 33.5, 12)$, $f_y(88.8, 33.5, 12)$, and $f_t(88.8, 33.5, 12)$ to be positive or negative? Explain.

2. The following surfaces, labeled a, b, and c, are graphs of a function f and its partial derivatives f_x and f_y. Identify each surface and give reasons for your choices.



Hint: Assume one of them is the graph of f; try to figure out its partial derivatives.

3. Find the partial derivatives of the function.

   (a) $z = y\cos(xy)$

   (b) $f(u, v) = (uv - v^3)^2$

   (c) $w = \ln(x + 2y + 3z)$

   (d) $u = \sin(x_1^2 + x_2^2 + \cdots + x_n^2)$

   Ans: (d) $\partial u/\partial x_i = 2x_i \cdot \cos(x_1^2 + x_2^2 + \cdots + x_n^2)$

4. Let $f(x, y, z) = xy^2z^3 + \arccos(x\sqrt{y}) + \sqrt{1 + xz}$. Find f_{xyz}, by using a different order of differentiation for each term.

   Ans: $6yz^2$

5. Show that each of the following functions is a solution of the wave equation $u_{tt} = a^2 u_{xx}$.

   (a) $u = \sin(kx)\sin(akt)$

   (b) $u = (x + at)^3 + (x - at)^6$

   (c) $u = \sin(x + at) + \ln(x - at)$

   (d) $u = f(x + at) + g(x - at)$

   where f and g are twice differentiable functions.

[3]MSU, the **land-grant research university**, has an elevation of 118 meters, or 387 feet.

14.4. Tangent Planes & Linear Approximations

Recall: As one zooms into a curve $y = f(x)$, the more the curve resembles a line. More specifically, the curve looks more and more like the **tangent line**. It is the same for surface: the surface looks more and more like the **tangent plane**

Some functions are difficult to evaluate at a point; the equation of the tangent plane (which is much simpler) can be used **to approximate the value of the function at a given point**.

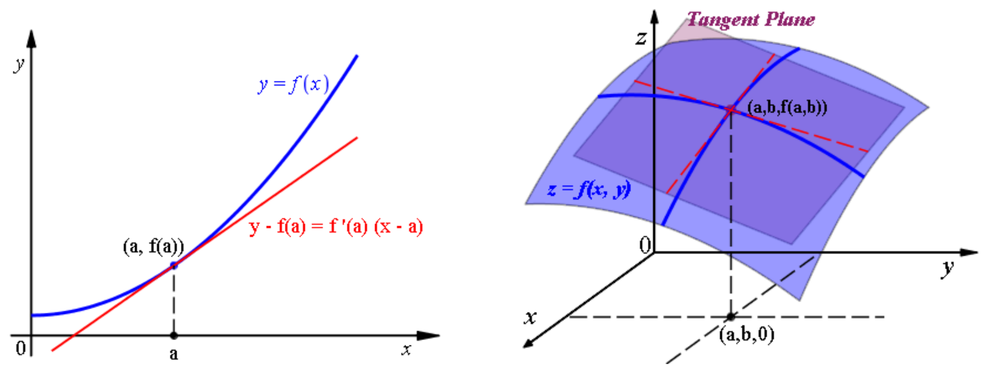


Figure 14.9: A tangent line and a tangent plane.

Tangent plane for $z = f(x, y)$ at (x_0, y_0, z_0): Any tangent plane passing through $P(x_0, y_0, z_0)$, $z_0 = f(x_0, y_0)$, has an equation of the form

$$A(x-x_0) + B(y-y_0) + C(z-z_0) = 0, \qquad \mathbf{n} =< A, B, C > .$$

By dividing the equation by $C\ (\neq 0)$ and letting $a = -A/C$ and $b = -B/C$, we can write it in the form

$$z - z_0 = a(x - x_0) + b(y - y_0). \tag{14.10}$$

Then, the intersection of the plane with $y = y_0$ must be the x**-directional tangent line** at (x_0, y_0, z_0), having the slope of $f_x(x_0, y_0)$:

$$\boxed{z - z_0 = a(x - x_0), \ \text{where} \quad y = y_0.}$$

Therefore $\boxed{a = f_x(x_0, y_0)}$. Similarly, we can conclude $\boxed{b = f_y(x_0, y_0)}$.

Summary 14.41. *Suppose that $f(x,y)$ has continuous partial derivatives. An equation of the **tangent plane** (equivalently, the **linear approximation**) to the surface $z = f(x,y)$ at the point $P(x_0, y_0, z_0)$ is*

$$z - z_0 = f_x(x_0, y_0)\,(x - x_0) + f_y(x_0, y_0)\,(y - y_0), \qquad \textbf{(14.11)}$$

where $z_0 = f(x_0, y_0)$.

Problem 14.42. Find an equation for the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$.

Solution.

Ans: $z = 4x + 2y - 3$

Linear approximation (linearization) of f at (a, b):

$$f(x,y) \approx L(x,y) := f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b). \qquad \textbf{(14.12)}$$

Problem 14.43. Give the linear approximation of $f(x,y) = xe^{xy}$ at $(1, 0)$. Then use this to approximate $f(1.1, -0.1)$.

Solution.

Ans: $L(x,y) = x + y$; $L(1.1, -0.1) = 1$, while $f(1.1, -0.1) = 0.9854\cdots$.

Differentiability for Functions of Multiple Variables

Recall: A function $y = f(x)$ is **differentiable** at a if

$$\lim_{\Delta x \to 0} \frac{f(a + \Delta x) - f(a)}{\Delta x} \quad \text{exists.} \quad (=: f'(a))$$

Thus, if f is differentiable at a, then $\dfrac{f(a + \Delta x) - f(a)}{\Delta x} = f'(a) + \varepsilon$ and

$$\Delta y \equiv f(a + \Delta x) - f(a) = f'(a)\Delta x + \varepsilon \Delta x, \tag{14.13}$$

where $\varepsilon \to 0$ as $\Delta x \to 0$. $\left(\because \; \frac{f(a+\Delta x) - f(a)}{\Delta x} = f'(a) + \varepsilon\right)$

Now, for $z = f(x, y)$, suppose that (x, y) changes from (a, b) to $(a + \Delta x, b + \Delta y)$. Then the corresponding **change** of z is

$$\Delta z = f(a + \Delta x, b + \Delta y) - f(a, b).$$

Definition 14.44. *A function $z = f(x, y)$ is* ***differentiable*** *at (a, b) if Δz can be expressed in the form*

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y, \tag{14.14}$$

where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0, 0)$.

It is sometimes hard to use Definition 14.44 directly to check the differentiability of a function.

Theorem 14.45. *If f_x and f_y* ***exist*** *near (a, b) and are* ***continuous*** *at (a, b), then $z = f(x, y)$ is differentiable at (a, b).*

Note: The above theorem implies that if **partial derivatives of f are continuous**, then **the slope of f exists for all directions**.

Problem 14.46. Let $f(x, y) = y + \sin(x/y)$. Explain why the function is differentiable at $(0, 3)$.

Differentials

Recall: For $y = f(x)$, let dx be the differential of x (an independent variable). The **differential** of y is then defined as

$$dy = f'(x)\,dx. \tag{14.15}$$

Note: $\boldsymbol{\Delta y}$ represents the change in height of the curve $y = f(x)$, while $\boldsymbol{dy}$ represents **the change in height of the tangent line**; when x changes by $\Delta x = dx$.

Definition 14.47. *For $z = f(x, y)$, we define **differentials** dx and dy to be independent variables. Then the **differential** dz is defined by*

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy, \tag{14.16}$$

*which is also called the **total differential**.*

Problem 14.48. Let $z = f(x, y) = x^2 + 3xy - y^2$.

(a) Find the differential dz.

(b) If (x, y) changes from $(2, 3)$ to $(2.1, 2.9)$, compare the values of Δz and dz.

Solution.

Ans: (a) $dz = (2x + 3y)dx + (3x - 2y)dy$; (b) $dz = 1.3$, $\Delta z = 1.27$

Problem 14.49. Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter, if the metal in the top and bottom is 0.1 cm thick and the metal in the side is 0.05 cm thick.

Solution. $V(r,z) = \pi r^2 z$. Therefore

$$dV \;=\; V_r\,dr + V_z\,dz \;=\; 2\pi r z\,dr + \pi r^2\,dz,$$

where $dr = 0.05$ and $dz = 2 \cdot 0.1 = 0.2$.

Ans: $dV = 2.8\pi = 8.796459431\cdots$ $(\Delta V = 9.0022337\cdots)$

Exercises 14.4

1. Find an equation of the tangent plane to the given surface at the specified point.
   (a) $z = \sin(2x+3y),\;\; (-3,2,0)$
   (b) $z = x^2 + 2y^2 - 3y,\;\; (1,-1,6)$

2. Explain why the function is **differentiable** at the given point. Then, find the linearization $L(x,y)$ of the function at that point.
   (a) $f(x,y) = 5 + x\ln(xy-1),\;\; (1,2)$
   (b) $f(x,y) = xy + \sin(y/x),\;\; (2,0)$

3. Given that f is a differentiable function with $f(5,2) = 4$, $f_x(5,2) = 1$, and $f_y(5,2) = -1$, use a linear approximation to estimate $f(4.9, 2.2)$.
   Ans: 3.7

4. Use differentials to estimate the amount of tin in a closed tin can with diameter 8 cm and height 16 cm if the tin is 0.05 cm thick.
   Ans: 8π

14.5. The Chain Rule and Implicit Differentiation

14.5.1. Chain rule

Recall: **Chain Rule for Functions of a Single Variable**
If $y = f(x)$ and $x = g(t)$, where f and g are differentiable, then y is a function of t, differentiable, and

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}. \tag{14.17}$$

$\boxed{\textbf{Theorem}}$ **14.50.** ***The Chain Rule (Case 1).*** *Suppose that $z = f(x,y)$ is a differentiable function, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t. Then z is a differentiable function of t and*

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}. \tag{14.18}$$

Observation: Let $z = f(x,y) = x\,y$ and x and y be functions of t:

$$z = f(x,y) = x\,y = x(t)\,y(t).$$

Then

$$\begin{aligned} \frac{dz}{dt} &= x'(t)\,y(t) + x(t)\,y'(t), \quad \text{(product rule)} \\ \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} &= y\,x'(t) + x\,y'(t). \end{aligned}$$

Problem **14.51.** If $z = x^2y + xy^3$, where $x = \cos t$ and $y = \sin t$, find dz/dt at $t = 0$.

Solution.

Ans: 1

Now, we will solve the above problem using the following script in Maple.

Maple script and answers

```
1 z := x*y^3+x^2*y:
2 x := cos(t):  y := sin(t):
3 zt := diff(z, t)
4     zt := -2 cos(t) sin(t) + cos(t) - sin(t) + 3 cos(t) sin(t)
5 simplify(%)
6     -4 cos(t) + 3 cos(t) + 5 cos(t) - 2 cos(t) - 1
7 eval(zt, t = 0)
8     1
```

Lines 4, 6, and 8 are answers from Maple.

Theorem 14.52. ***The Chain Rule (Case 2).*** *Suppose that $z = f(x, y)$ is a differentiable function, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t. Then*

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}, \quad \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}. \tag{14.19}$$

Problem 14.53. If $z = e^x \sin(y)$, where $x = st^2$ and $y = s^2 t$, find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$.

Solution.

Ans:
$$\begin{aligned} z_s &= t^2 \mathrm{e}^{st^2} \sin(s^2 t) + 2\,\mathrm{e}^{st^2} st \cos(s^2 t) \\ z_t &= 2\,st\mathrm{e}^{st^2} \sin(s^2 t) + \mathrm{e}^{st^2} s^2 \cos(s^2 t) \end{aligned}$$

Functions of Three and More Variables:

Theorem 14.54. ***The Chain Rule (General Version).*** *Suppose that u is a differentiable function of n variables, $x_1, x_2, \ldots, x_n$, each of which has m variables, $t_1, t_2, \ldots, t_m$. Then for each $i \in \{1, 2, \ldots, m\}$,*

$$\frac{\partial u}{\partial t_i} = \frac{\partial u}{\partial x_1}\frac{\partial x_1}{\partial t_i} + \frac{\partial u}{\partial x_2}\frac{\partial x_2}{\partial t_i} + \cdots + \frac{\partial u}{\partial x_n}\frac{\partial x_n}{\partial t_i}.$$

Problem 14.55. Write the chain rule for $w = f(x, y, z, t)$, where $x = x(u, v), y = y(u, v), z = z(u, v)$, and $t = t(u, v)$. That is, find $\frac{\partial w}{\partial u}$ and $\frac{\partial w}{\partial v}$.

Problem 14.56. If $g(s, t) = f\left(s^2 - t^2, t^2 - s^2\right)$ and f is differentiable, show that g satisfies the equation

$$t\frac{\partial g}{\partial s} + s\frac{\partial g}{\partial t} = 0.$$

Solution. Let $x = s^2 - t^2$ and $y = t^2 - s^2$.
Then $g_s = f_x\, x_s + f_y\, y_s$ and $g_t = f_x\, x_t + f_y\, y_t$.

Problem 14.57. If $z = x^3 + x^2 y$, where $x = s + 2t - u$ and $y = stu$, find the values of z_s, z_t, and z_u, when $s = 2$, $t = 0$, $u = 1$.

Solution.

Ans: $z_s = 3$, $z_t = 8$, and $z_u = -3$

14.5.2. Implicit differentiation

Consider $F(x, y) = 0$, where y is a function of x, i.e., $y = f(x)$. Then,

$$F_x \frac{dx}{dx} + F_y \frac{dy}{dx} = 0.$$

Thus,we have

$$y' = -\frac{F_x}{F_y}. \tag{14.20}$$

Problem 14.58. Find y' if $x^3 + y^3 = 6xy$.

Solution. Let $F = x^3 + y^3 - 6xy$. Then, use (14.20).

Ans: $y' = -(x^2 - 2y)/(y^2 - 2x)$

Note: You can solve the above problem using the technique you learned earlier in Calculus I. That is, applying ***x*-derivative** to $x^3+y^3=6xy$ reads

$$3x^2+3y^2\,y'=6y+6xy'.$$

Thus

$$3y^2\,y'-6xy'=-3x^2+6y \;\Rightarrow\; y'=-\frac{3x^2-6y}{3y^2-6x}.$$

Claim 14.59. Let $z=f(x,y)$ and $F(x,y,z)=0$.
Then $F_x\dfrac{\partial x}{\partial x}+F_y\dfrac{\partial y}{\partial x}+F_z\dfrac{\partial z}{\partial x}=0$ and $F_x\dfrac{\partial x}{\partial y}+F_y\dfrac{\partial y}{\partial y}+F_z\dfrac{\partial z}{\partial y}=0$. Thus

$$\frac{\partial z}{\partial x}=-\frac{F_x}{F_z} \quad \text{and} \quad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}. \tag{14.21}$$

Problem 14.60. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$, if

$$x^3+y^3+z^3+6xyz=1. \tag{14.22}$$

Solution.

Ans: $z_x=-\frac{3x^2+6yz}{3z^2+6xy}=-\frac{x^2+2yz}{z^2+2xy}$. See Figure 14.8, p. 427, for a figure of (14.22).

Exercises 14.5

1. Use the Chain Rule to find dz/dt or dw/dt.

   (a) $z = \cos x \sin y; \quad x = t^3, \;\; y = 1/t$
   (b) $w = (x + y^2 + z^3)^2; \quad x = 1 + 2t, \;\; y = -2t, \;\; z = t^2$

2. Suppose f is a differentiable function of x and y, and $g(u, v) = f(u + \cos v,\; u^2 + 1 + \sin v)$. Use the table of values to find $\boldsymbol{g_u(0, 0)}$ and $\boldsymbol{g_v(0, 0)}$.

	f	g	f_x	f_y
$(0,0)$	1	2	-1	10
$(0,1)$	3	5	10	5
$(1,1)$	2	7	20	2

Ans: $g_u(0, 0) = 20$ & $g_v(0, 0) = 2$

3. Use the Chain Rule to find the indicated partial derivatives.

   (a) $z = x^2 + y^4; \quad x = s + 2t - 3u, \;\; y = stu; \quad \dfrac{\partial z}{\partial s}, \; \dfrac{\partial z}{\partial t}, \; \dfrac{\partial z}{\partial u}$
   when $s = 3$, $t = 1$, and $u = 1$

   Ans: $z_s(3, 1, 1) = 112$ & $z_u(3, 1, 1) = 312$

   (b) $w = xy + yz + zx; \quad x = r\cos\theta, \;\; y = r\sin\theta; \quad \dfrac{\partial w}{\partial r}, \; \dfrac{\partial w}{\partial \theta}, \; \dfrac{\partial w}{\partial z}$
   when $r = 2$, $\theta = \pi/2$, and $z = 1$

   Ans: $w_z = 2$

4. Use the formulas in (14.21) to find $\partial z/\partial x$ and $\partial z/\partial y$, where z is function of (x, y).

   (a) $x^2 + 2y^2 + 3z^2 - 4 = 0$
   (b) $e^z = xy + z$

   Ans: (b) $z_x = y/(e^z - 1)$

14.6. Directional Derivatives and the Gradient Vector

14.6.1. Directional Derivatives

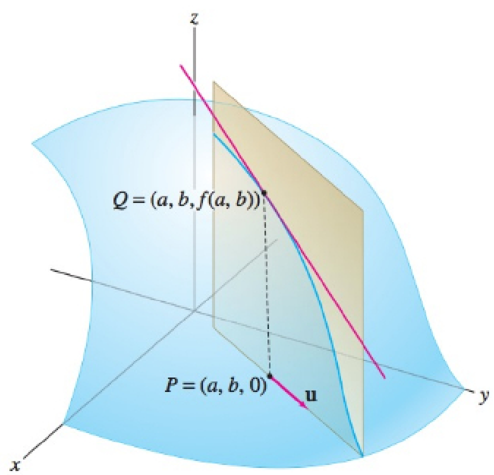


Figure 14.10

Recall: If $z = f(x, y)$, then the partial derivatives f_x and f_y represent the rates of change of z in the x- and y-directions, that is, in the directions of the unit vectors **i** and **j**.

Note: It would be nice to be able to find the slope of the tangent line to a surface S in the direction of an arbitrary unit vector $\mathbf{u} = \langle a, b \rangle$.

Definition 14.61. *The* ***directional derivative*** *of f at (x_0, y_0) in the direction of a unit vector $\mathbf{u} = \langle a, b \rangle$ is*

$$D_{\mathbf{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}, \tag{14.23}$$

if the limit exists.

Note that $(x_0 + ha, y_0 + hb) = (x_0, y_0) + h\langle a, b \rangle = (x_0, y_0) + h\mathbf{u}$ and

$$\begin{aligned} f(x_0 + ha, y_0 + hb) - f(x_0, y_0) &= f(x_0 + ha, y_0 + hb) - f(x_0, y_0 + hb) \\ &+ f(x_0, y_0 + hb) - f(x_0, y_0) \end{aligned}$$

Thus

$$\begin{aligned} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h} &= a\,\frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0 + hb)}{ha} \\ &+ b\,\frac{f(x_0, y_0 + hb) - f(x_0, y_0)}{hb}, \end{aligned}$$

which converges to "$a\, f_x(x_0, y_0) + b\, f_y(x_0, y_0)$" as $h \to 0$.

Theorem 14.62. *If f is a differentiable function of x and y, then f has a* ***directional derivative*** *in the direction of any unit vector $\mathbf{u} = \langle a, b\rangle$ and*

$$\begin{aligned} D_{\mathbf{u}} f(x,y) &= f_x(x,y)\,a + f_y(x,y)\,b \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \langle a, b\rangle \\ &= \langle f_x(x,y), f_y(x,y)\rangle \cdot \mathbf{u}. \end{aligned} \tag{14.24}$$

Problem 14.63. Find the directional derivative $D_{\mathbf{u}} f(x,y)$, if $f(x,y) = x^3 + 2xy + y^4$ and $\mathbf{u}$ is the unit vector given by the angle $\theta = \frac{\pi}{4}$. What is $D_{\mathbf{u}} f(2,3)$?

Solution. $\mathbf{u} = \langle \cos(\pi/4), \sin(\pi/4)\rangle = \left\langle 1/\sqrt{2}, 1/\sqrt{2}\right\rangle$.

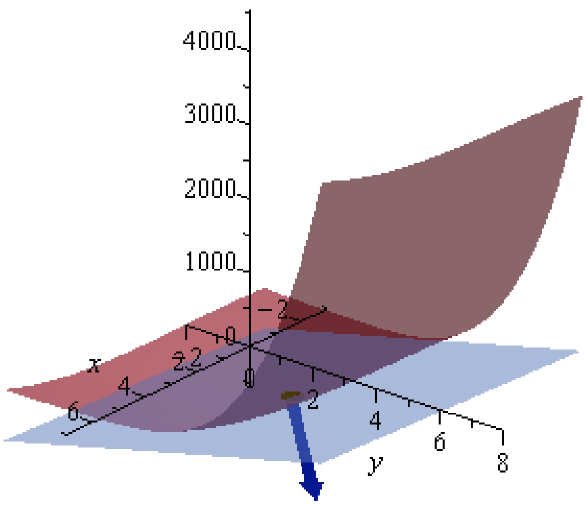


Figure 14.11

Ans: $65\sqrt{2}$

Note: 1. The only reason we are restricting the directional derivative to the unit vector is because we care about the rate of change in f per unit distance. Otherwise, the magnitude is irrelevant.
2. If the unit vector $\mathbf{u}$ makes an angle θ with the positive x-axis, then $\mathbf{u} = \langle \cos\theta, \sin\theta \rangle$. Thus

$$D_{\mathbf{u}} f(x,y) = f_x(x,y)\cos\theta + f_y(x,y)\sin\theta. \tag{14.25}$$

Self-study 14.64. Find the directional derivative of $f(x,y) = x + \sin(xy)$ at the point $(1,0)$ in the direction given by the angle $\theta = \pi/3$.

Solution.

Ans: $(1+\sqrt{3})/2$

Problem 14.65. If $f(x,y,z) = x^2 - 2y^2 + z^4$, find the directional derivative of f at $(1,3,1)$ in the direction of $\mathbf{v} = \langle 2, -2, -1 \rangle$.

Solution.

Ans: 8

Gradient Vector

Definition 14.66. *Let f be a differentiable function of two variables x and y. Then the* ***gradient*** *of f is the vector function*

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \frac{\partial f}{\partial x}\,\mathbf{i} + \frac{\partial f}{\partial y}\,\mathbf{j}. \tag{14.26}$$

Problem 14.67. If $f(x,y) = \sin(x) + e^{xy}$, find $\nabla f(x,y)$ and $\nabla f(0,1)$.

Solution.

Ans: $\langle 2, 0\rangle$

Note: With this notation of the gradient vector, we can rewrite

$$D_{\mathbf{u}} f(x,y) = \nabla f(x,y)\cdot \mathbf{u} = f_x(x,y)a + f_y(x,y)b, \quad \text{where } \mathbf{u} = \langle a, b\rangle\,. \tag{14.27}$$

Problem 14.68. Find the directional derivative of $f(x,y) = x^2y^3 - 4y$ at the point $(2,-1)$ and in the direction of the vector $\mathbf{v} = \langle 3, 4\rangle$.

Solution.

Ans: 4

14.6.2. Maximizing the Directional Derivative

Note that

$$D_{\mathbf{u}}f = \nabla f \cdot \mathbf{u} = |\nabla f|\,|\mathbf{u}|\cos\theta = |\nabla f|\cos\theta \le |\nabla f|, \tag{14.28}$$

where θ is the angle between ∇f and $\mathbf{u}$; the maximum occurs when $\theta = 0$.

Theorem 14.69. *Let f be a differentiable function of two or three variables. Then*

$$\max_{\mathbf{u}} D_{\mathbf{u}}f(\mathbf{x}) = |\nabla f(\mathbf{x})| \tag{14.29}$$

and it occurs when $\mathbf{u}$ has the same direction as $\nabla f(\mathbf{x})$.

Problem 14.70. Let $f(x, y) = xe^y$.

(a) Find the rate of change of f at $P(1, 0)$ in the direction from P to $Q(-1, 2)$.

(b) In what direction does f have the maximum rate of change? What is the maximum rate of change?

Solution.

Ans: (a) 0; (b) $\sqrt{2}$

Remark 14.71. Let $\mathbf{u} = \dfrac{\nabla f(\mathbf{x})}{|\nabla f(\mathbf{x})|}$, the unit vector in the gradient direction. Then

$$D_{\mathbf{u}}f(\mathbf{x}) = \nabla f(\mathbf{x}) \cdot \mathbf{u} = \nabla f(\mathbf{x}) \cdot \frac{\nabla f(\mathbf{x})}{|\nabla f(\mathbf{x})|} = |\nabla f(\mathbf{x})|. \tag{14.30}$$

This implies that the directional derivative is maximized in the gradient direction.

Claim 14.72. The gradient direction is the direction where the function changes fastest, more precisely, increases fastest!

The Gradient Vector of Level Surfaces

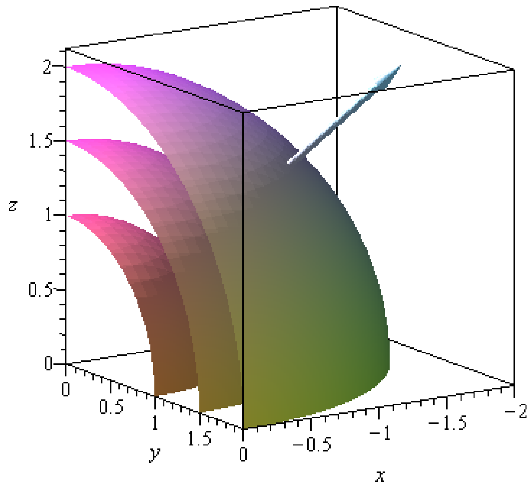


Figure 14.12: Level surfaces $x^2+y^2+z^2=k^2$, where $k=1,1.5,2$, and the gradient vector at $P(-1,1,\sqrt{2})$, when $k=2$.

Suppose S is a surface with equation

$$\boxed{F(x,y,z)=k}$$

and $P(x_0,y_0,z_0) \in S$. Let C be any curve that lies on the surface S, passes through P, and is described by a continuous vector function

$$\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle\,. \tag{14.31}$$

Then, any point $\langle x(t),y(t),z(t)\rangle$ must satisfy

$$F(x(t),y(t),z(t))=k. \tag{14.32}$$

Apply the Chain Rule to have

$$\frac{d}{dt}F = F_x\frac{dx}{dt}+F_y\frac{dy}{dt}+F_z\frac{dz}{dt}=\nabla F\cdot \mathbf{r}'(t)=0.$$

In particular, letting $t=t_0$ be such that $\mathbf{r}(t_0)=\langle x_0,y_0,z_0\rangle$,

$$\nabla F(x_0,y_0,z_0)\cdot \mathbf{r}'(t_0)=0, \tag{14.33}$$

where $\mathbf{r}'(t_0)$ is the tangent vector at $P(x_0,y_0,z_0)$.

Summary 14.73. ***(Gradient Vector)***
Given a level surface $F(x,y,z)=k$, the gradient vector $\nabla F(x,y,z)$ is ***normal*** *to the surface and pointing the* ***fastest increasing direction****.*

- *It is similarly true for* ***level curves****.*

Tangent Plane to a Level Surface

Suppose S is a surface given as $F(x, y, z) = k$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ is on S. Then the **tangent plane** to S at $\mathbf{x}_0$ is

$$\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = F_x(\mathbf{x}_0)(x-x_0)+F_y(\mathbf{x}_0)(y-y_0)+F_z(\mathbf{x}_0)(z-z_0) = 0. \quad \textbf{(14.34)}$$

The **normal line** to S at $\mathbf{x}_0$ is

$$\frac{x-x_0}{F_x(\mathbf{x}_0)} = \frac{y-y_0}{F_y(\mathbf{x}_0)} = \frac{z-z_0}{F_z(\mathbf{x}_0)}. \quad \textbf{(14.35)}$$

Problem 14.74. Find the equations of the tangent plane and the normal line at $P(-1, 1, 2)$ to the ellipsoid

$$x^2 + y^2 + \frac{z^2}{4} = 3.$$

Solution.

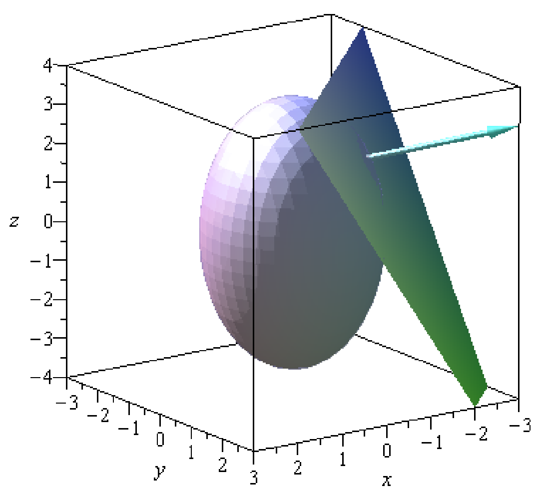


Figure 14.13

Ans: $-2x - 6 + 2y + z = 0$; $\frac{x+1}{-2} = \frac{y-1}{2} = \frac{z-2}{1}$

Exercises 14.6

1. Find the directional derivative of f at the point P in the direction indicated by either the angle θ or a vector v.

   (a) $f(x,y) = x\sin(xy), \quad P(0,1), \quad \theta = \pi/4$
   (b) $f(x,y,z) = y^2e^{xyz}, \quad P(0,1,-1), \quad \mathbf{v} =< -1,2,2 >$

   Ans: (b) $5/3$

2. Find the maximum rate of change of f at the given point and the direction in which it occurs.

   (a) $f(x,y) = \sin(xy), \quad (0,1)$
   (b) $f(x,y,z) = \dfrac{z}{x+y}, \quad (1,1,4)$

   Ans: (b) $|\nabla f(1,1,4)| = 3/2, \quad \nabla f(1,1,4) =< -1,-1,1/2 >$

> **Note**: We know that a differentiable function f increases most rapidly in the direction of ∇f. Thus, it is natural to claim that the function ***decreases most rapidly*** in the direction opposite to the gradient vector, that is, $-\nabla f$.

3. Find the direction in which the function $f(x,y,z) = x^2+y^2+z^2$ decreases fastest at the point $(1,1,1)$.

4. Find directions (unit vectors) in which the directional derivative of $f(x,y) = x^2+xy^2$ at the point $(1,2)$ has value 0.

   Ans: $\mathbf{u} = \pm\frac{<2,-3>}{\sqrt{13}}$

5. Find the equations of (i) the tangent plane and (ii) the normal line to the given surface at the specified point.

   (a) $(x-1)^2+(y-2)^2+(z-3)^2 = 3, \quad (2,1,4)$
   (b) $xy+yz+zx-5 = 0, \quad (1,1,2)$

   Ans: (b) $3(x-1)+3(y-1)+2(z-2) = 0$ & $\frac{x-1}{3} = \frac{y-1}{3} = \frac{z-2}{2}$

14.7. Maximum and Minimum Values

Recall: To find the **absolute maximum and minimum values** of **a continuous function f on a closed interval $[a, b]$**:

1. Find values of f at the **critical points** of f in (a, b).
2. Find values of f at the **end points** of the interval.
3. The largest is the absolute maximum value;
   the smallest is the absolute minimum value.

Recall: **(Second Derivative Test for $\mathrm{y = f(x)}$)**
Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$, then f has a **local minimum** at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$, then f has a **local maximum** at c.

14.7.1. Local extrema

Definition 14.75. *Let f be a function of two variables x and y.*

- *It has a* ***local minimum*** *at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b).*
- *It has a* ***local maximum*** *at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).*

Theorem 14.76. ***First Derivative Test***
If f has a **local extremum** *at (a, b) and the first order partial derivatives exist, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$, that is, $\nabla f(a, b) = 0$.*

Self-study 14.77. Find the **critical points** of

$$f(x, y) = 2x^3 - 3x^2 + y^2 + 4y + 1.$$

Ans: $(0, -2)$, $(1, -2)$

Theorem 14.78. ***Second Derivative Test***
Suppose that the second order partial derivatives of f are ***continuous*** *near (a,b) and suppose that $\nabla f(a,b) = 0$. Let*

$$D = D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2.$$

- *If $D > 0$ and $f_{xx}(a,b) > 0$, then $f(a,b)$ is a local minimum.*
- *If $D > 0$ and $f_{xx}(a,b) < 0$, then $f(a,b)$ is a local maximum.*
- *If $D < 0$, then $f(a,b)$ is a saddle point.*

Note:

1. If $D = 0$, then no conclusion can be drawn from this test.
2. $D = \det \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix} = f_{xx}f_{yy} - f_{xy}f_{yx} = f_{xx}f_{yy} - (f_{xy})^2$

   The matrix is called the **Hessian matrix** of f, whose determinant is the **Gaussian curvature**, the product of the principal curvatures.
3. Let $D > 0$. Then, $f_{xx}(a,b) \gtrless 0$ is equivalent to $f_{yy}(a,b) \gtrless 0$.

Problem 14.79. Find all local extrema of $f(x,y) = x^4 + y^4 - 4xy + 1$.

Solution.

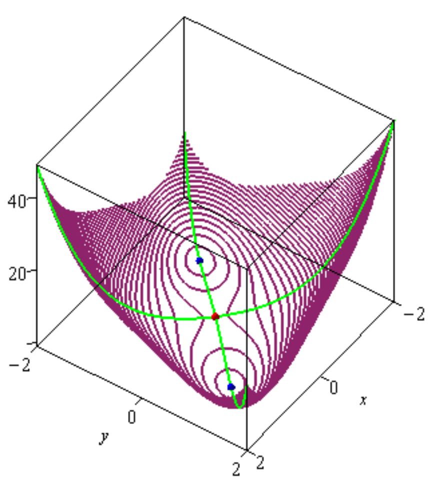


Figure 14.14

Ans: local min: $(\pm 1, \pm 1)$; saddle point: $(0,0)$

Problem 14.80. Find the shortest distance from the point $(6, 3, 2)$ to the plane $2x + 2y - z + 2 = 0$.

Solution. ***Hint***: (a) You may use the formula $D = |ax_0 + by_0 + cz_0 + d|/\sqrt{a^2 + b^2 + c^2}$, where D be the distance. (b) **Optimization**: Define to minimize

$$f(x, y) = D^2 = (x - 6)^2 + (y - 3)^2 + (z - 2)^2 = (x - 6)^2 + (y - 3)^2 + (2x + 2y)^2$$

Ans: $\min D = 6$, when $(x, y, z) = (2, -1, 4) \in P$

Problem 14.81. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the plane $3x + 2y + z = 6$.

Solution. ***Hint***: Maximize $V = xyz$, subject to $3x + 2y + z = 6$. Thus $V = xy(6 - 3x - 2y)$. Try to find the maximum by setting $\nabla V = 0$.

Ans: $(x, y, z) = (2/3, 1, 2)$; $V = xyz = 4/3$

14.7.2. Absolute extrema

Theorem 14.82. ***(Existence)***.
If f is continuous on a closed and bounded set $D \subset \mathbb{R}^2$, then f attains an ***absolute minimum value*** *$f(x_0, y_0)$ and an* ***absolute maximum value*** *$f(x_1, y_1)$ at some points $(x_i, y_i) \in D$, $i = 0, 1$.*

Strategy 14.83. To find absolute extrema,

1. Find critical points and values of f at those critical points.
2. Find the extreme values that occur on the boundary.
3. Compare all of those values for the largest and smallest values.

Problem 14.84. Find the absolute extrema of $f(x, y) = x^2 - 2xy + 2y$ on the rectangle $R = \{(x, y) \mid 0 \le x \le 3,\ 0 \le y \le 2\}$.
Solution.

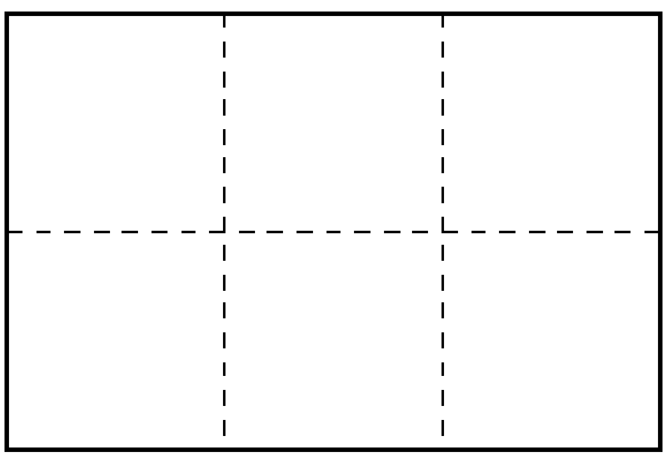


Figure 14.15: $R = [0, 3] \times [0, 2]$

Ans: abs.min=0; abs.max=9

Exercises 14.7

1. (i) Find the local maxima and minima and saddle points of the function.
   (ii) CAS Use Maple's **plot3d** and **contourplot** functions to verify them.

   (a) $f(x, y) = x^3 - 3xy^2$
   (b) $f(x, y) = (2x^2 + y^2)e^{-x^2-y^2}$

   (**Note**: You may use Mathematica, if you want.)

2. Find the *absolute* maximum and minimum values of f on D.

   (a) $f(x, y) = x^2 + y^2 - 2x$; D is the closed triangular region with vertices $(2, 0)$, $(0, 2)$, and $(0, -2)$.
   Ans: max: $f(0, \pm 2) = 4$; min: $f(1, 0) = -1$
   (b) $f(x, y) = 4x^2 + y^4$; $D = \{(x, y) | x^2 + y^2 \le 1\}$.
   Ans: max: $f(\pm 1, 0) = 4$; min: $f(0, 0) = 0$

3. Find three positive numbers whose sum is 60 and whose product is maximum.
   Hint: The problem can read: $\max_{(x,y,z)} xyz$, subject to $x + y + z = 60$. Thus for example it can be reformulated as: $\max_{(x,y)} xy(60 - x - y)$, with each component being positive. From this, you may conclude $x = y$.

4. Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex in the plane $2x + 5y + z = 30$. ***Clue***: Try to use the hint given for Problem 14.81.

14.8. Method of Lagrange Multipliers

Recall: (Section 14.6)

- Given **a level curve $f(x,y)=k$**, the **gradient vector $\nabla f(x,y)$** is
  - **normal to the curve** and
  - pointing the **fastest increasing direction**.
- It is similarly true for **level surfaces**.

Eqauality-Constrained Optimization

In this section, we consider Lagrange's method to solve the problem of the form

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{subj.to} \quad g(\mathbf{x}) = c. \tag{14.36}$$

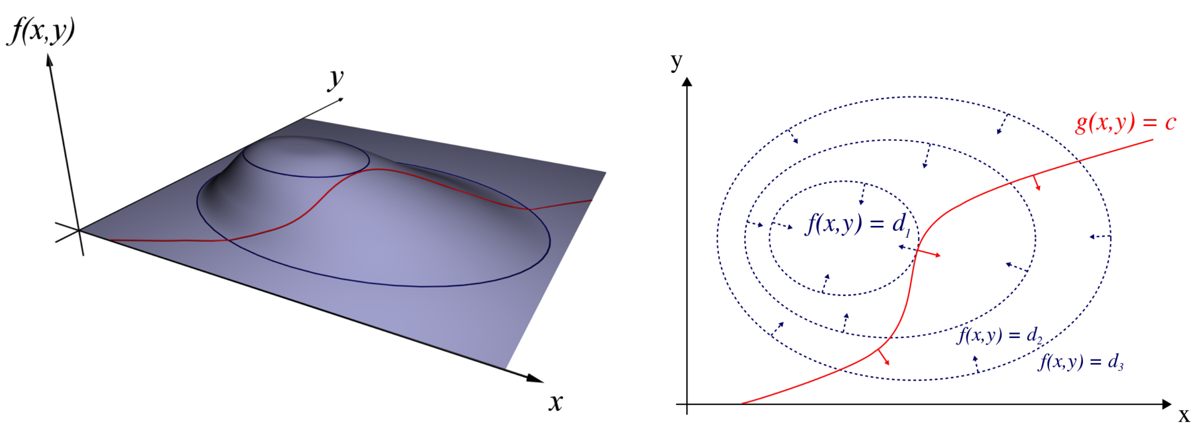


Figure 14.16: The method of Lagrange multipliers in $\mathbb{R}^2$: $\boxed{\nabla f \,/\!/\, \nabla g\text{, at maximum}}$.

Strategy **14.85.** **(Method of Lagrange multipliers)**

For the maximum and minimum values of $f(x,y)$ **subject to** $g(x,y)=c$,

(a) Find all values of (x,y) and λ such that

$$\boxed{\nabla f(x,y) = \lambda \nabla g(x,y) \;\text{ and }\; g(x,y)=c}. \tag{14.37}$$

(b) Evaluate f at all these points, to find the maximum and minimum.

Problem 14.86. (Revisit of Problem 14.81, p. 454). Find the volume of the largest rectangular box in the first octant with three faces in the coordinate planes and one vertex on the plane $3x + 2y + z = 6$, using the method of Lagrange multipliers.

Solution.

Ans: $4/3$

Problem 14.87. A topless rectangular box is made from 12m^2 of cardboard. Find the dimensions of the box that maximizes the volume of the box.

Solution. Maximize $V = xyz$ subj.to $2xz + 2yz + xy = 12$.

Ans: 4 $(x = y = 2z = 2)$

Problem 14.88. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the circle $x^2 + y^2 = 1$.

Solution. $\nabla f = \lambda \nabla g \Longrightarrow \begin{bmatrix} 2x \\ 4y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Therefore, $\begin{cases} 2x = 2x\,\lambda & ① \\ 4y = 2y\,\lambda & ② \\ x^2 + y^2 = 1 & ③ \end{cases}$

From ①, $x = 0$ or $\lambda = 1$.

Ans: min: $f(\pm 1, 0) = 1$; max: $f(0, \pm 1) = 2$

Problem 14.89. Find the extreme values of $f(x,y) = x^2 + 2y^2$ on the disk $x^2 + y^2 \le 1$.

Solution. ***Hint***: You may use Lagrange multipliers when $x^2 + y^2 = 1$.

Ans: min: $f(0,0) = 0$; $f(0, \pm 1) = 2$

Two Constraints

Consider the problem of the form

$$\max_{\mathbf{x}} f(\mathbf{x}) \quad \text{subj.to} \quad g(\mathbf{x}) = c \ \text{ and } \ h(\mathbf{x}) = d. \tag{14.38}$$

Then, at extrema we must have

$$\nabla f \ \in \ \text{Plane}(\nabla g, \nabla h) := \{c_1 \nabla g + c_2 \nabla h\}. \tag{14.39}$$

Thus (14.38) can be solved by finding all values of (x, y, z) and (λ, μ) such that

$$\boxed{\begin{array}{rcl} \nabla f(x,y,z) &=& \lambda \nabla g(x,y,z) + \mu \nabla h(x,y,z) \\ g(x,y,z) &=& c \\ h(x,y,z) &=& d \end{array}} \tag{14.40}$$

Problem 14.90. Find the maximum value of the function $f(x, y, z) = z$ on the curve of the intersection of the cone $2x^2 + 2y^2 = z^2$ and the plane $x + y + z = 4$.

Solution. Letting $g = 2x^2 + 2y^2 - z^2 = 0$ ④ and $h = x + y + z = 4$ ⑤, we have

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 4x \\ 4y \\ -2z \end{bmatrix} + \mu \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \implies \left\{ \begin{array}{rcrl} 0 &=& 4\lambda x + \mu & ① \\ 0 &=& 4\lambda y + \mu & ② \\ 1 &=& -2\lambda z + \mu & ③ \end{array} \right.$$

From ① and ②, we conclude $x = y$; using ④, we have $z = \pm 2x$.

Ans: 2

Exercises 14.8

1. Use Lagrange multipliers to find extreme values of the function subject to the given constraint.

   (a) $f(x,y) = xy; \quad x^2 + 4y^2 = 2$

   (b) $f(x,y) = x + y + 2z; \quad x^2 + y^2 + z^2 = 6$

   Ans: max: $f(1,1,2) = 6$; min: $f(-1,-1,-2) = -6$

2. Find extreme values of f subject to both constraint.

$$f(x,y,z) = x^2 + y^2 + z^2; \quad x - y = 3, \;\; x^2 - z^2 = 1.$$

Ans: $f(1,-2,0) = 5$

Note: The value just found for Problem 2 is the minimum. Why? See the figure below.

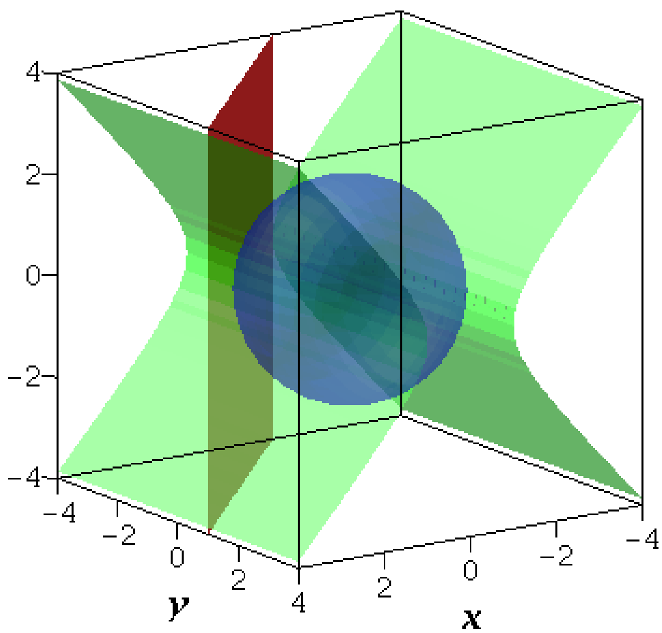


Figure 14.17: implicitplot3d. red: $x - y = 3$; green: $x^2 - z^2 = 1$; blue: $f(x,y,z) = 5$.

3. Use Lagrange multipliers to solve Problem 3 in Exercise 14.7. (See p. 456.)

4. Use Lagrange multipliers to solve Problem 4 in Exercise 14.7.

CHAPTER **15**

Multiple Integrals

The **multiple integral** is a definite integral of a function of more than one real variable, for example, $f(x,y)$ or $f(x,y,z)$. Integrals of a function of two variables over a region in $\mathbb{R}^2$ are called **double integrals**, and integrals of a function of three variables over a region of $\mathbb{R}^3$ are called **triple integrals**. In this chapter, you will learn double integrals and triple integrals in rectangular coordinates, polar coordinates, cylindrical coordinates, and spherical coordinates. Also, you will learn how to perform integration by changing variables between or inside coordinates.

Contents of Chapter 15

15.1. Double Integrals over Rectangles

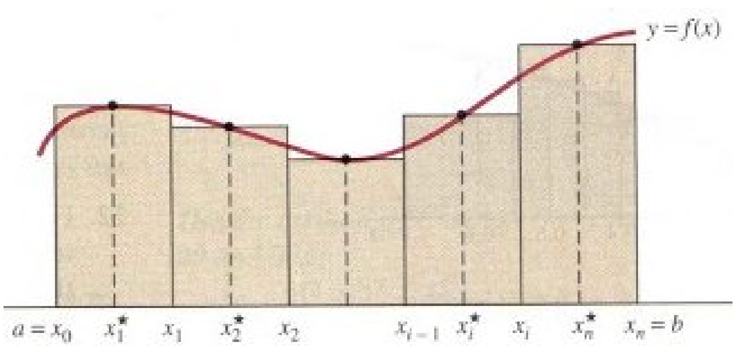


Figure 15.1: Riemann Sum.

Recall: **(Review on Definite Integrals)**.

- We defined the integral in terms of **Riemann Sum**.
  - We first find the area underneath the curve $y = f(x)$ by dividing the area into rectangles.
  - Then the exact area can be found by evaluating

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x, \quad \Delta x = \frac{b-a}{n}.$$

- We also can get the **definite integral**, using the **Fundamental Theorem of Calculus** (Part 2):

$$\int_a^b f(x)\,dx = F(b) - F(a), \tag{15.1}$$

  where F is a function such that $F' = f$ (antiderivative).

15.1.1. Volumes as Double Integrals

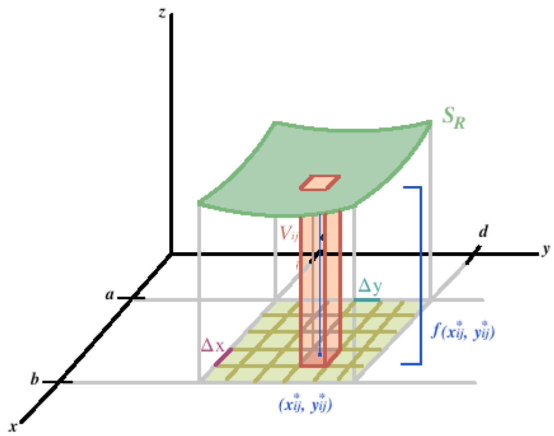


Figure 15.2

Let $\boldsymbol{R = [a,b] \times [c,d]}$ be a rectangle. Define

$$\Delta x = (b-a)/m, \quad \Delta y = (d-c)/n,$$

for some $m, n > 0$. Let

$$\begin{aligned} x_i &= a + i\Delta x, \quad i = 0, 1, \cdots, m, \\ y_j &= c + j\Delta y, \quad j = 0, 1, \cdots, n, \end{aligned}$$

and

$$R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j].$$

Let $S_R = \{(x,y,z) \mid 0 \le z \le f(x,y), (x,y) \in R\}$ define the solid that lies above R. Let $\Delta A = \Delta x\, \Delta y$ denote the area of each R_{ij}. Then we can express this volume of S_R as

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A, \tag{15.2}$$

where (x_{ij}^*, y_{ij}^*) is a **sample point** in each division R_{ij}.

Definition 15.1. *The **double integral** of f over the rectangle R is*

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\, \Delta A. \tag{15.3}$$

The **double integral** is **the limit of Riemann sums**.

We can simplify this if we choose each sample point to be the point in the **upper right corner** of each sub-rectangle, $(x_{ij}^*, y_{ij}^*) = (x_i, y_j)$:

$$\iint_R f(x,y)\, dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f(x_i, y_j)\, \Delta A. \tag{15.4}$$

Problem 15.2. Estimate the volume of the solid that lies above the square $R = [0,2] \times [0,2]$ and below the elliptic paraboloid $z = 16 - x^2 - 2y^2$. Divide R into four equal squares and choose the sample point to be the **upper right corner** of each square R_{ij}. Approximate the Volume.

Solution.

Ans: 34

Problem 15.3. (Midpoint rule). Estimate the volume of the solid that lies above the square $R = [0,2] \times [1,2]$ and below the function $f(x,y) = 5x^2 - 4y$. Divide R into four equal rectangles and choose the sample point to be the **midpoint** of each rectangle R_{ij}. Approximate the volume.

Solution. *We will find the exact volume in Problem 15.9 below.*

Ans: 1

15.1.2. Iterated Integrals

Note: We defined the **double integral** as **the limit of Riemann sums**.

- However, taking these Riemann sums is a bit of a pain.
- We will overcome the difficulty, using two partial integrals.

Suppose that f is a function of two variables that is integrable on the rectangle $R = [a, b] \times [c, d]$.

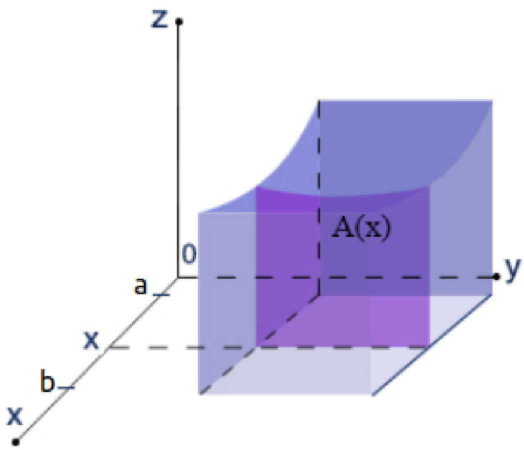


Figure 15.3: $A(x)$.

Definition 15.4. *We define*

$$A(x) = \int_c^d f(x, y)\, dy \qquad (15.5)$$

as the ***partial integral with respect to*** y*. We evaluate this integral by treating* x *as a constant, and integrate* $f(x, y)$ *with respect to* y*.*

Definition 15.5. *We define*

$$B(y) = \int_a^b f(x, y)\, dx \qquad (15.6)$$

as the ***partial integral with respect to*** x*. We evaluate this integral by treating* y *as a constant, and integrate* $f(x, y)$ *with respect to* x*.*

Note: The *Fundamental Theorem of Calculus, Part 2*, Equation (15.1) on p. 464, can be used to evaluate the partial integrals.

Definition 15.6. *The **iterated integral** is defined as follows:*

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx = \int_a^b A(x)\,dx. \tag{15.7}$$

*In other words, we work this integral **from the inside out**.*

Problem 15.7. Evaluate the integrals

$$\text{(a)}\ \int_0^3 \int_1^2 x^2 y\,dy\,dx \qquad \text{and} \qquad \text{(b)}\ \int_1^2 \int_0^3 x^2 y\,dx\,dy.$$

Solution. $R = [0,3] \times [1,2]$.

Ans: $27/2$

Theorem 15.8. ***(Fubini's Theorem).*** *If f is continuous on the rectangle $R = \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, c \le y \le d\}$, then*

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy. \tag{15.8}$$

Note: The **double integral** and the **iterated integral** were defined ***separately and differently***.

Problem 15.9. (Revisit of Problem 15.3). Evaluate the double integral $\iint\limits_R (5x^2 - 4y)\, dA$,, where $R = \{(x, y) \in \mathbb{R}^2 \mid 0 \le x \le 2, 1 \le y \le 2\}$.

Solution.

Ans: $4/3$

Problem 15.10. Evaluate $\iint\limits_R y \sin(xy)\, dA$, where $R = [1, 2] \times [0, \pi]$.

Solution. Let's try the iterated integrals with x-first and y-first.

Ans: 0

Separable functions $f(x,y) = g(x)\,h(y)$:
Let $R = [a,b] \times [c,d]$. Then

$$\begin{aligned}\iint\limits_R f(x,y)\,dA &= \int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \Big(\int_a^b g(x)\,\underline{h(y)}\,dx\Big)\,dy \\ &= \int_c^d h(y)\underline{\Big(\int_a^b g(x)\,dx\Big)}\,dy \\ &= \Big(\int_a^b g(x)\,dx\Big)\int_c^d h(y)\,dy,\end{aligned}$$

where the underlined (in maroon) are treated as constants.

$$\boxed{\iint\limits_R g(x)h(y)\,dA = \int_a^b g(x)\,dx \cdot \int_c^d h(y)\,dy, \quad R = [a,b] \times [c,d]. \qquad \textbf{(15.9)}}$$

Problem 15.11. Evaluate $\iint_R e^{x+3y}\,dA$, where $R = [0,3] \times [0,1]$.
Solution.

Ans: $(e^3 - 1)^2/3$

Average Value

Recall: The **average value** of a function f of one variable defined on an interval $[a,b]$ is

$$f_{\text{ave}} = \frac{1}{b-a}\int_a^b f(x)\,dx,$$

where $b-a$ is the length of the interval.

Definition 15.12. *In a similar fashion, we define the* ***average value*** *of f of two variables defined on R to be*

$$f_{\text{ave}} = \frac{1}{A(R)}\iint\limits_R f(x,y)\,dA, \tag{15.10}$$

where $A(R)$ is the area of R.

Problem 15.13. Find the average value of $f(x,y) = x^2 + \sin(2y)$ over $R = [0,3]\times[0,\pi]$.

Solution. Use symmetry, for a simpler calculation!

Ans: 3

Exercises 15.1

1. Estimate the volume of the solid that lies below the surface $z = x^2 + y$ and above the rectangle
$$R = \{(x, y) \mid 0 \leq x \leq 4,\ \ 0 \leq y \leq 6\}.$$
Use a Riemann sum with $m = 2$, $n = 3$, and the *Midpoint Rule*.

Ans: $48 \cdot 4 = 192$

2. Let V be the volume of the solid that lies under the surface $z = 30 - 4x - y^2$ and above the rectangle $R = \{(x, y) \mid 2 \leq x \leq 6,\ \ 0 \leq y \leq 2\}$. Use the lines $x = 4$ and $y = 1$ to divide R into four subrectangles. Let L and U be the Riemann sums computed respectively using lower left corners and upper right corners. Without using the actual numbers V, L, and U, arrange them in increasing order and describe your reasoning.

3. Evaluate the double integral by first identifying it as the volume of a solid.

   (a) $\displaystyle\iint_R (x + 1)\, dA, \quad R = \{(x, y) \mid 0 \leq x \leq 2,\ \ 0 \leq y \leq 2\}$

   (b) $\displaystyle\iint_R (4 - 2y)\, dA, \quad R = [0, 1] \times [0, 1]$

4. Calculate the iterated integral.

   (a) $\displaystyle\int_1^3 \int_0^2 (6xy^2 - 12x^2)\, dx\, dy$

   (b) $\displaystyle\int_0^2 \int_1^3 (6xy^2 - 12x^2)\, dy\, dx$

   (c) $\displaystyle\int_0^1 \int_0^2 2\pi xy \sin(\pi xy^2)\, dy\, dx$

   (d) $\displaystyle\int_0^2 \int_0^1 2\pi xy \sin(\pi xy^2)\, dx\, dy$

Ans: (a) 40, (c) 1

5. Calculate the double integral.

   (a) $\displaystyle\iint_R y \sec^2(x)\, dA, \quad R = [0, \pi/4] \times [0, 4]$

   (b) $\displaystyle\iint_R xe^{-xy}\, dA, \quad R = [0, 2] \times [0, 1]$

Ans: (a) 8; (b) $1 - e^{-2}$

6. Find the volume of the solid in the first octant bounded by the cylinder $z = 9 - y^2$ and the plane $x = 2$.

Ans: 36

15.2. Double Integrals over General Regions

15.2.1. Iterated Integrals over General Regions

We know how to find **the volume of the solid under a surface**, when the projection of the solid down to the xy-plane is **a rectangular region**.

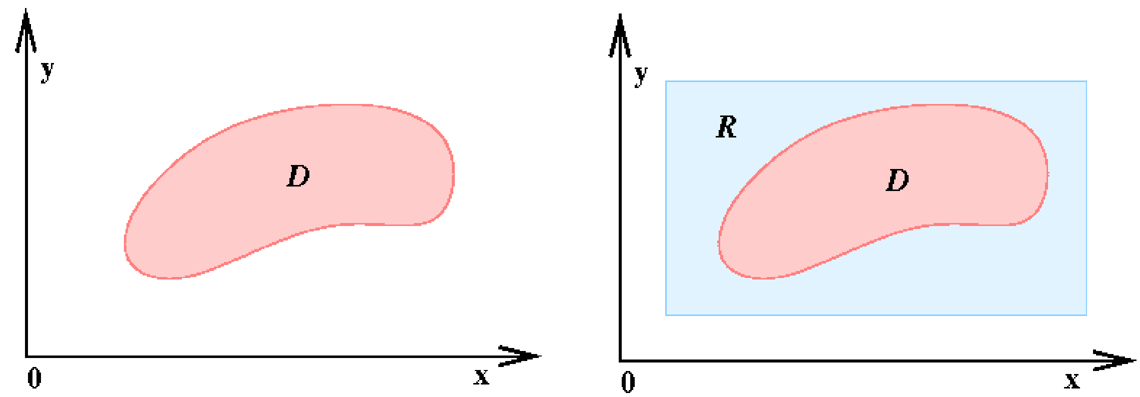


Figure 15.4: A general region D and its surrounding rectangle R.

Let $D \subset \mathbb{R}^2$ be **a bounded region of general shape** as in Figure 15.4.

- For a bounded function f defined over D, what we want to find is

$$\iint_D f(x,y)\,dA.$$

- **Expand the domain to a surrounding rectangle R** and define

$$F(x,y) = \begin{cases} f(x,y) & \text{if } (x,y) \in D \\ 0 & \text{if } (x,y) \in R \backslash D. \end{cases} \tag{15.11}$$

- Then,

$$\iint_D f(x,y)\,dA = \iint_R F(x,y)\,dA, \tag{15.12}$$

which implies the following.

- The integral $\displaystyle\iint_D f(x,y)\,dA$ exists, for a **general bounded region D**.
- The **iterated integral** can be applied to get the double integral.

Quesiton. What if the region D is not rectangular but defined as **the boundary between two functions**?

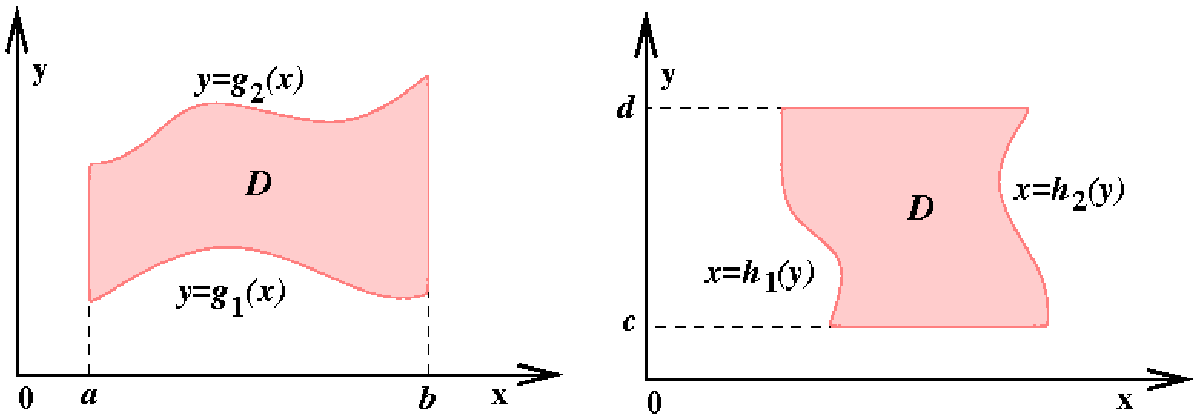


Figure 15.5: General regions D: Type 1 (left) and Type 2 (right).

Let the region D be given as

$$\begin{aligned} D_1 &= \{(x,y) \in \mathbb{R}^2 \mid a \le x \le b, g_1(x) \le y \le g_2(x)\} \\ D_2 &= \{(x,y) \in \mathbb{R}^2 \mid h_1(y) \le x \le h_2(y), c \le y \le d\}, \end{aligned}$$

Then

$$\begin{aligned} \iint\limits_{D_1} f(x,y)\,dA &= \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx \\ \iint\limits_{D_2} f(x,y)\,dA &= \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy \end{aligned} \tag{15.13}$$

Strategy 15.14. Double integral over general regions D:

1. ***Visualize*** to recognize the region.
2. Decide the ***order of integration***.
3. If the calculation becomes complicated, ***try the other order***.

Problem 15.15. Evaluate $\displaystyle\iint_D 2xy\,dA$, where D is the region bounded by the line $y = x - 2$ and the parabola $x = y^2$.

Solution. First, visualize the region.

Ans: $45/4$

Problem 15.16. Find the volume of the solid that lies under the plane $z = 1 + 2y$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. Try for both orders.

Ans: $28/5$

Note: Here, the main concern is **how to access the domain D**; the iterated integration must access points in D, **once–and–only–once**.

15.2.2. Properties of Double Integrals

Double integrals share properties with **definite integrals** of functions of one variable.

Proposition 15.17. ***(Properties of Double Integrals)***
Let f and g be continuous functions defined in D and $c \in \mathbb{R}$. Then

① $\displaystyle\iint_D [f(x,y)+g(x,y)]\,dA = \iint_D f(x,y)\,dA + \iint_D g(x,y)\,dA$

② $\displaystyle\iint_D c\,f(x,y)\,dA = c\iint_D f(x,y)\,dA$

③ $\displaystyle\iint_D f(x,y)\,dA \ge \iint_D g(x,y)\,dA,$ ***if*** $f(x,y) \ge g(x,y),\ \forall\,(x,y) \in D$

④ $\displaystyle\iint_D f(x,y)\,dA = \iint_{D_1} f(x,y)\,dA + \iint_{D_2} f(x,y)\,dA,$ ***when*** $D = D_1 \dot{\bigcup} D_2$

⑤ $\displaystyle\iint_D 1\,dA = A(D)$

⑥ $\displaystyle m \cdot A(D) \le \iint_D f(x,y)\,dA \le M \cdot A(D),$ ***if*** $m \le f(x,y) \le M,\ \forall\,(x,y) \in D$

Problem 15.18. Show that ⑤ $\iint_D 1\,dA = A(D)$, where $A(D)$ denotes the area of the region D.

Hint: Consider a solid cylinder whose base is D and whose height is 1.

Problem 15.19. Use Property ⑥ in Proposition 15.17 to estimate the integral $I = \iint_D e^{\sin x \cos y} dA$, where D is the disk with center the origin and radius 2.

Solution.

Ans: $4\pi/e \le I \le 4\pi e$

Let's solve some exra problems.

Problem 15.20. Evaluate $\displaystyle\iint_D (x+2y)\, dA$, where D is the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$.

Solution.

Ans: $\frac{32}{15}$

Problem 15.21. Evaluate $\displaystyle\int_0^1 \int_x^1 \sin(y^2)\,dy\,dx$.

Solution. Visualize the region and try to change the order of integration.

Ans: $(1-\cos 1)/2$

Problem 15.22. Evaluate the integral $\displaystyle\int_0^2 \int_{y/2}^1 2\,e^{y/x}\,dx\,dy$ by reverting the order of integration.

Solution.

Ans: $e^2 - 1$

Exercises 15.2

1. Evaluate the double integral, by setting up an iterated integral in the easier order.

   (a) $\displaystyle\iint_D 2e^{-x^2}\,dA, \quad D = \{(x,y) \mid 0 \le x \le 2,\ 0 \le y \le x\}$

   (b) $\displaystyle\iint_D x\,dA$, D is bounded by $y = x+2$ and $y = x^2$

   (c) $\displaystyle\iint_D y\sin\pi x\,dA$, D is bounded by $x = 0$, $x = y^2$, and $y = 2$

   Ans: (a) $1 - e^{-4}$; (c) $2/\pi$

2. Evaluate the volume of the solid that lies under the surface $z = x(y+2)$ and above the triangle with vertices $P(1,1)$, $Q(3,1)$, and $R(1,3)$.

   Ans: 12

3. Sketch the region of the integral and change the order of integration.

   (a) $\displaystyle\int_0^1 \int_0^{y^2} f(x,y)\,dx\,dy$ (b) $\displaystyle\int_1^e \int_0^{\ln x} f(x,y)\,dy\,dx$

   Ans: (b) $\int_0^1 \int_{e^y}^e f(x,y)\,dx\,dy$

4. Evaluate the integral by reversing the order of integration:

   (a) $\displaystyle\int_0^1 \int_{x^2}^1 \sqrt{y}\cos(y^2)\,dy\,dx$ (b) $\displaystyle\int_0^4 \int_0^{\sqrt{4-y}} e^{12x-x^3}\,dx\,dy$

   Ans: (a) $\frac{\sin 1}{2}$

5. In evaluating a double integral over a region D, a sum of iterated integrals was obtained as follows:

$$\iint_D f(x,y)\,dA = \int_0^1 \int_0^y f(x,y)\,dx\,dy + \int_1^2 \int_0^{2-y} f(x,y)\,dx\,dy.$$

   (a) Sketch the region D.
   (b) Express the double integral as a ***single*** iterated integral with **reversed** order of integration.

15.3. Double Integrals in Polar Coordinates

We have spent most of our lives in the Cartesian/Rectangular coordinate system, which was invented by none other than *René Descartes*, who was because he thought. Sometimes, however, functions (and consequently integrals) become simpler when expressed in different coordinate systems. There are many different coordinate systems. Here, we will focus on one that was invented by *Sir Isaac Newton* – the **polar coordinate system**.

Polar Coordinates

Definition 15.23. ***(Polar point).*** *Points in polar coordinate system are defined by two parameters (r, θ), where r is the distance the point is from the origin and θ is the angle between the polar axis (positive x-axis) and the line that connects the point to the origin.*

Since a picture is worth a thousand words, here is a picture describing what was just described:

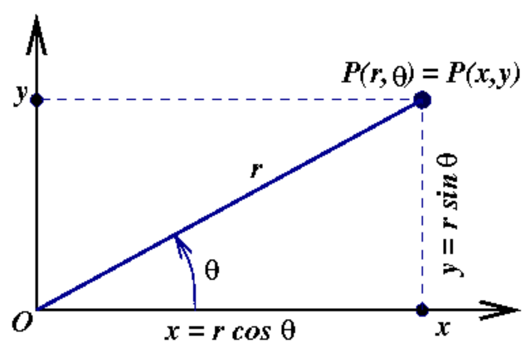


Figure 15.6: Point in Rectangular/Cartesian and Polar coordinates.

Naturally, there is a conversion from the Polar Coordinates to the Rectangular Coordinate system and vice versa. That conversion looks like:

$(x, y)_R \leftarrow (r, \theta)_P$	$(r, \theta)_P \leftarrow (x, y)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$

(15.14)

Frequently Used Trigonometric Formulas

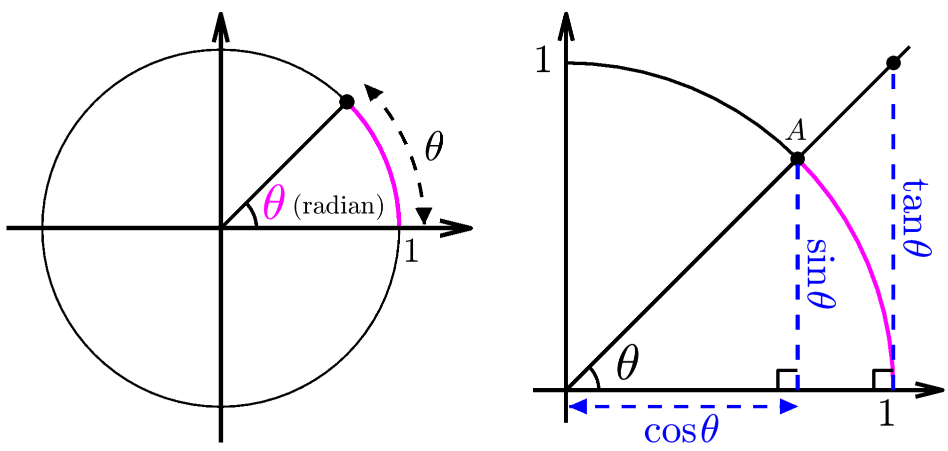


Figure 15.7: A definition of the **angle** and trigonometric functions.

$$
\begin{aligned}
&\sin^2 x + \cos^2 x = 1 && 1 + \tan^2 x = \sec^2 x \\
&\sin 2x = 2 \sin x \cos x && \cos 2x = \cos^2 x - \sin^2 x \\
&\sin^2 x = \frac{1 - \cos 2x}{2} && \cos^2 x = \frac{1 + \cos 2x}{2}
\end{aligned}
\tag{15.15}
$$

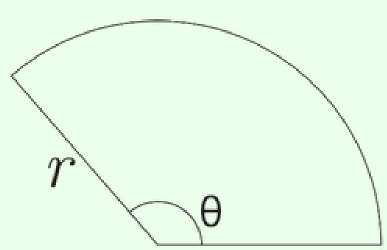


Figure 15.8

Sectors: arc length and area

$$
\begin{aligned}
&\text{Arc length:} && \ell = r\theta \\
&\text{Area:} && A = \frac{1}{2} r\ell = \frac{1}{2} r^2 \theta
\end{aligned}
\tag{15.16}
$$

15.3.1. Polar Rectangles and Iterated Integrals

Consider a **polar rectangle**:

$$R = \{(r, \theta) \mid a \le r \le b, \ \ \alpha \le \theta \le \beta\}.$$

Let $\Delta r = (b-a)/m$ and $\Delta\theta = (\beta-\alpha)/n$, for some m, n, and

$$\begin{aligned} r_i &= a + i\Delta r, \quad i = 0, 1, \cdots, m, \\ \theta_j &= \alpha + j\Delta\theta, \quad j = 0, 1, \cdots, n. \end{aligned}$$

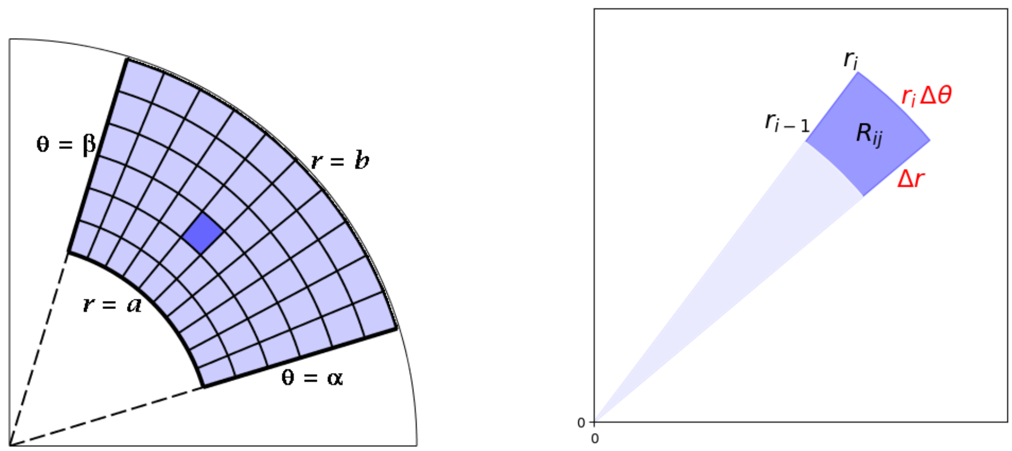


Figure 15.9: Dividing the polar rectangle $R = ([a, b] \times [\alpha, \beta])_P$: (left) polar subrectangles and (right) zoom-in of $R_{ij} = ([r_{i-1}, r_i] \times [\theta_{j-1}, \theta_j])_P$.

The area of R_{ij} is

$$\Delta A_{ij} = \frac{1}{2} r_i^2 \Delta\theta - \frac{1}{2} r_{i-1}^2 \Delta\theta = \frac{1}{2}(r_i + r_{i-1})(r_i - r_{i-1})\Delta\theta = r_i^* \Delta r \Delta\theta, \qquad (15.17)$$

where $r_i^* = (r_i + r_{i-1})/2$.

$\boxed{\text{Theorem}}$ **15.24.** ***(Polar version of iterated integral).*** *If f is continuous on the polar rectangle R given by $0 \le a \le r \le b, \alpha \le \theta \le \beta$, where $0 \le \beta - \alpha \le 2\pi$, then*

$$\iint\limits_R f(x, y)\, dA = \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta. \qquad (15.18)$$

Note: ① Do not forget the "$\boldsymbol{r}$" before the $\boldsymbol{dr\,d\theta}$!
② It follows from Figure 15.9 that $\boldsymbol{\Delta A_{ij} \approx \Delta r \cdot r_i \Delta\theta = r_i \Delta r \Delta\theta}$.

Problem **15.25.** Evaluate $\iint\limits_R (3x+4y^2)\,dA$, where R is the region in the upper half-plane bounded by the circles $x^2+y^2=1$ and $x^2+y^2=4$.
Solution.

Ans: $15\pi/2$

Problem **15.26.** Find the volume of the solid bounded by the plane $z=0$ and the paraboloid $z=1-x^2-y^2$.
Solution. Volume $V=\iint\limits_D (1-x^2-y^2)\,dA$, where $D=\{(x,y)\mid x^2+y^2\le 1\}$.

Ans: $\pi/2$

15.3.2. General Polar Regions

Theorem 15.27. ***(Polar version of (15.13), p. 474)***
If f is continuous on a polar region of the form

$$D = \{(r, \theta) \mid \alpha \le \theta \le \beta, h_1(\theta) \le r \le h_2(\theta)\},$$

then

$$\iint\limits_D f(x, y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta. \tag{15.19}$$

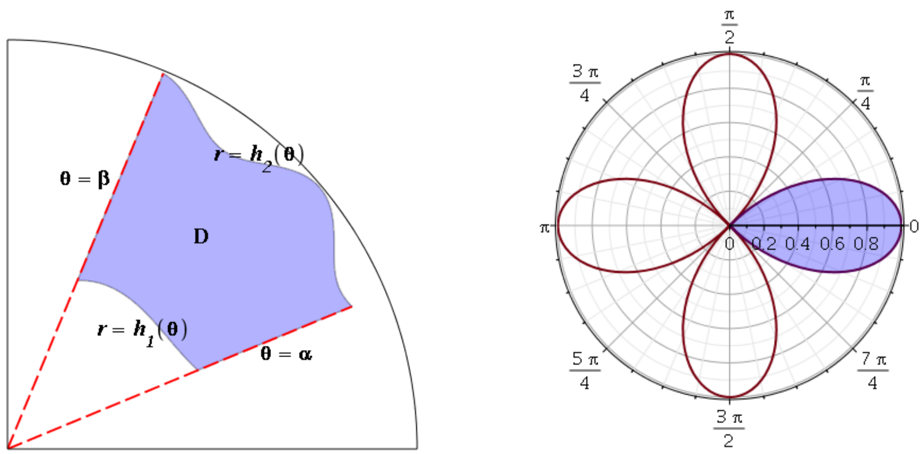


Figure 15.10: A general polar region and the four-leaved rose.

Problem 15.28. Plot the **four-leaved rose** $r = \cos(2\theta)$.

Solution. ***Hint***: First draw a plot in the **θr-coordinates** and convert it to the **xy-coordinates**.

Problem 15.29. Use a double integral to find the area enclosed by one loop of the four-leaved rose $r = \cos(2\theta)$.

Solution. $A(D) = \iint\limits_D dA = \int_{-\pi/4}^{\pi/4} \int_0^{\cos 2\theta} r\,dr\,d\theta.$

Ans: $\pi/8$

Problem 15.30. Find the volume of the solid that lies under the paraboloid $z = x^2 + y^2$, above the xy-plane, and inside the cylinder $x^2 + y^2 = 2x$.

- First, find what the **polar region** looks like.
  - That is to say, translate $x^2 + y^2 = 2x$ into polar coordinates and see what that region looks like. (Also, you may refer to $(x-1)^2+y^2 = 1$.)
- Then, look at $z = x^2 + y^2$ as a polar function; use it as your integrand.
- Don't forget the r in **"$r\,dr\,d\theta$"**!

Solution.

Ans: $\frac{3}{2}\pi$

Problem 15.31. (A variant of Problem 15.30)

Evaluate the double integral $\int_0^2 \int_0^{\sqrt{2x-x^2}} \sqrt{x^2+y^2}\,dy\,dx$, by recognizing the region and converting it to polar coordinates.

Solution. ***Hint***: $D = \{\theta = 0..\pi/2,\ \ r = 0..2\cos\theta\}$

Ans: 16/9

Volume of n-Ball: The **unit interval** $[-1, 1]$ can be rewritten as

$$B_1 \stackrel{\text{def}}{=} \{x \mid x^2 \le 1\} \subset \mathbb{R}. \tag{15.20}$$

Similarly, the **unit circle** and the **unit sphere (of radius 1)** read

$$B_2 \stackrel{\text{def}}{=} \{(x,y) \mid x^2+y^2 \le 1\} \subset \mathbb{R}^2 \text{ and } B_3 \stackrel{\text{def}}{=} \{(x,y,z) \mid x^2+y^2+z^2 \le 1\} \subset \mathbb{R}^3. \tag{15.21}$$

In general, an **n-dimensional ball (or n-Ball) of radius r** is defined as

$$B_{n,r} = \{(x_1, x_2, \cdots, x_n) \mid x_1^2 + x_2^2 + \cdots + x_n^2 \le r^2\} \subset \mathbb{R}^n. \tag{15.22}$$

It is possible to define **volume of n-Ball of radius** r, $V_{n,r}$; in $\mathbb{R}$ it is **length**, in $\mathbb{R}^2$ it is **area**, in $\mathbb{R}^3$ it is **ordinary volume**, and in $\mathbb{R}^n$, $n \ge 4$, it is called a **hypervolume**. For example,

$$V_{1,r} = V(B_{1,r}) = 2r, \quad V_{2,r} = V(B_{2,r}) = \pi r^2, \quad V_{3,r} = V(B_{3,r}) = \frac{4}{3}\pi r^3. \tag{15.23}$$

Note that $\boldsymbol{V_{n,r} = V_{n,1} \cdot r^n,\ n \ge 1}$.

Challenge 15.32. *Let $B_n = B_{n,1}$ and $V_n = V(B_{n,1})$. Use* ***polar coordinates*** *to find V_n, the volume of the unit n-Ball B_n, $n \ge 4$.*

Solution. See the figure.

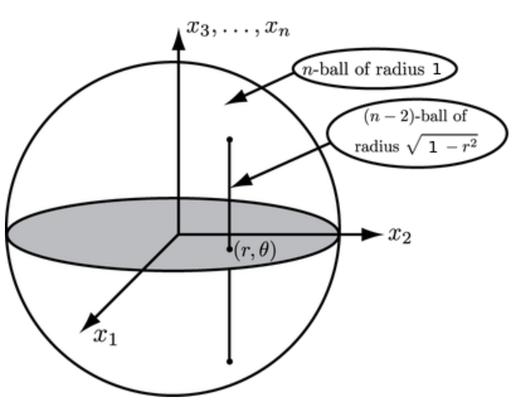


Figure 15.11: The unit n-Ball, $B_{n,1}$.

$$\begin{aligned} V_n &= \int_0^{2\pi} \int_0^1 \left[V_{n-2}\big(\sqrt{1-r^2}\big)^{n-2}\right] r\,dr\,d\theta \\ &= V_{n-2} \int_0^{2\pi} \int_0^1 (1-r^2)^{(n-2)/2}\, r\,dr\,d\theta \\ &= V_{n-2} \cdot 2\pi \cdot \int_0^1 (1-r^2)^{(n-2)/2}\, r\,dr \end{aligned}$$

Ans: $\boldsymbol{V_n = \frac{2\pi}{n} V_{n-2}}$. (You will solve this problem differently in P.6, p. 722.)

$$V_1 = \mathbf{2}, \quad V_2 = \pi, \quad V_3 = \frac{2\pi}{3} V_1 = \frac{\mathbf{4}\pi}{\mathbf{3}}, \quad V_4 = \frac{2\pi}{4} V_2 = \frac{\pi^{\mathbf{2}}}{\mathbf{2}}, \cdots$$

Exercises 15.3

1. Use polar coordinates to evaluate the double integral, or the volume of the solid.

   (a) $\displaystyle\iint_D e^{x^2+y^2}\,dA$, where D is the region bounded by the semi-circle $x = \sqrt{1-y^2}$ and the y-axis.

   (b) The solid that lies above the cone $z = \sqrt{x^2+y^2}$ and below the sphere $x^2+y^2+z^2 = 8$.

   Ans: (a) $(e-1)\pi/2$; (b) $\frac{32(\sqrt{2}-1)\pi}{3}$

2. A swimming pool is circular with 60–ft diameter. The depth is constant along east-west lines and increases linearly from 2 ft at the east end to 8 ft at the west end. Find the volume of water in the pool, using a double integral in polar coordinates. ***Hint***: $V = \iint_D (5 + \frac{x}{10})\,dA$, where D is the circle of radius 30 and centered at the origin.

3. Use polar coordinates to evaluate

$$\iint_{D_a} e^{-x^2-y^2}\,dA, \tag{15.24}$$

   where D_a is the disk of radius a centered at the origin.

   Ans: $\pi(1-e^{-a^2})$

4. We may define the improper integral (over the entire plane $\mathbb{R}^2$)

$$I := \iint_{\mathbb{R}^2} e^{-x^2-y^2}\,dA = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-x^2-y^2}\,dx\,dy = \lim_{a\to\infty}\iint_{D_a} e^{-x^2-y^2}\,dA. \tag{15.25}$$

   (a) Use the result from the previous problem (Problem 3, Exercises 15.3) to conclude

$$\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}. \tag{15.26}$$

   (b) Let $\sigma > 0$. Use the change of variable $x = \sigma t$ to evaluate

$$\int_{-\infty}^{\infty} e^{-x^2/\sigma^2}\,dx. \tag{15.27}$$

15.4. Applications of Double Integrals

Objectives. Find the **mass** and **center of mass** of a planar lamina and **moments of inertia**, using double integrals. Then, apply them for **probability** and **mean** values.

Density and Mass

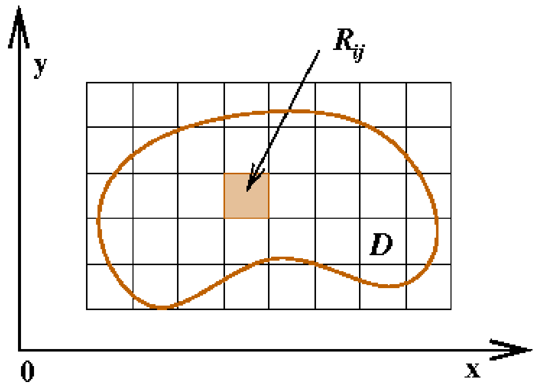


Figure 15.12

Let a lamina occupy a region D in xy-plane. Then its **density** is defined as

$$\rho(x,y) = \lim_{\Delta A \to 0} \frac{\Delta m}{\Delta A}, \tag{15.28}$$

where Δm and ΔA the mass and the area of a small rectangle that contains (x, y). Thus, the mass of the lamina over D approximates

$$m \approx \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A.$$

By increasing the number of subrectangles, we obtain the **total mass** of the lamina

$$m = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \rho(x_{ij}^*, y_{ij}^*)\Delta A = \iint_D \rho(x,y)\, dA. \tag{15.29}$$

Problem 15.33. Find the mass of the triangular lamina with vertices $(0,0)$, $(2,2)$, and $(0,4)$, given that the density at (x,y) is $\rho(x,y) = 2x + y$.

Solution.

Ans: $40/3$

Definition 15.34. *The **moment** of a particle about an axis is the product of its mass and its **directed distance** from the axis. Say,* $M_x = m \cdot y$, $M_y = m \cdot x$.

Theorem 15.35. *The **moments (first moments)** of the entire lamina about* x*- and* y*-axes:*

$$M_x = \iint\limits_D y\, \rho(x,y)\, dA, \quad M_y = \iint\limits_D x\, \rho(x,y)\, dA. \tag{15.30}$$

*When we define the **center of mass*** $(\overline{x}, \overline{y})$ *so that* $m\,\overline{x} = M_y$ *and* $m\,\overline{y} = M_x$*, then*

$$\overline{x} = \frac{M_y}{m} = \frac{1}{m} \iint_D x\, \rho(x,y)\, dA, \ \ \overline{y} = \frac{M_x}{m} = \frac{1}{m} \iint_D y\, \rho(x,y)\, dA. \tag{15.31}$$

Problem 15.36. (Revisit of Problem 15.33). Find the center of mass for the triangular lamina with vertices $(0,0)$, $(2,2)$, and $(0,4)$, given that the density at (x,y) is $\rho(x,y) = 2x + y$.

Solution. We know $m = 40/3$.

Ans: $(\overline{x}, \overline{y}) = (4/5,\ 11/5)$

Probability

Recall: The **probability density function** f of a continuous random variable X is a function such that

$$f(x) \geq 0, \ \ \forall\, x \in \mathbb{R}, \ \ \text{and} \ \ \int_{-\infty}^{\infty} f(x)\, dx = 1.$$

The **probability** that X lies between a and b is

$$P(a \leq X \leq b) = \int_a^b f(x)\, dx.$$

Definition 15.37. *The **joint density function** of a pair of random variables X and Y is a function f such that*

$$f(x, y) \geq 0, \ \ \forall\,(x, y) \in \mathbb{R}^2, \ \ \textit{and} \ \ \iint\limits_{R^2} f(x, y)\, dA = 1.$$

*The **probability** that (X, Y) lies in a region D is*

$$P((X, Y) \in D) = \iint\limits_{D} f(x, y)\, dA.$$

Problem 15.38. If the joint density function for X and Y is given by

$$f(x, y) = \begin{cases} k(3x - x^2)\,(2y - y^2), & \text{if } (x, y) \in [0, 3] \times [0, 2], \\ 0, & \text{otherwise}. \end{cases}$$

find the constant k. Then, find $P(X \leq 2, Y \geq 1)$.

Solution.

Ans: $k = 1/6$; $P(X \leq 2, Y \geq 1) = 10/27$

Expected Values of X and Y

Recall: If f is a probability density function of a random variable X, then its **mean** is

$$\mu = \int_{-\infty}^{\infty} x\, f(x)\, dx.$$

Definition 15.39. *Let $f(x, y)$ be a joint density function of random variables X and Y. We define the X**-mean** and Y**-mean**, also called the **expected values**, of X and Y, to be*

$$\mu_1 = \iint\limits_{\mathbb{R}^2} x\, f(x, y)\, dA, \quad \mu_2 = \iint\limits_{\mathbb{R}^2} y\, f(x, y)\, dA.$$

Problem 15.40. Let $f(x, y) = \begin{cases} \frac{4-2x^2-2y^2}{3\pi}, & \text{if } x^2 + y^2 \le 1, \\ 0, & \text{otherwise}. \end{cases}$

(a) Verify f is a joint density function.

(b) Find $P(X \le 0, Y \ge 0)$.

(c) Find the expected values of X and Y.

Solution.

Ans: (b) $1/4$; (c) $\mu_1 = \mu_2 = 0$

Exercises 15.4

1. Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ.

   (a) D is the triangle with vertices $(0,0)$, $(4,0)$, and $(2,2)$; $\rho(x,y) = y$

   (b) D is the part of the disk $x^2 + y^2 \le 4$ in the first quadrant; ρ is proportional to its distance from the origin ***Hint***: Set $\rho(x,y) = k\sqrt{x^2+y^2}$ and use polar coordinates for the integrals.

   Ans: (a) $m = 8/3$, $(\overline{x}, \overline{y}) = (2,1)$; (b) $m = 4k\pi/3$, $(\overline{x}, \overline{y}) = (3/\pi, 3/\pi)$

2. CAS Use a computer algebra system (Maple, Mathematica, etc.) to find the mass and center of mass of the lamina that occupies the region D and has the given density function.

   (a) $D = \{(x,y) \mid 0 \le x \le ye^{-y},\ \ 0 \le y \le 1\}$; $\quad \rho(x,y) = (1+x^2)\cos y$

   Ans: $m \approx 0.2167$, $\ (\overline{x}, \overline{y}) \approx (0.1507, 0.5697)$

   (b) D is the region closed by the right loop of the four-leaved rose $r = \cos 2\theta$ (as shown in Figure 15.10 on page 485); $\rho(x,y) = \sqrt{x^2+y^2}$

3. Suppose X and Y are random variable with joint density function

$$f(x,y) = \begin{cases} k(x+1)y, & \text{if } 0 \le x \le 2,\ \ 0 \le y \le 1, \\ 0, & \text{otherwise}. \end{cases}$$

   (a) Find the value of the constant k.

   (b) Find $P(x \le 1, y \le 1)$

   (c) Find $P(x - y \ge 1)$

   (d) Find X-mean and Y-mean.

   Ans: (a) $k = 1/2$; (c) $11/48$

15.5. Surface Area

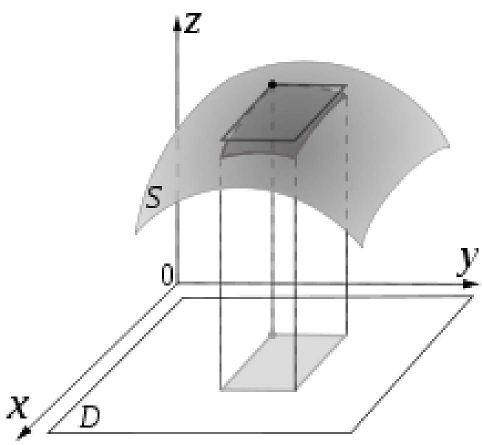


Figure 15.13

We may define the **surface area** of S to be

$$A(S) = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \Delta T_{ij}, \qquad (15.32)$$

where

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}|.$$

Here,

$$\begin{aligned} \mathbf{a} &= \langle \Delta x, 0, f_x(\mathbf{x}_{ij})\Delta x \rangle, \\ \mathbf{b} &= \langle 0, \Delta y, f_y(\mathbf{x}_{ij})\Delta y \rangle. \end{aligned}$$

Since

$$\mathbf{a} \times \mathbf{b} = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \Delta x & 0 & f_x \Delta x \\ 0 & \Delta y & f_y \Delta y \end{bmatrix} = \langle -f_x, -f_y, 1 \rangle \, \Delta x \Delta y, \qquad (15.33)$$

we have ($\Delta A = \Delta x \Delta y$)

$$\Delta T_{ij} = |\mathbf{a} \times \mathbf{b}| = \sqrt{f_x^2 + f_y^2 + 1}\, \Delta A. \qquad (15.34)$$

Definition 15.41. *The* ***surface area*** *of S with $z = f(x,y)$, $(x,y) \in D$, where ∇f is continuous, is*

$$A(S) = \iint_D \sqrt{f_x(x,y)^2 + f_y(x,y)^2 + 1}\, dA. \qquad (15.35)$$

Recall: For $y = f(x)$, $x \in [a,b]$, the **arc length** is obtained as

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx. \qquad (15.36)$$

Note: The surface area will be considered again when we learn Parametric Surfaces and Their Areas; see §16.6.2.

Problem 15.42. Find the area of the part of the paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.
Solution. (See Problem 16.89 on p. 588.)

Ans: $\frac{\pi}{6}(37\sqrt{37} - 1)$

Problem 15.43. Find the area of the part of the surface $z = xy$ that lies within the cylinder $x^2 + y^2 = 1$.
Solution.

Ans: $\frac{2\pi}{3}(2\sqrt{2} - 1)$

Exercises 15.5

1. Find the area of the surface.

   (a) The part of the plane $2x + y + 5z = 10$ that lies in the first octant

   (b) The part of the sphere $x^2 + y^2 + z^2 = 2$ that lies above the plane $z = 1$

   Ans: (b) $2\sqrt{2}\pi\,(\sqrt{2} - 1)$

2. Find the area of the finite part of the paraboloid $z = x^2 + y^2$ cut of by the plane $z = 9$.

3. CAS Use your calculator (or, a computer algebra system) to estimate the area of the surface correct to four decimal places.

   The part of the surface $z = \sin(x^2 + y^2)$ that lines in the cylinder $x^2 + y^2 = 4$.

   Hint: If you use Maple for numeric integration for $\int_a^b f(x)\,dx$, the command looks:
   `int(f(x),x=a..b,numeric)`

   Ans: 27.7291

15.6. Triple Integrals

The Limit of Riemann Sums

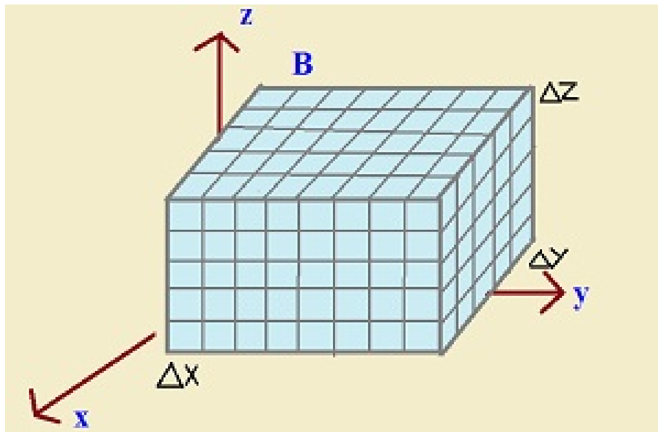


Figure 15.14: A rectangular box.

- Let's begin with **a function of three variables** defined on a **rectangular box**:

$$w = f(x, y, z), \quad (x, y, z) \in B := [a, b] \times [c, d] \times [r, s].$$

- In defining a triple integral, the first step is to divide B into sub-boxes.
- For some positive integers $\ell, m, n > 0$,

$$\Delta x = \frac{b-a}{\ell}, \quad \Delta y = \frac{d-c}{m}, \quad \Delta z = \frac{s-r}{n}.$$

  Let B_{ijk} be the (ijk)-th sub-box:

$$B_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{\ell-1}, z_\ell];$$

  each sub-box has volume $\Delta V = \Delta x \Delta y \Delta z$.

Definition 15.44. *The **triple integral** of f over the box B is*

$$\iiint_B f(x, y, z)\, dV = \lim_{\ell,m,n\to\infty} \sum_{i=1}^{\ell} \sum_{j=1}^{m} \sum_{k=1}^{n} f(\mathbf{x}_{ijk}^*)\Delta V. \qquad (15.37)$$

where $\mathbf{x}_{ijk}^* = (x_i^*, y_j^*, z_k^*) \in B_{ijk}$.

Theorem 15.45. ***(Fubini's Theorem for Triple Integrals)***. *If f is continuous on $B = [a,b] \times [c,d] \times [r,s]$, then*

$$\iiint\limits_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_r^s f(x,y,z)\,dz\,dy\,dx; \tag{15.38}$$

the integration order can be changed for five other choices.

Problem 15.46. Evaluate the triple integral $\iiint_B xyz^2\,dV$, where $B = [0,1] \times [-1,2] \times [0,3]$.

Solution.

Ans: $27/4$

Triple Integral over a General Bounded Region E:

Strategy 15.47. To evaluate a given triple integral over E:

1. Recognize (visualize in your brain) the domain E.
2. Separate the domain, e.g., $E = D \times [u_1(x,y), u_2(x,y)]$, $D \subset \mathbb{R}^2$.
   Then, $\iiint_E f(x,y,z)\,dV = \iint_D \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dA$.

The principle: You must find **a scheme** to **cover the whole domain** E (without missing, without duplicating).

Let's go on a journey!!

Problem 15.48. Evaluate $\iiint_E z\,dV$, where E is the solid tetrahedron bounded by the four planes $x = 0$, $y = 0$, $z = 0$, and $x + y + z = 1$.

Solution. $E = D \times [0, 1 - x - y]$, where D is the unit right triangle in the xy-plane.

Ans: $1/24$

Problem 15.49. Evaluate $\iiint_E \sqrt{x^2+y^2}\,dV$, where E is the region bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solution. $E = D \times [x^2 + y^2, 4]$, where D is the disk of center the origin and radius 2.

Ans: $128\pi/15$

Applications of Triple Integrals

Claim 15.50. Let $f(x, y, z) = 1$ for all points in E. Then triple integral of f over E represents the **volume** of E:

$$V(E) = \iiint_E 1\, dV. \tag{15.39}$$

Problem 15.51. Use the triple integral to find the volume of the tetrahedron T bounded by the four planes $x = 0$, $\boldsymbol{y = x}$, $z = 0$, and $x + y + z = 2$.

Solution.

Ans: $2/3$

Changing the Order of Integration

Problem 15.52. Write a couple of other iterated integrals that are equivalent to

$$\int_0^1 \int_y^1 \int_0^y f(x, y, z)\, dx\, dz\, dy$$

Hint: Change the order for adjacent two variables in the integral, keeping the other the same. For example, start with $x \leftrightarrow z$ or $z \leftrightarrow y$.

Solution.

Exercises 15.6

1. Evaluate the iterated integral.

   (a) $\displaystyle\int_0^2 \int_0^1 \int_0^{\ln x} xe^{-y}\, dy\, dx\, dz$ (b) $\displaystyle\int_0^{\pi} \int_0^2 \int_0^{\sqrt{4-z^2}} z\cos x\, dy\, dz\, dx$

   Ans: (a) -1; (b) 0

2. Evaluate the triple integral.

   (a) $\displaystyle\iiint_E e^{z/x}\, dV, \quad E = \{(x, y, z) \mid 0 \le x \le 1,\ x \le y \le 1,\ 0 \le z \le x\}$

   (b) $\displaystyle\iiint_E y\, dV$, E is determined by the paraboloid $y = x^2 + z^2$ and the plane $y = 4$

   Ans: (a) $(e-1)/6$; (b) $64\pi/3$

3. Fill the lower and upper bounds appropriately for the triple integral.

$$\int_0^1 \int_z^1 \int_y^1 f(x,y,z)\, dx\, dy\, dz = \int_{\boxed{1}}^{\boxed{2}} \int_{\boxed{3}}^{\boxed{4}} \int_{\boxed{5}}^{\boxed{6}} f(x,y,z)\, dx\, dz\, dy$$

$$= \int_{\boxed{7}}^{\boxed{8}} \int_{\boxed{9}}^{\boxed{10}} \int_{\boxed{11}}^{\boxed{12}} f(x,y,z)\, dz\, dx\, dy$$

   Ans: 5: y; 6: 1; 7: 0; 8: 1 11: 0; 12: y

15.7. Triple Integrals in Cylindrical Coordinates

Recall: **(Equation (15.14))**. The conversion between the **Polar Coordinates** and the Rectangular Coordinate system reads

$(x,y)_R \leftarrow (r,\theta)_P$	$(r,\theta)_P \leftarrow (x,y)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$

(15.40)

Definition 15.53. *In the **cylindrical coordinate system**, a **point** P in the 3D space is represented as an ordered triple (r,θ,z), where r and θ are polar coordinates of the projection of P onto the xy-plane and z is the directed distance from the xy-plane to P.*

Definition 15.54. *The conversion between the **Cylindrical Coordinates** and the Rectangular Coordinate system gives*

$(x,y,z)_R \leftarrow (r,\theta,z)_C$	$(r,\theta,z)_C \leftarrow (x,y,z)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

(15.41)

Note: The triple integral with a Cylindrical Domain E can be carried out by first separating the domain like

$$E = D \times [u_1(x,y), u_2(x,y)], \quad \text{where } D \text{ is a } \textbf{polar region}.$$

Problem 15.55. (a) Plot the point with the cylindrical coordinates $(2, \frac{2\pi}{3}, 1)_C$ and find its rectangular coordinates.
(b) Find cylindrical coordinates of the point with rectangular coordinates $(3, -3, 7)_R$.
Solution.

Ans: (a) $(-1, \sqrt{3}, 1)_R$; (b) $(3\sqrt{2}, -\pi/4, 7)_C$.

Problem 15.56. Evaluate $\displaystyle\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_{\sqrt{x^2+y^2}}^{2}(x^2+y^2)\,dz\,dy\,dx.$
Hint. Change the triple integral into cylindrical coordinates.
Solution.

Ans: $16\pi/5$

Problem 15.57. Find the volume of the solid that lies within both the cylinder $x^2 + y^2 = 1$ and the sphere $x^2 + y^2 + z^2 = 4$.

Solution.

Ans: $\frac{4\pi}{3}(8 - 3\sqrt{3})$

Exercises 15.7

1. Identify the surface whose equation is given.

   (a) $r^2 + 4z^2 = 4$ (b) $r = 2\cos\theta$

   Hint: (b) It can be rewritten as $r^2 = 2r\cos\theta$, which in turn reads $x^2 + y^2 = 2x$.

2. Evaluate $\displaystyle\int_0^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^{2} x\, dz\, dy\, dx$, by changing the triple integral into cylindrical coordinates.

   Ans: $8/3$

3. Use cylindrical coordinates to find the **volume** of the solid E that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2 + y^2 + z^2 = 2$.

   Ans: $\frac{4}{3}\pi(\sqrt{2} - 1)$

4. Use cylindrical coordinates to evaluate $\displaystyle\iiint_E y\, dV$, where E is the solid that lies between the cylinders $x^2+y^2 = 1$ and $x^2+y^2 = 9$, above the xy-plane, and below the plane $z = y+3$.

   Ans: 20π

15.8. Triple Integrals in Spherical Coordinates

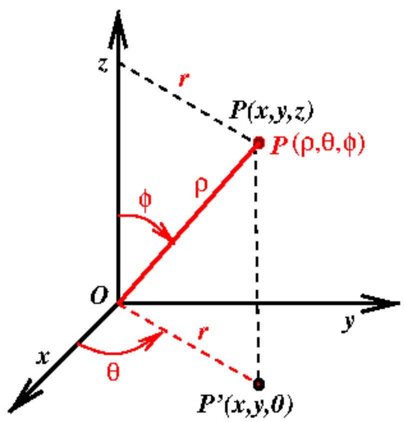


Figure 15.15: Spherical coordinates of P.

Definition 15.58. *The* ***spherical coordinates*** (ρ, θ, ϕ) *of a point* P *is shown in Figure 15.15, where* $\rho = |\overline{OP}| = \sqrt{x^2 + y^2 + z^2}$*,* θ *is the angle from the* x*-axis to the line segment* $\overline{OP'}$*, and* ϕ *is the angle between the positive* z*-axis and the line segment* $\overline{OP}$*.*

Note: By observing the definition, we can see the following inequalities:

$$\rho \geq 0 \qquad 0 \leq \theta \leq 2\pi, \qquad 0 \leq \phi \leq \pi.$$

For a **convenient conversion formula**, consider first

$$\mathbf{z} = \boldsymbol{\rho} \cos \boldsymbol{\phi}, \quad \mathbf{r} = \boldsymbol{\rho} \sin \boldsymbol{\phi}, \tag{15.42}$$

then use

$$\boldsymbol{x} = \boldsymbol{r} \cos \boldsymbol{\theta}, \quad \boldsymbol{y} = \boldsymbol{r} \sin \boldsymbol{\theta}.$$

Definition 15.59. *The conversion between the* ***Spherical Coordinates*** *and the Rectangular Coordinate system gives*

$(x, y, z)_R \leftarrow (\rho, \theta, \phi)_S$	$(\rho, \theta, \phi)_S \leftarrow (x, y, z)_R$
$\boldsymbol{x} = \boldsymbol{r} \cos \boldsymbol{\theta} = \rho \sin \phi \cos \theta$	$\rho^2 = x^2 + y^2 + z^2$
$\boldsymbol{y} = \boldsymbol{r} \sin \boldsymbol{\theta} = \rho \sin \phi \sin \theta$	$\cos \phi = \dfrac{z}{\rho}$
$z = \rho \cos \phi$	$\cos \theta = \dfrac{x}{\rho \sin \phi}$

(15.43)

Problem 15.60. (a) Plot the point with the spherical coordinates $(2, \pi/4, \pi/3)_S$ and find its rectangular coordinates.
(b) Find spherical coordinates of the point with rectangular coordinates $(0, 2\sqrt{3}, -2)_R$.

Solution.

Ans: (a) $(\sqrt{3/2}, \sqrt{3/2}, 1)_R$; (b) $(4, \pi/2, 2\pi/3)_S$

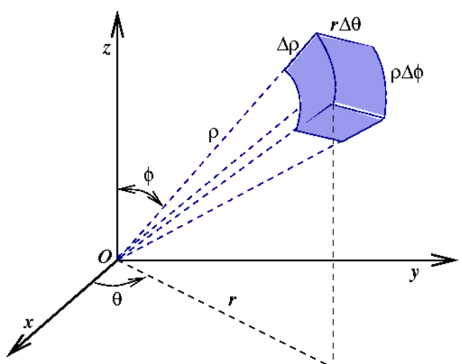


Figure 15.16: A small spherical wedge E_{ijk}, of volume $\Delta V_{ijk} \approx r\,\rho\,\Delta\rho\,\Delta\theta\,\Delta\phi$.

Triple Integral with Spherical Coordinates

In the spherical coordinate system, the counter part of a rectangular box is a **spherical wedge**

$$E = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\},$$

where $a \ge 0$, $\beta - \alpha \le 2\pi$, and $d - c \le \pi$. We divide smaller spherical wedges $\{E_{ijk}\}$ by means of equally spaced ρ_i, θ_j, ϕ_k. Figure 15.16 shows that E_{ijk} is approximately a rectangular box, of which the volume approximates

$$\Delta V_{ijk} \approx \Delta\rho \cdot r\Delta\theta \cdot \rho\Delta\phi = \mathbf{r}\rho\,\Delta\rho\,\Delta\theta\,\Delta\phi = \rho^{\mathbf{2}} \sin\phi\,\Delta\rho\,\Delta\theta\,\Delta\phi. \qquad (15.44)$$

$\boxed{\textbf{Theorem}}$ **15.61.** ***(Triple Integral on Spherical Wedges)*.**

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_\alpha^\beta \int_a^b f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi) \times \underline{\rho^2\,\sin\phi}\,d\rho\,d\theta\,d\phi, \tag{15.45}$$

where E is a spherical wedge given by

$$E = \{(\rho,\theta,\phi) \in \mathbb{R}^3 \mid a \le \rho \le b, \alpha \le \theta \le \beta, c \le \phi \le d\}.$$

Note: *The scaling factor* $\boxed{\rho^2\,\sin\phi = r\rho}$

Problem 15.62. Evaluate $\iiint_B e^{\left(x^2+y^2+z^2\right)^{3/2}}\,dV$, where B is the unit ball $B = \left\{(x,y,z) \mid x^2+y^2+z^2 \le 1\right\}$.

Solution.

Ans: $\frac{4}{3}\pi(e-1)$

Theorem 15.63. ***(Spherical Fubini's Theorem)***. *We can extend Theorem 15.61 to regions defined by*

$$E = \{(\rho, \theta, \phi) \in \mathbb{R}^3 \mid g_1(\theta, \phi) \le \rho \le g_2(\theta, \phi), \alpha \le \theta \le \beta, c \le \phi \le d\},$$

in such a way:

$$\iiint_E f(x, y, z)\, dV = \int_c^d \int_\alpha^\beta \int_{g_1(\theta,\phi)}^{g_2(\theta,\phi)} f\left(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi\right)\, \underline{\rho^2\, \sin\phi}\, d\rho\, d\theta\, d\phi. \quad (15.46)$$

Problem 15.64. Use spherical coordinates to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = z$.

Solution. Sphere: $\rho^2 = \rho\cos\phi \Rightarrow \rho = \cos\phi$.

Cone: $\rho\cos\phi = r = \rho\sin\phi \Rightarrow \cos\phi = \sin\phi$. So, $\phi = \pi/4$. Therefore,

$$V = \int_0^{\pi/4} \int_0^{2\pi} \int_0^{\cos\phi} \underline{\rho^2 \sin\phi}\, d\rho\, d\theta\, d\phi$$

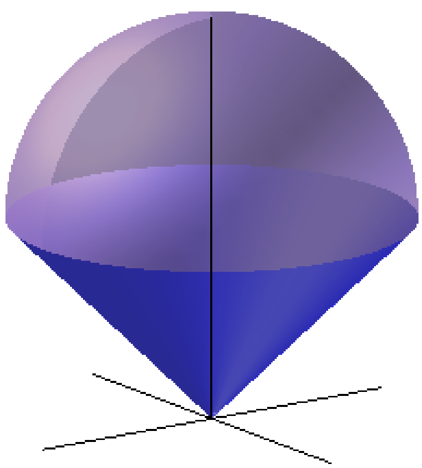


Figure 15.17

Ans: $\pi/8$

Exercises 15.8

1. Write the equation in spherical coordinates.

   (a) $x^2 + y^2 + z^2 = 1$
   (b) $z = x^2 + y^2$
   (c) $2x^2 + 2y^2 + z^2 = 4$
   (d) $z = x^2 - y^2$

   Hint: (c) $2x^2 + 2y^2 + z^2 = (x^2 + y^2 + z^2) + (x^2 + y^2)$

2. Sketch the solid whose volume is given by the integral; evaluate the integral.

   (a) $\displaystyle\int_0^{\pi/4}\int_0^{\pi}\int_0^{2} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$
   (b) $\displaystyle\int_0^{\pi/2}\int_0^{2\pi}\int_0^{\cos\phi} \rho^2 \sin\phi \, d\rho \, d\theta \, d\phi$

   Ans: (b) $\pi/6$

3. Use spherical coordinates to to find the volume of the solid that lies above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 1$.

4. Use spherical coordinates to evaluate $\displaystyle\iiint_B x\, e^{\left(x^2+y^2+z^2\right)^2}\, dV$, where B is the portion of the unit ball $x^2 + y^2 + z^2 \leq 1$ that lies in the first octant.

   Ans: $(e-1)\pi/16$

15.9. Change of Variables in Multiple Integrals

15.9.1. The Key Idea

We have done **changes of variables** several times in the past. Dating as far back when we learned integration with the "**u-substitution**", we started using changes of variables (we made $u = g(x)$.) Indeed,

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du, \qquad g:[a,b]\to[g(a),g(b)]. \tag{15.47}$$

Another way of the **change of variables** is

$$\int_a^b f(x)\,dx = \int_c^d f(x(u))\frac{dx}{\underline{du}}\,du, \tag{15.48}$$

where $x = x(u):[c,d]\to[a,b]$.

Example 15.65. Evaluate $\int_0^2 xe^{x^2}\,dx$.

Solution. ① $u = x^2 \Rightarrow du = 2\,x\,dx \Rightarrow x\,dx = \dfrac{du}{2}$; $u(0) = 0$, $u(2) = 4$. Therefore

$$\int_0^2 xe^{x^2}\,dx = \int_0^4 e^u\,\frac{du}{2} = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}e^u\Big|_0^4 = \frac{1}{2}(e^4-1).$$

② Another way: $\boxed{\boldsymbol{x = x(u) = \sqrt{u}}}$ $\Rightarrow$ $\dfrac{dx}{du} = \dfrac{1}{2\sqrt{u}}$. Therefore

$$\int_0^2 xe^{x^2}\,dx = \int_0^4 \sqrt{u}\,e^u\,\underline{\frac{1}{2\sqrt{u}}}\,du = \frac{1}{2}\int_0^4 e^u\,du = \frac{1}{2}(e^4-1).$$

Example 15.66. A change of variable is also useful in multiple integrals, as in double integrals in polar coordinates. For a polar region R, we have used the conversion:

$$x = r\cos\theta, \quad y = r\sin\theta,$$

which is a **transformation** from the $r\theta$-plane to the xy-plane. Then,

$$\iint_R f(x,y)\,dA = \iint_Q f(r\cos\theta, r\sin\theta)\,\underline{r}\,dr\,d\theta, \tag{15.49}$$

where Q is the region in the $r\theta$-plane.

Goal: The goal of this section is to write a general form for a change of variables, which turns the integral easier.

Definition 15.67. *A change of variables is a* ***transformation*** $T : Q \to R$ *(from the* uv*-plane to the* xy*-plane),* $T(u, v) = (x, y)$*, where* x *and* y *are related to* u *and* v *by the equations*

$$x = g(u, v), \quad y = h(u, v). \qquad [\textbf{or}, \ \mathbf{r}(u, v) = \langle g(u, v), h(u, v)\rangle]$$

We usually take these transformations to be C^1***-Transformation****, meaning* g *and* h *have continuous first-order partial derivatives, and* ***one-to-one****.*

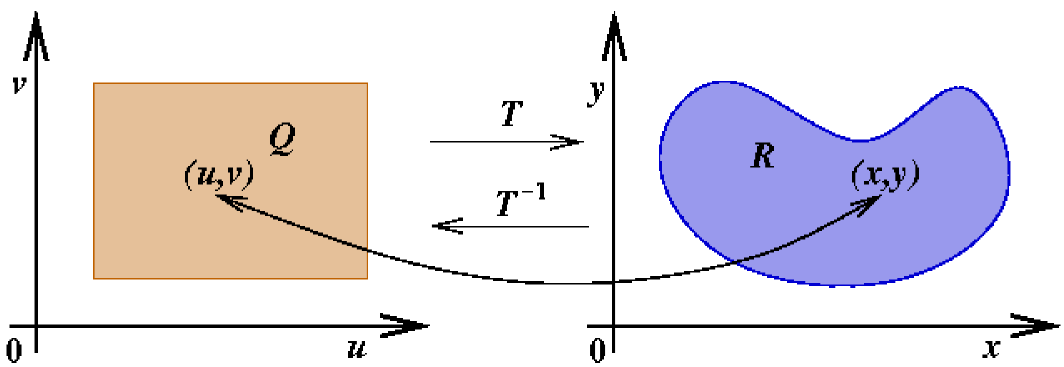


Figure 15.18: Transformation: $R = T(Q)$, the **image** of T.

Problem 15.68. A transformation is defined by $\mathbf{r}(u, v) = \langle 2u - v, u + v\rangle$. Find the image of the unit square $Q = \{(u, v) \mid 0 \le u \le 1,\ 0 \le v \le 1\}$.

Solution.

Ans: A rectangle of vertices $(0, 0)$, $(2, 1)$, $(1, 2)$, $(-1, 1)$.

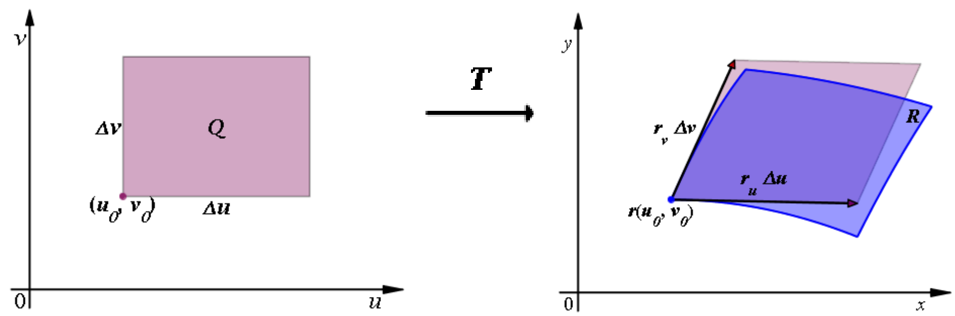


Figure 15.19: A small rectangle in the uv-plane and its image of T in the xy-plane.

Now, let's see how a change of variables affects a double integral.

- See Figure 15.19, where $T : Q \to R$ given by

$$\mathbf{r}(u, v) = \langle x, y \rangle = \langle g(u, v), h(u, v) \rangle . \tag{15.50}$$

- The **tangent vectors** at $\mathbf{r}(u_0, v_0)$ w.r.t the u- and v-directions are

$$\mathbf{r}_u(u_0, v_0) = \langle g_u, h_u \rangle (u_0, v_0), \quad \mathbf{r}_v(u_0, v_0) = \langle g_v, h_v \rangle (u_0, v_0).$$

- We can approximate the image region $R = T(Q)$ by a parallelogram determined by the *scaled* tangent vectors. Therefore,

$$\Delta A = A(R) \approx |(\mathbf{r}_u \Delta u) \times (\mathbf{r}_v \Delta v)| = |\mathbf{r}_u \times \mathbf{r}_v| \, \Delta u \, \Delta v. \tag{15.51}$$

- Computing the cross product, we obtain

$$\mathbf{r}_u \times \mathbf{r}_v = \det \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{bmatrix} = \det \begin{bmatrix} x_u & y_u \\ x_v & y_v \end{bmatrix} \mathbf{k} \tag{15.52}$$

Definition 15.69. *The **Jacobian of** $T : x = g(u, v),\ y = h(u, v)$ is*

$$\frac{\partial(x, y)}{\partial(u, v)} \overset{\text{def}}{=\!=} \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u \, y_v - x_v \, y_u. \tag{15.53}$$

Summary 15.70. *For a differentiable transformation $T : Q \to R$ given by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v)\rangle$,*

$$\boldsymbol{\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\, \Delta v.} \tag{15.54}$$

Theorem 15.71. *Suppose that T is a C^1-**transformation** whose **Jacobian** is nonzero, and suppose that T maps a region Q in the uv-plane onto a region R in the xy-plane. Let f be a continuous function on R. Suppose also that T is an **one-to-one** transformation except perhaps along the boundary of the regions. Then*

$$\boldsymbol{\iint\limits_R f(x,y)\, dA = \iint\limits_Q f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv.} \tag{15.55}$$

Example 15.72. (Transformation to polar coordinates). The transformation from $Q = [a,b] \times [\alpha,\beta]$ in the $r\theta$-plane to R in the xy-plane is given by

$$T: \;\; x = g(r,\theta) = r\cos\theta, \;\; y = h(r,\theta) = r\sin\theta.$$

The Jacobian of T is

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix} = r\cos^2\theta + r\sin^2\theta = r.$$

Thus Theorem 15.24 (p. 483) gives

$$\begin{aligned} \iint\limits_R f(x,y)\, dA &= \iint\limits_Q f(r\cos(\theta), r\sin(\theta)) \left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\, d\theta \\ &= \int_\alpha^\beta \int_a^b f(r\cos(\theta), r\sin(\theta))\, r\, dr\, d\theta. \end{aligned} \tag{15.56}$$

15.9.2. Multiple Integrals by Change of Variables

Summary 15.73. In order to evaluate the double integral $\iint_R f(x,y)\,dA$ by change of variables, you should *first* find a **region *Q*** and an **one-to-one transformation *T*** such that

$$T(Q) = R. \tag{15.57}$$

We will call the region Q a **predomain** of R. The **scaling factor** from the predomain Q to the domain R is ***the absolute value of the determinant of the Jacobian matrix***.

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right|.$$

Problem 15.74. Evaluate $\displaystyle\iint_R (x+y)\,dA$, where R is the quadrilateral region with vertices given by $(0,0)$, $(3,-3)$, $(6,0)$, and $(3,3)$, using the transformation $x = u+3v$ and $y = u-3v$.

Solution.

Ans: 54

Problem 15.75. Evaluate the integral $\iint\limits_R e^{(x+y)/(x-y)}\,dA$, where R is the trapezoidal region with vertices $(1,0), (2,0), (0,-2)$, and $(0,-1)$.

Solution.

Ans: $\frac{3}{4}(e - e^{-1})$

Problem 15.76. Evaluate $\iint_R \sin(x^2 + 4y^2)\, dA$, where R is the region in the first quadrant bounded by $x^2 + 4y^2 = 4$.

Solution. Consider the transformation: $x = 2u$, $y = v$.

Ans: $\frac{\pi}{8}(1 - \cos 4)$

Triple Integrals

Definition 15.77. ***(Higher order Jacobian).*** *The* ***Jacobian*** *of T, given by*

$$x = g(u, v, w),\;\; y = h(u, v, w),\;\; z = k(u, v, w),$$

is the following determinant:

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \det \begin{bmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{bmatrix}. \tag{15.58}$$

Theorem 15.78. *Under hypotheses similar to those in Theorem 15.71, we have the following formula for triple integrals:*

$$\iiint\limits_R f(x, y, z)\, dV = \iiint\limits_Q f\left(x\left(u, v, w\right), y\left(u, v, w\right), z\left(u, v, w\right)\right) \left|\frac{\partial(x, y, z)}{\partial(u, v, w)}\right| du\, dv\, dw. \tag{15.59}$$

Self-study 15.79. Show that when dealing with spherical coordinates,

$$dV = \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi. \tag{15.60}$$

Recall. $x = \rho \sin\phi \cos\theta, y = \rho \sin\phi \sin\theta$, and $z = \rho\cos\phi$.

Exercises 15.9

1. Use the given transformation to evaluate the integral.

   (a) $\displaystyle\iint\limits_R y^2\,dA$, where R is the region bounded by $4x^2+9y^2=36$; $(x,y)=(3u,2v)$

   (b) $\displaystyle\iint\limits_R (3x-y)\,dA$, where R is the triangular region with the three vertices $(0,0)$, $(2,1)$, and $(1,3)$; $(x,y)=(2u+v,u+3v)$

   Hint: (a) $\iint_R y^2\,dA=\iint_Q 4v^2\cdot 6\,du\,dv$, where $Q=\{(u,v)\mid u^2+v^2\le 1\}$
   Hint: (b) $\iint_R(3x-y)\,dA=\iint_Q 5u\cdot 5\,du\,dv$; figure out Q by yourself

   Ans: (a) 6π; (b) $25/6$

2. Make an appropriate change of variables to evaluate the integral

   $$\iint\limits_R \sin(x^2+4y^2)\,dA,$$

   where R is the region in the first quadrant bounded by the ellipse $x^2+4y^2=1$.

3. Make an appropriate change of variables to evaluate $\displaystyle\iint\limits_R 2(x-y)e^{x^2-y^2}\,dA$, where R is the rectangle enclosed by the lines: $x-y=0,\ x-y=1,\ x+y=0,\ x+y=2$.

4. Make an appropriate change of variables to evaluate the integral $\displaystyle\iint\limits_R e^{x+y}\,dA$, where R is given by the inequality $|x|+|y|\le 1$.

   Ans: $e-e^{-1}$

Chapter 16

Integrals and Vector Fields

In this chapter, we study the calculus of vector fields. In particular, you will learn

Subjects	Applications
Line integral	Work done by a force vector field in moving an object along a curve
Surface integral	The rate of fluid flow across a surface
Fundamental theorem of calculus, in 2/3-D	Green's theorem, Stokes's theorem, and Divergence theorem

Contents of Chapter 16

16.1. Vector Fields

16.1.1. Definitions

Definition 16.1. *If D is a region in $\mathbb{R}^2$, a* ***(2D) vector field*** *on D is a function $\boldsymbol{F}$ that assigns to each point $(x, y) \in D$ a two-dimensional vector $\boldsymbol{F}(x, y)$. If D is a solid region in $\mathbb{R}^3$, a* ***(3D) vector field*** *on D is a function $\boldsymbol{F}$ that assigns to each point $(x, y, z) \in D$ a three-dimensional vector $\boldsymbol{F}(x, y, z)$.*

Expressions for vector fields:

$$\begin{aligned} \boldsymbol{F}(x, y) &= \langle P(x, y),\, Q(x, y) \rangle \\ &= P(x, y)\,\mathbf{i} + Q(x, y)\,\mathbf{j}, \\ \boldsymbol{F}(x, y, z) &= \langle P(x, y, z),\, Q(x, y, z),\, R(x, y, z) \rangle \\ &= P(x, y, z)\,\mathbf{i} + Q(x, y, z)\,\mathbf{j} + R(x, y, z)\,\mathbf{k}. \end{aligned}$$

Example 16.2. $\boldsymbol{F}(x, y) = \langle x,\, x - y \rangle$ is a vector field in $\mathbb{R}^2$. $\boldsymbol{G}(x, y, z) = x^2\,\mathbf{i} + y^2\,\mathbf{j} + z^2\,\mathbf{k}$ is a vector field in $\mathbb{R}^3$. Let's sketch $\boldsymbol{F}$.

(x, y)	$\boldsymbol{F}(x, y) = \langle x,\, x - y \rangle$
$(0, 0)$	
$(1, 0)$	
$(1, 1)$	
$(0, 1)$	

Problem 16.3. Let $\boldsymbol{F}(x, y) = \langle -y,\, x\rangle$. Describe $\boldsymbol{F}$ by sketching some of the vectors $\boldsymbol{F}(x, y)$.

Solution.

(x, y)	$\boldsymbol{F}(x, y) = \langle -y,\, x\rangle$
$(1, 0)$	
$(0, 1)$	
$(-1, 0)$	
$(0, -1)$	

Note:

- $\mathbf{x} \cdot \boldsymbol{F}(\mathbf{x}) = \langle x,\, y\rangle \cdot \langle -y,\, x\rangle = -xy + xy = 0.$
  Thus, $\boldsymbol{F}(\mathrm{x}) = \langle -y,\, x\rangle$ is perpendicular to the position vector x.
- $|\boldsymbol{F}(\mathbf{x})| = \sqrt{y^2 + x^2} = |\mathbf{x}|.$
  Therefore, $\boldsymbol{F}(\mathrm{x})$ has the same magnitude as x.

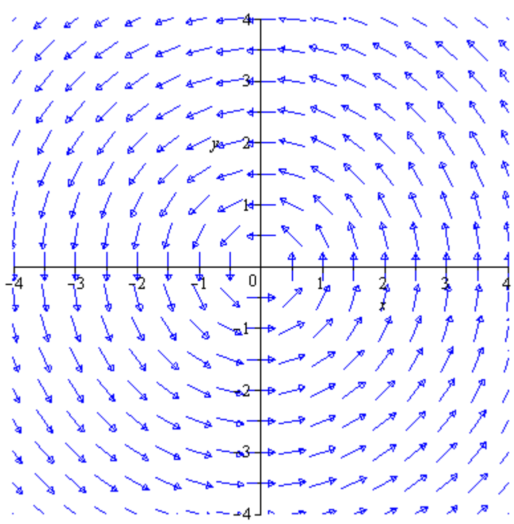


Figure 16.1: The vector field $\boldsymbol{F} = \langle -y, x\rangle$, showing directions only.

Vector fields in $\mathbb{R}^3$

Problem 16.4. Sketch the vector field on $\mathbb{R}^3$ given by $\boldsymbol{F}(x, y, z) = z\,\mathbf{k} = \langle 0,\, 0,\, z\rangle$.

Example 16.5.

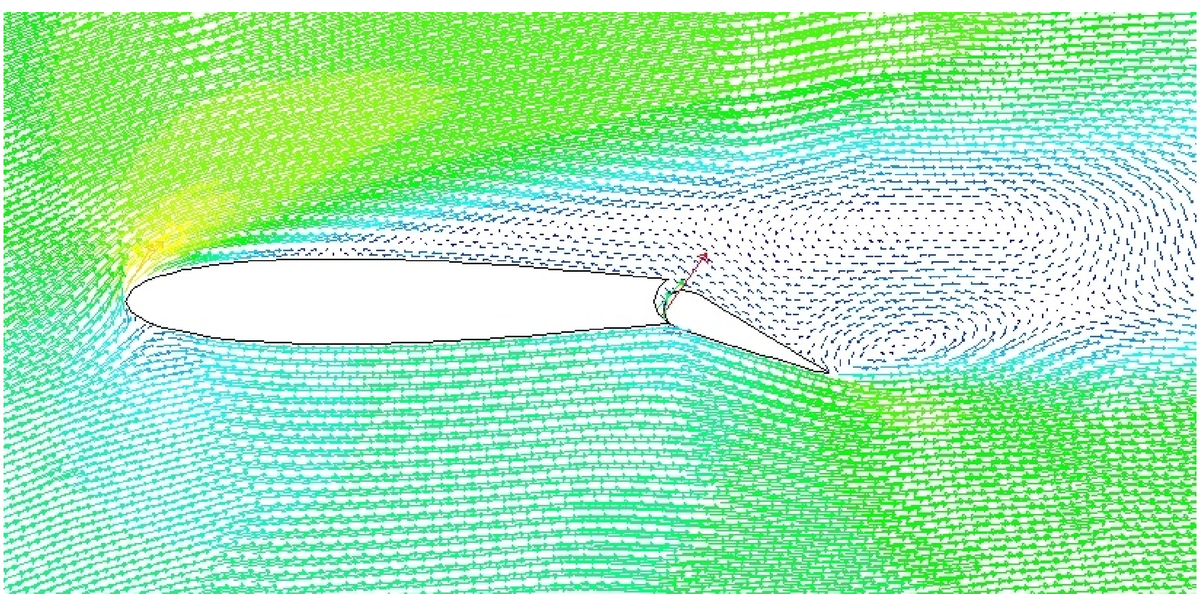


Figure 16.2: Airfoil simulation, showing the velocity field.

16.1.2. Conservative Vector Fields and Potential Functions

- Suppose that $f(x, y)$ is a differentiable function on D. Earlier we defined the **gradient** ∇f of f:

$$\nabla f(x, y) = \langle f_x(x, y),\ f_y(x, y)\rangle \ =\ f_x\,\mathbf{i} + f_y\,\mathbf{j}.$$

  We now see that ∇f is a two-dimensional vector field on D.
- Similarly, if $f(x, y, z)$ is a differentiable function on a solid $D \subset \mathbb{R}^3$, then $\nabla f(x, y, z)$ is a three-dimensional vector field on D.

From now on, we will refer to the gradient of a function f as the **gradient vector field** of f.

Problem 16.6. Find the gradient vector field of

$$f(x, y) = x^2 y - y^3.$$

Solution.

Ans: $\langle 2xy, x^2 - 3y^2\rangle$

Definition 16.7. *A vector field* $\boldsymbol{F}$ *is* ***conservative*** *if there is a differentiable function* f *such that*

$$\nabla f = \boldsymbol{F}.$$

The function f *is called a* ***potential function*** *of* $\boldsymbol{F}$*, or simply* ***potential****.*

Claim 16.8. Gradient fields are, always, conservative.

Problem 16.9. (Continuation of Problem 16.6). Let $\boldsymbol{F}(x,y) = \langle 2xy,\, x^2 - 3y^2 \rangle$. Then $\boldsymbol{F}$ is conservative.

Solution. Let's try to find f such that $\nabla f = \boldsymbol{F}$.

Ans: $f(x,y) = x^2y - y^3 + K$

Note: Not every vector field is conservative, and it is not difficult to give an example of a vector field that is **nonconservative**.

Example 16.10. Show that the vector field $\boldsymbol{F}(x,y) = (x^2 + y)\,\mathbf{i} + y^3\,\mathbf{j}$ is not conservative.

Proof. **Assume that $\boldsymbol{F}$ is conservative**. Then, there exists f such that $\nabla f = \langle f_x, f_y \rangle = \boldsymbol{F}$:

$$f_x = x^2 + y, \qquad f_y = y^3.$$

Then

$$f_{xy} = 1 \quad \text{and} \quad f_{yx} = 0. \tag{16.1}$$

Since both mixed partials are constants, they are continuous everywhere. Thus, by the Clairaut's theorem, we must have

$$f_{xy} = f_{yx}.$$

However, in (16.1), they are not equal. Contradiction! □

We will study properties of **conservative vector fields** in Section 16.3 below, in detail.

Problem 16.11. At time $t = 1$, a particle is located at $(1, 3)$. When it moves in a velocity field $\mathbf{v}(x, y) = \langle xy - 2,\, y^2 - 10\rangle$, find its approximate location at $t = 1.05$.

Solution. ***Clue***: $\mathbf{r}(t) \approx \mathbf{r}(t_0) + \mathbf{r}'(t_0) \cdot (t - t_0)$, where $\mathbf{r}'$ is the velocity vector.

Ans: $\langle 1.05, 2.95\rangle$

Exercises 16.1

1. Match the vector fields F with the plots labeled (I)–(IV). Give reasons for your choices.

   (a) $\boldsymbol{F} = \langle e^x, 5y \rangle$
   (b) $\boldsymbol{F} = \langle \sin(x+y), x \rangle$
   (c) $\boldsymbol{F} = \langle x+y, y \rangle$
   (d) $\boldsymbol{F} = \langle x, -y \rangle$

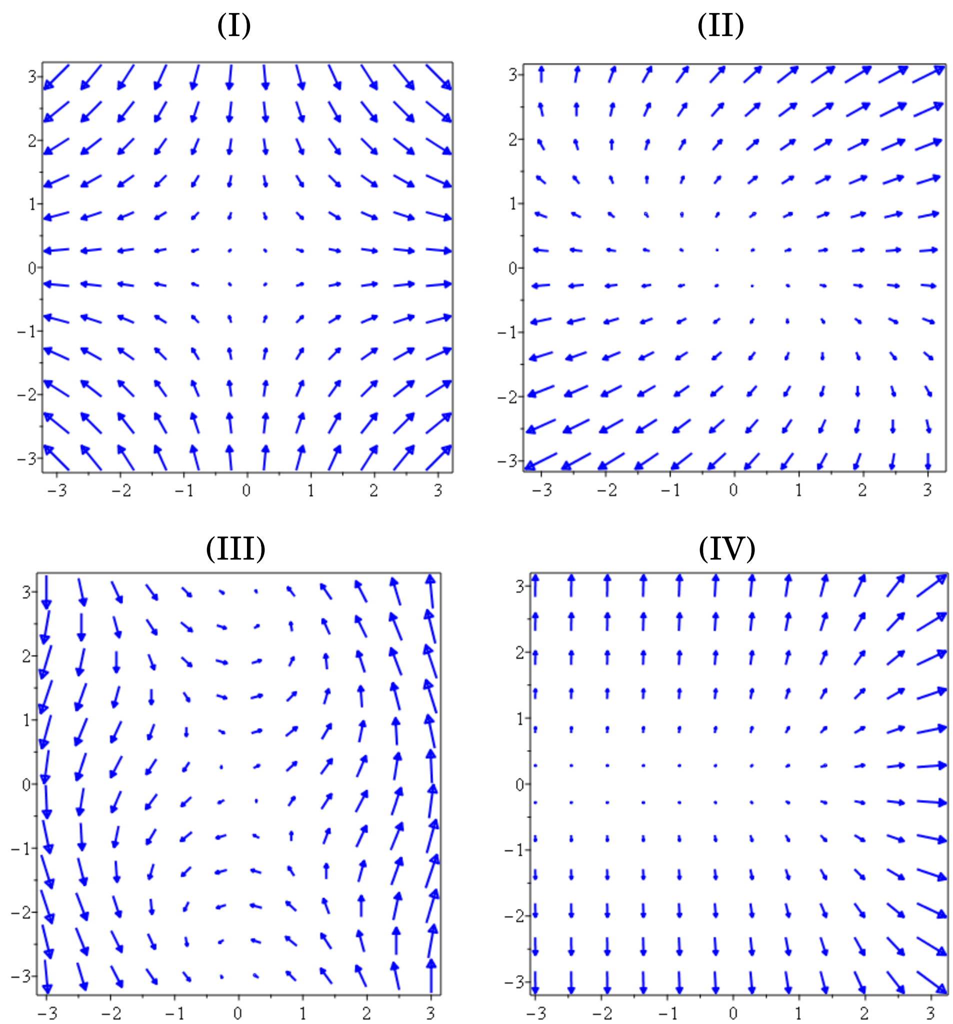


Figure 16.3: Maple `fieldplot`.

Hint: Let's see Figure (III), for example; arrows are directing up for $x > 0$ and down for $x < 0$, which implies that the second component of $\boldsymbol{F}$ is closely related to x. Now, what can you say about Figure (IV)? Arrows never look the west direction, which implies that the first component of $\boldsymbol{F}$ is nonnegative.

2. $\boxed{\text{CAS}}$ Use a CAS (`fieldplot` in Maple and `PlotVectorField` in Mathematica) to plot

$$\boldsymbol{F}(x,y) = (y^3 - xy^2)\,\mathbf{i} + (2xy - 2x^2)\,\mathbf{j}.$$

   Explain the appearance by finding the set of points (x, y) such that $\boldsymbol{F}(x,y) = 0$. (Attach the figure.)

3. Find the gradient vector field ∇f and sketch it.

   (a) $f(x,y) = \dfrac{(x-y)^2}{2}$ (b) $f(x,y) = \dfrac{x^3 - y^3}{3}$

4. Match the functions f with their gradient vector fields plotted with labels (I)–(IV). Give reasons for your choices.

   (a) $f(x,y) = xe^y$ (b) $f(x,y) = x^2 + y^2$

   (c) $f(x,y) = x(x-2y)$ (d) $f(x,y) = \cos(x^2 + y^2)$

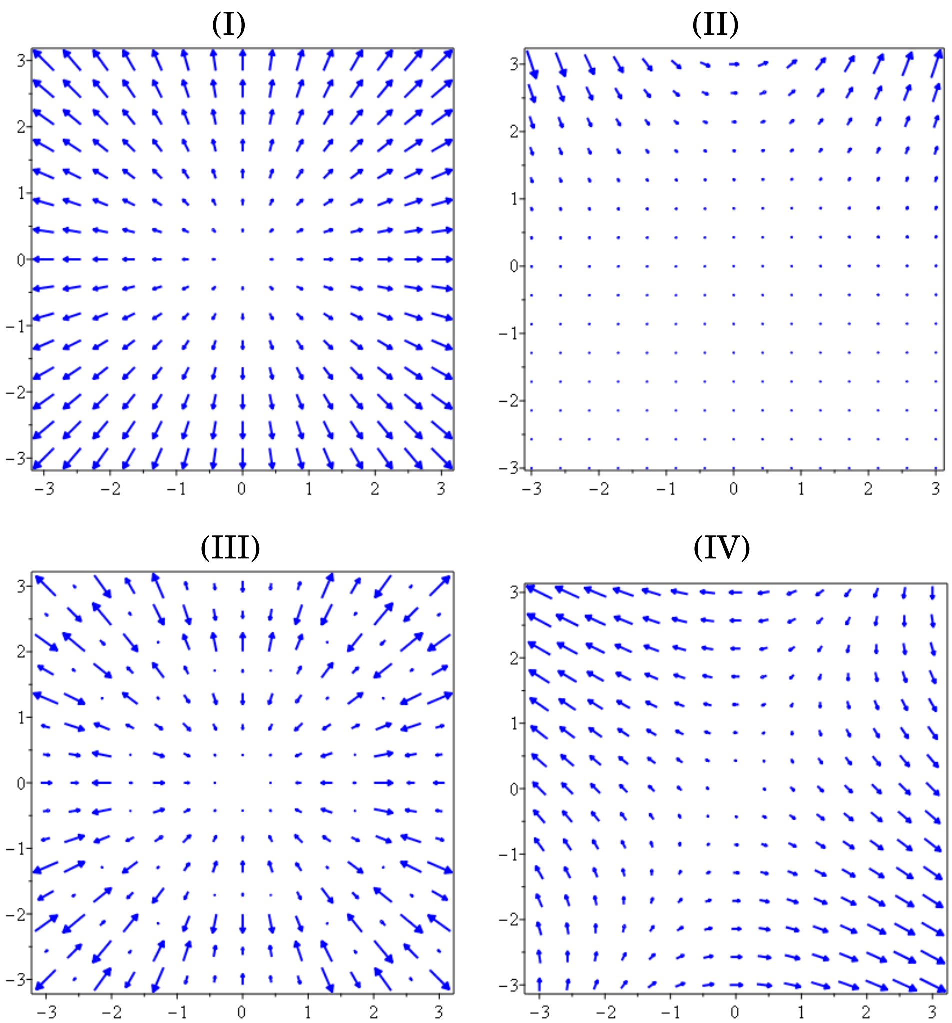


Figure 16.4: Maple `fieldplot` for ∇f.

Ans: (a) II

16.2. Line Integrals

Recall: In single-variable calculus, if a **force $f(x)$** is applied to an object to **move it along a straight line** from $x = a$ to $x = b$, then the amount of **work** done is given by the integral

$$W = \int_a^b f(x)\,dx \quad \Big(= \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\Delta x \Big). \tag{16.2}$$

Up to this point, our intervals of integration were always either bijective function or a closed interval $[a, b]$. In this section, we will be integrating over a **parametrized curve** instead of a nice interval as before.

Goal: To integrate functions along a curve, as opposed to along an interval.

Definition 16.12. *A **plane curve** C is given by the **vector equation***

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle, \quad a \le t \le b, \tag{16.3}$$

*or equivalently, by the **parametric equations***

$$x = g(t),\ \ y = h(t), \quad a \le t \le b. \tag{16.4}$$

Recall: You have learned

$$\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2} = \sqrt{\left(\frac{\Delta x_i}{\Delta t}\right)^2 + \left(\frac{\Delta y_i}{\Delta t}\right)^2}\,\Delta t$$

and therefore

$$\begin{aligned} ds &= \lim_{n\to\infty} \Delta s_i = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt \\ &= \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = |\mathbf{r}'(t)|\,dt. \end{aligned}$$

Thus the **arc length** of C can be computed as

$$L = \int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt.$$

16.2.1. Line Integrals for Scalar Functions in the Plane

Now, suppose that a force is applied to **move an object along a path** traced by a curve C. If the amount of force is given by $f(x, y)$, then the **amount of work** done must be given by the integral

$$W = \int_C f(x, y)\, ds, \tag{16.5}$$

where s is the arc length element, i.e., $ds = \sqrt{dx^2 + dy^2}$.

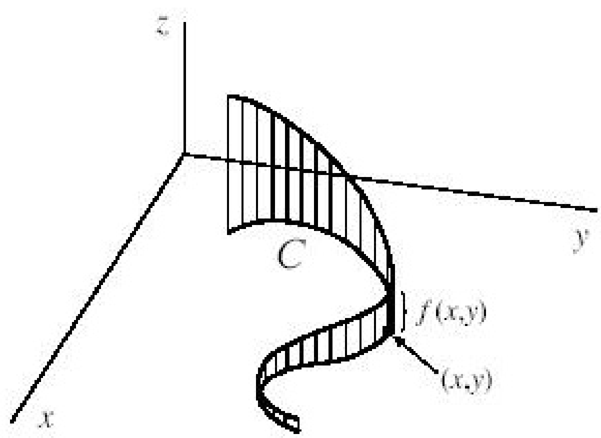


Figure 16.5: A function defined on a curve C.

Assumption. The curve C is smooth, i.e., $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq 0$.

Definition 16.13. *If f is defined on a smooth curve C given by (16.3), then **line integral of f along C** is*

$$\int_C f(x, y)\, ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\, \Delta s_i, \tag{16.6}$$

if this limit exists. Here $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$.

The line integral defined in (16.6) can be rewritten as

$$\begin{aligned}\textstyle\int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\,\sqrt{(x'(t))^2+(y'(t))^2}\,dt \\ &= \int_a^b f(x(t),y(t))\,|\mathbf{r}'(t)|\,dt.\end{aligned} \tag{16.7}$$

Problem 16.14. Evaluate $\displaystyle\int_C (2+x^2y)ds$, where C is upper half of the unit circle $x^2+y^2=1$.

Solution. ***Clue***: Find the parametric equation for C and then follow the formula (16.7).

Ans: $2\pi + \frac{2}{3}$

Definition 16.15. *C is a **piecewise smooth curve** if it is a union of a finite number of smooth curves $C_1,\ C_2,\ \cdots,\ C_n$. That is,*

$$C = C_1 \cup C_2 \cup \cdots \cup C_n.$$

In the case, we define the integral of f along C as the sum of the integrals of f along each of the smooth pieces of C:

$$\int_C f(x,y)\,ds = \int_{C_1} f(x,y)\,ds + \int_{C_2} f(x,y)\,ds + \cdots + \int_{C_n} f(x,y)\,ds. \quad (16.8)$$

Problem 16.16. Evaluate $\int_C 2x\,ds$, where C consists of the arc C_1 of the parabola $y = x^2$ from $(0,0)$ to $(1,1)$ followed by the vertical line segment C_2 from $(1,1)$ to $(1,2)$.

Solution. ***Clue***: Begin with parametric representation of C_1 and C_2. For example, $C_1 : x = t,\ y = t^2,\ 0 \le t \le 1$ and $C_2 : x = 1,\ y = t,\ 1 \le t \le 2$.

Ans: $\frac{1}{6}(5\sqrt{5} - 1) + 2$

Application to Physics: To compute the **mass of a wire** that is shaped like a plane curve C, where the density of the wire is given by a function $\rho(x,y)$ defined at each point (x,y) on C, we can evaluate the line integral

$$m = \int_C \rho(x,y)\,ds. \tag{16.9}$$

Thus **the center of mass of the wire** is the point $(\overline{x},\overline{y})$, where

$$\overline{x} = \frac{1}{m}\int_C x\rho(x,y)\,ds, \quad \overline{y} = \frac{1}{m}\int_C y\rho(x,y)\,ds. \tag{16.10}$$

Problem 16.17. A wire takes the shape of the semicircle, $x^2+y^2=1,\ \ y\ge 0$, and its density is proportional to the distance from the line $y=1$. Find the center of mass of the wire.

Solution. ***Clue***: First parametrize the wire and use $\rho(x,y)=k(1-y)$.

Ans: $(\overline{x},\overline{y}) = \left(0, \frac{4-\pi}{2(\pi-2)} \approx 0.38\right)$, where $m = k(\pi-2)$

16.2.2. Line Integrals with Respect to Coordinate Variables

Definition 16.18. ***Line integrals of f along C with respect to x and y are defined as***

$$\begin{aligned} \int_C f(x,y)\,dx &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*, y_i^*)\Delta x_i, \\ \int_C f(x,y)\,dy &= \lim_{n\to\infty}\sum_{i=1}^n f(x_i^*, y_i^*)\Delta y_i. \end{aligned} \tag{16.11}$$

Note that

$$x = x(t),\;\; y = y(t),\;\; \Rightarrow\;\; dx = x'(t)dt,\;\; dy = y'(t)dt.$$

Thus

$$\begin{aligned} \int_C f(x,y)\,dx &= \int_a^b f(x(t), y(t))x'(t)\,dt, \\ \int_C f(x,y)\,dy &= \int_a^b f(x(t), y(t))y'(t)\,dt. \end{aligned} \tag{16.12}$$

Remark 16.19. It frequently happens that line integral with respect x and y occur together. When this happens, it is customary to abbreviate by writing

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy.$$

Let $\boldsymbol{F}(x,y) = \langle P(x,y),\, Q(x,y)\rangle$ and $\mathbf{r} = \langle x, y\rangle = \langle x(t), y(t)\rangle$ represent the curve C. Then, since $d\mathbf{r} = \langle dx, dy\rangle$, we can rewrite the above as

$$\int_C P(x,y)\,dx + Q(x,y)\,dy = \int_C \boldsymbol{F}\cdot d\mathbf{r}, \tag{16.13}$$

which is a line integral of vector fields. We will consider it in detail in § 16.2.3 below (p. 544).

Problem 16.20. Evaluate $\displaystyle\int_C y^2\,dx + x\,dy$, where

$$\begin{cases} \text{(a)} \quad C = C_1: & \text{the line segment from } (-5,-3) \text{ to } (0,2) \\ \text{(b)} \quad C = C_2: & \text{the arc of } x = 4 - y^2 \text{ from } (-5,-3) \text{ to } (0,2) \end{cases}$$

Solution. ***Clue***: $C_1 : \mathbf{r}(t) = (1-t)\mathbf{r}_0 + t\mathbf{r}_1,\ 0 \le t \le 1$ and $C_2 : x = 4 - t^2,\ y = t,\ -3 \le t \le 2.$

Ans: (a) $-\frac{5}{6}$ (b) $40\frac{5}{6}$

Orientation of curves: It is important to note that the value of line integrals with respect to x or y (or z, in 3-D) depends on the **orientation of** C, unlike line integrals with respect to the arc length s. If the curve is traced in reverse (that is, from the terminal point to the initial point), then the sign of the line integral is reversed as well. We **denote by $-C$ the curve with its orientation reversed**. We then have

$$\int_{-C} P\,dx = -\int_C P\,dx, \quad \int_{-C} Q\,dy = -\int_C Q\,dy. \tag{16.14}$$

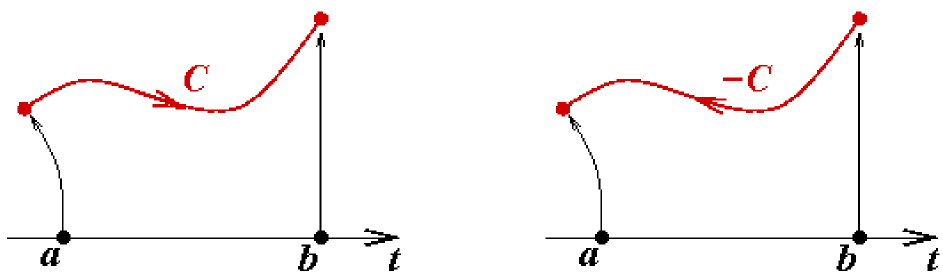


Figure 16.6: Curve C and its reversed curve $-C$.

Note: For line integrals with respect to the arc length s,

$$\int_{-C} f\,ds = \int_C f\,ds. \tag{16.15}$$

Problem 16.21. (Variant of Problem 16.20(a)): The reversed curve $-C_1$ is the line segment from $(0,2)$ to $(-5,-3)$:

$$\mathbf{r}(t) = (1-t)\langle 0,2\rangle + t\langle -5,-3\rangle = \langle -5t, -5t+2\rangle, \ \ 0 \le t \le 1.$$

Thus we must have $\displaystyle\int_{-C_1} y^2\,dx + x\,dy = \frac{5}{6}$.

Solution.

Line Integrals in Space

First, the definition for the line integral (with respect to arc length) can be generalized as follows.

Definition 16.22. *Suppose that C is a smooth space curve given by*

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$$

*Then the **line integral of f along C** is defined in a similar manner as in Definition 16.13:*

$$\int_C f(x,y,z)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*, z_i^*)\,\Delta s_i. \tag{16.16}$$

It can be evaluated using a formula similar to (16.7):

$$\begin{aligned} \int_C f(x,y,z)\,ds &= \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt \\ &= \int_a^b f(x(t),y(t),z(t))\,\sqrt{(x')^2+(y')^2+(z')^2}\,dt. \end{aligned} \tag{16.17}$$

Note:

- When $f(x,y,z) \equiv 1$,
$$\int_C ds = \int_a^b |\mathbf{r}'(t)|\,dt = L: \ \text{ arc length}$$
- When $\boldsymbol{F} = \langle P, Q, R\rangle$,
$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz.$$

Problem 16.23. Evaluate $\displaystyle\int_C y\sin z\,ds$, where C is the **circular helix** given by $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $0 \le t \le 2\pi$.

Solution. ***Hint***: You may use one of formulas: $\sin^2 t = (1-\cos 2t)/2$, $\cos^2 t = (1+\cos 2t)/2$.

Ans: $\sqrt{2}\pi$

Problem 16.24. Evaluate $\displaystyle\int_C z\,dx + x\,dy + y\,dz$, where C is given by $x = t^2,\ \ y = t^3,\ \ z = t^2, \quad 0 \le t \le 1.$

Solution.

Ans: $\frac{3}{2}$

16.2.3. Line Integrals of Vector Fields

Recall: Earlier we have found that the **work** done by a constant force $\boldsymbol{F}$, in moving an object from a point P to another point Q in the space, is

$$W = \boldsymbol{F}\cdot\boldsymbol{D}, \tag{16.18}$$

where $\boldsymbol{D} = \overrightarrow{PQ}$, the **displacement vector**. □

In general: Let C be a smooth space curve given by

$$\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$$

Then **the work done by a force $\boldsymbol{F}$** in moving an object along the curve C is

$$W = \lim_{n\to\infty}\sum_{i=1}^{n} \boldsymbol{F}(x_i^*, y_i^*, z_i^*)\cdot[\boldsymbol{T}(x_i^*, y_i^*, z_i^*)\,\Delta s_i] = \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds, \tag{16.19}$$

where $\mathbf{r}(t_i) = (x_i, y_i, z_i)$, $\Delta s_i = |\mathbf{r}(t_i) - \mathbf{r}(t_{i-1})|$, and $\boldsymbol{T}$ is the **unit tangential vector**

$$\boldsymbol{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}. \tag{16.20}$$

Since $ds = |\mathbf{r}'(t)|\,dt$, we have

$$W = \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds = \int_a^b \boldsymbol{F}\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,|\mathbf{r}'(t)|\,dt = \int_a^b \boldsymbol{F}\cdot\mathbf{r}'(t)\,dt = \int_C \boldsymbol{F}\cdot d\mathbf{r}. \tag{16.21}$$

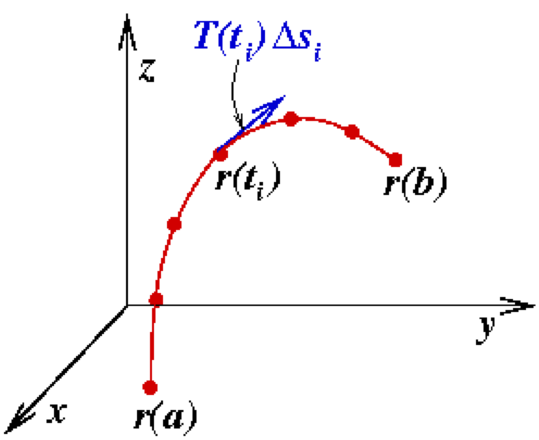


Figure 16.7

Definition 16.25. *Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t), a \le t \le b$. Then the **line integral of $\boldsymbol{F}$ along** C is*

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} \overset{\text{def}}{=\!=} \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds = \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt. \tag{16.22}$$

*We say that work is the line integral with respect to arc length of the **tangential component** of force.*

Note:

- The last term in (16.22) gives a **calculation formula**.
- Although ① $\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, \mathrm{ds}$ and ② integrals with respect to arc length are unchanged when orientation is reversed, it is still true that
$$\int_{-C} \boldsymbol{F} \cdot d\mathbf{r} = -\int_C \boldsymbol{F} \cdot d\mathbf{r}.$$
Why?

Problem 16.26. Evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$, where $\boldsymbol{F}(x, y, z) = xy\,\mathbf{i} + yz\,\mathbf{j} + zx\,\mathbf{k}$ and C is given by $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle, \;\; 0 \le t \le 1$.

Solution.

Ans: $\frac{27}{28}$

Remark 16.27. (Equivalent to Definition 16.25, p. 545).
Let $\boldsymbol{F} = \langle P, Q, R \rangle$. Then

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C P\,dx + Q\,dy + R\,dz. \tag{16.23}$$

Problem 16.28. Let $\boldsymbol{F}(x,y) = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle$ and C the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$.

(a) Use a graph of $\boldsymbol{F}$ and C to guess whether $\int_c \boldsymbol{F} \cdot d\mathbf{r}$ is positive, negative, or zero.

(b) Evaluate the integral.

Solution. ***Hint***: (b) $C : \mathbf{r}(t) = \langle t, 1+t^2 \rangle$, $-1 \le t \le 1$; use Eqn. (16.22).

Ans: 0

Exercises 16.2

1. Evaluate the line integral, using the formula $\int_C f(x,y)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$.

   (a) $\displaystyle\int_C x^2y\,ds$, where C is given by $\mathbf{r}(t) =< \cos 2t, \sin 2t >$, $0 \le t \le \pi/4$

   (b) $\displaystyle\int_C 2xye^{xyz}\,ds$, where C is the line segment from $(0,0,0)$ to $(2,1,2)$

   Ans: (a) $1/3$; (b) $e^4 - 1$

2. Let $\boldsymbol{F}$ be the vector field shown in the Figure 16.8.

   (a) If C_1 is the horizontal line segment from $P(3,2)$ to $Q(-3,2)$, determine whether $\displaystyle\int_{C_1} \boldsymbol{F}\cdot d\mathbf{r}$ is positive, negative, or zero.

   (b) Let C_2 be the clockwise-oriented circle of radius 3 centered at the origin. Determine whether $\displaystyle\int_{C_2} \boldsymbol{F}\cdot d\mathbf{r}$ is positive, negative, or zero.

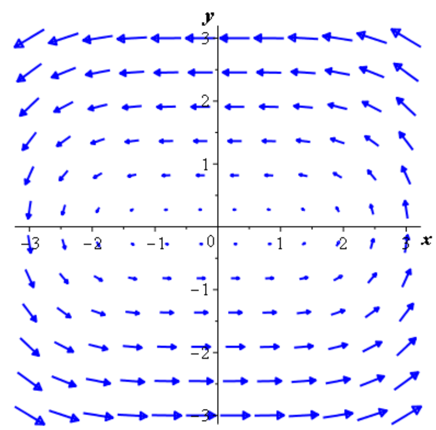


Figure 16.8

3. Use (16.22) to evaluate the line integral $\displaystyle\int_C \boldsymbol{F}\cdot d\mathbf{r}$, where C is parameterized by $\mathbf{r}(t)$.

   (a) $\boldsymbol{F}(x,y) = x^2y^3\,\mathbf{i} + x^3y^2\,\mathbf{j}$,
   $\mathbf{r}(t) = (t^3 - 2t)\,\mathbf{i} + (t^3 + 2t)\,\mathbf{j}$, $\quad 0 \le t \le 1$

   (b) $\boldsymbol{F}(x,y,z) = \langle -y, x, xy\rangle$,
   $\mathbf{r}(t) = \langle \cos t, \sin t, t\rangle$, $\quad 0 \le t \le \pi$

   Ans: (a) -9; (b) π

4. A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $y \ge 0$. If the linear density of the wire is $\rho(x,y) = ky$, find the **mass** and **center of mass** of the wire. ***Hint***: $C : \mathbf{r}(t) = \langle 2\cos t, 2\sin t\rangle,\ 0 \le t \le \pi$

   Ans: $8k$, $(0, \pi/2)$

16.3. The Fundamental Theorem for Line Integrals

Recall: Part 2 of **Fundamental Theorem of Calculus** (FTC2) is

$$\int_a^b f'(x)\,dx = f(b) - f(a). \tag{16.24}$$

Goal: It would be nice to get a generalization of the FTC2 (16.24) to line integrals.

16.3.1. Conservative Vector Fields

Theorem 16.29. *Suppose that* $\boldsymbol{F}$ *is continuous, and is a* ***conservative vector field****; that is,* $\boldsymbol{F} = \nabla f$ *for some* f*. Then*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{16.25}$$

Proof. By the Chain rule and the FTC2,

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\mathbf{r} &= \int_a^b \nabla f(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt = \int_a^b \frac{d}{dt}[(f\circ\mathbf{r})(t)]\,dt\\ &= (f\circ\mathbf{r})(t)\big|_a^b = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad \square\end{aligned}$$

Theorem 16.29 is the **Fundamental Theorem for Line Integrals**, which is a generalization of the FTC2. The function f is called a **potential function** of $\boldsymbol{F}$, or simply **potential**.

Problem 16.30. Let $\boldsymbol{F}(x,y) = \big\langle 3 + 2xy^2, 2x^2y - 4 \big\rangle$.

(a) Find a function f such that $\nabla f = \boldsymbol{F}$.

(b) Evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$, where $C:\ \mathbf{r}(t) = \langle \cos t, 2\sin t \rangle\,,\ \ 0 \le t \le \pi$.

Solution.

Ans: (a) $f(x,y) = 3x + x^2y^2 - 4y + K$ (b) -6

Problem 16.31. (Revisit of Problem 16.28). Let $\boldsymbol{F}(x,y) = \dfrac{x}{\sqrt{x^2+y^2}}\,\mathbf{i} + \dfrac{y}{\sqrt{x^2+y^2}}\,\mathbf{j}$ and C the parabola $y = 1 + x^2$ from $(-1,2)$ to $(1,2)$. Find a potential of $\boldsymbol{F}$ and evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$.

Solution.

Ans: $f(x,y) = \sqrt{x^2+y^2}$ and $\int_C \boldsymbol{F} \cdot d\mathbf{r} = 0$.

Independence of Path

Definition 16.32. *We say the line integral* $\displaystyle\int_C \boldsymbol{F}\cdot d\mathbf{r}$ *is* ***independent of path*** *if*

$$\int_{C_1} \boldsymbol{F}\cdot d\mathbf{r} = \int_{C_2} \boldsymbol{F}\cdot d\mathbf{r},$$

for any two paths C_1 *and* C_2 *that have the same initial and terminal points.*

Observation 16.33. In general, $\int_{C_1} \boldsymbol{F}\cdot d\mathbf{r} \neq \int_{C_2} \boldsymbol{F}\cdot d\mathbf{r}$. (See Problem 16.20, p. 540.) However, Theorem 16.29 says that when $\boldsymbol{F} = \nabla f$,

$$\int_{C_1} \boldsymbol{F}\cdot d\mathbf{r} = \int_{C_1} \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = \int_{C_2} \nabla f\cdot d\mathbf{r} = \int_{C_2} \boldsymbol{F}\cdot d\mathbf{r}.$$

Thus line integrals of **conservative** fields are **independent of path**.

Conservativeness ⇒ Independence of path

Definition 16.34. *A curve* C *is* ***closed*** *if its terminal point coincides with its initial point, that is,* $\mathrm{r}(b) = \mathrm{r}(a)$. *A* ***simple curve*** *is a curve that does not intersect itself.*

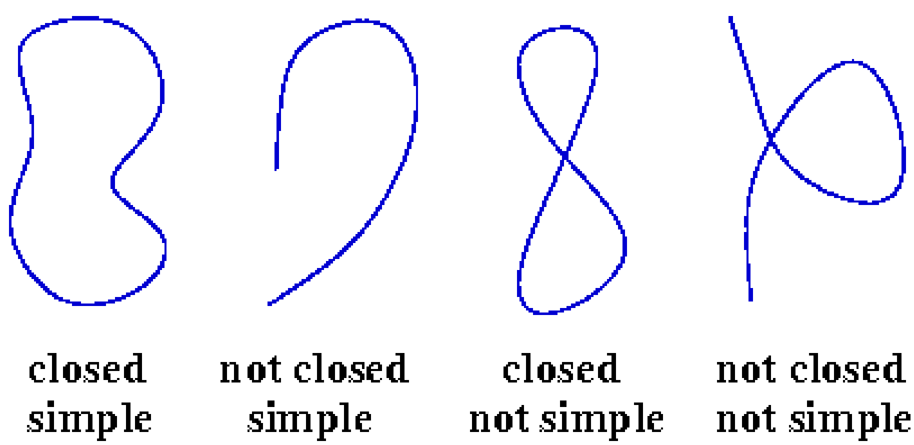


Figure 16.9

Theorem 16.35. *$\int_C \boldsymbol{F} \cdot d\mathbf{r}$ is independent of path in D if and only if $\int_C \boldsymbol{F} \cdot d\mathbf{r} = 0$ for every closed path C in D.*

Proof. ($\Rightarrow$) For a closed curve C, choose two points A and B to decompose C into two parts: $C = C_1 \cup C_2$. Then

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} + \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} - \int_{-C_2} \boldsymbol{F} \cdot d\mathbf{r} = 0,$$

because C_1 and $-C_2$ have the same initial and terminal points.

($\Leftarrow$) Let C_1 and C_2 have the same initial and terminal points. Then

$$0 = \int_{C_1 \cup (-C_2)} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} + \int_{-C_2} \boldsymbol{F} \cdot d\mathbf{r} = \int_{C_1} \boldsymbol{F} \cdot d\mathbf{r} - \int_{C_2} \boldsymbol{F} \cdot d\mathbf{r},$$

where the first equality comes from the assumption. $\square$

Independence of path $\Longleftrightarrow$ Zero Line Integral on closed paths

Pictorial Definitions

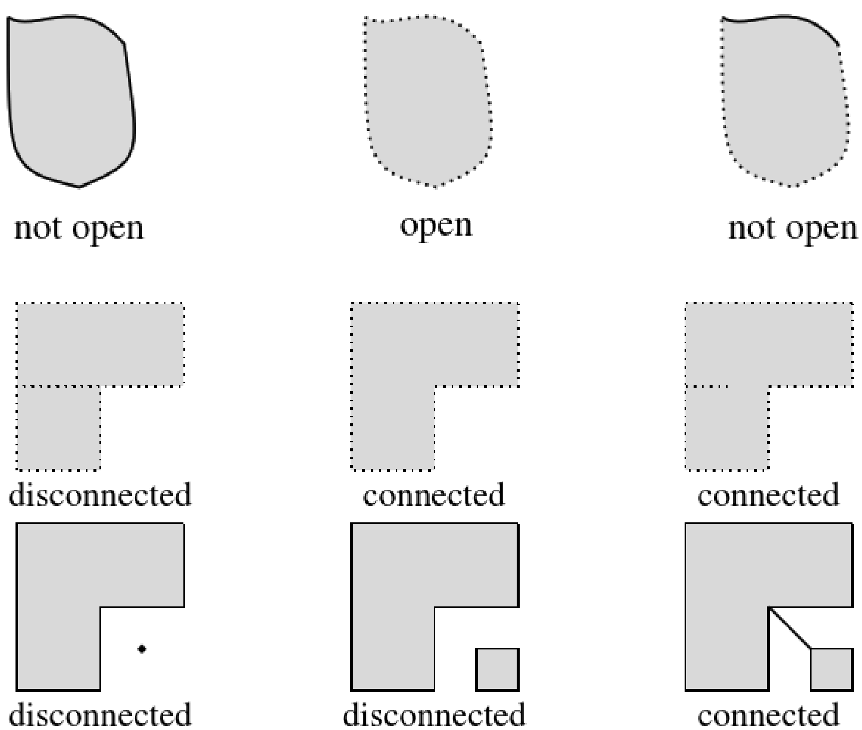


Figure 16.10: Pictorial definitions for D.

Definition 16.36. *A set D is said to be* ***open*** *if every point P in D has a disk with center P that is contained wholly and solely in D.*
Note. *D cannot contain any boundary points.*

Definition 16.37. *A set D is said to be* ***connected*** *if for every two points P and Q in D, there exists a path which connects P to Q.*

Theorem 16.38. *Suppose that the line integral of a vector field* $\boldsymbol{F}$ *is* ***independent of path*** within an ***open connected region*** *D, then* $\boldsymbol{F}$ *is a* ***conservative*** vector field on D.

Proof. (sketch). Choose an arbitrary point $(a,b) \in D$ and define

$$f(x,y) = \int_{(a,b)}^{(x,y)} \boldsymbol{F} \cdot d\mathbf{r}.$$

Since this line integral is independent of path, we can define $f(x,y)$ using any path between (a,b) and (x,y). By choosing a path that ends with a horizontal line segment from (x_1,y) to (x,y) contained entirely in D, $x_1 < x$, we can show that

$$\partial f/\partial x(x,y) = \partial/\partial x \Big[\int_{(a,b)}^{(x_1,y)} \boldsymbol{F} \cdot d\mathbf{r} + \int_{(x_1,y)}^{(x,y)} \boldsymbol{F} \cdot d\mathbf{r} \Big] = 0 + \partial/\partial x \int_{x_1}^{x} \boldsymbol{F} \cdot \langle dx, 0 \rangle = P.$$

Similarly, we can prove that $\partial f/\partial y(x,y) = Q$. $\square$

It follows from Observation 16.33 and Theorem 16.38:

Corollary 16.39. *In an open connected region,* $\boldsymbol{F}$ *is* ***conservative*** *if and only if its line integral is* ***independent of path****.*

Conservativeness $\Longleftrightarrow$ Independence of path

16.3.2. Clairaut's Theorem for Conservative Vector Fields

Section 16.3.1:
Conservativeness $\iff$ Zero Line Integral on closed paths
$\iff$ Independence of path (D: open connected)

Theorem 16.40. ***(Clairaut's Theorem for conservative vector fields).*** *If $\boldsymbol{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$ is a conservative vector field, where P and Q have continuous first-order partial derivatives on D, then*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{16.26}$$

throughout the domain D.

Quesiton. Does (16.26) imply conservativeness of $\boldsymbol{F}$?

Ans: No, in general. But, almost!

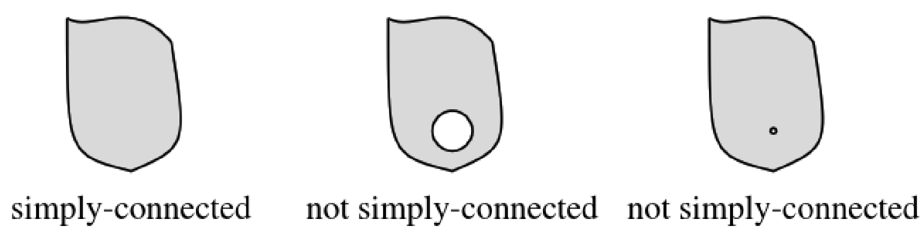


Figure 16.11: Simply-connectedness of D.

Definition 16.41. *D is a **simply-connected region** if it is connected and every simple closed curve contains only points in D.*

Theorem 16.42. *Let $\boldsymbol{F} = \langle P, Q\rangle$ be a vector field on an open simply-connected region D. If P and Q have **continuous first-order partial derivatives** throughout D,*

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}, \tag{16.27}$$

*then $\boldsymbol{F}$ is **conservative**.*

When the vector field $\boldsymbol{F} = \langle P, Q \rangle$ is differentiable over D

Conservativeness $\iff \dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$

Self-study 16.43. Determine whether or not the vector field $\boldsymbol{F}(x, y) = \langle 3 + 2xy, x^2 + x - 3y^2 \rangle$ is conservative.

Solution. ***Hint***: Check if $P_y = Q_x$ is satisfied.

Ans: no

Problem 16.44. Determine whether or not the vector field $\boldsymbol{F}(x, y) = \langle e^y + y \cos x,\ xe^y + \sin x \rangle$ is conservative.

Solution.

Ans: yes

Potential Functions

Recall: When $\boldsymbol{F}$ is conservative, we know from (16.25) on p.548 that

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)), \tag{16.28}$$

which is easy to evaluate when the potential f is known.

Problem 16.45. Given $\boldsymbol{F}(x,y) = \langle e^y + y\cos x,\ xe^y + \sin x\rangle$,

(a) Find a potential.

(b) Evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$, where C is parameterized as

$$\mathbf{r}(t) = \langle e^t \cos t, e^t \sin t\rangle,\ \ 0 \le t \le \pi.$$

Solution.

Ans: (a) $f(x,y) = xe^y + y\sin x + K$ (b) $-e^{\pi} - 1$

Problem 16.46. Let $\boldsymbol{F}(x, y, z) = \langle y^2,\, 2xy + e^{3z},\, 3ye^{3z}\rangle$. Find f such that $\nabla f = \boldsymbol{F}$.

Solution.

Ans: $f = xy^2 + ye^{3z} + K$

Recall: **The Fundamental Theorem for Line Integrals**

(Theorem 16.29, p. 548) Suppose that $\boldsymbol{F}$ is continuous, and is a **conservative vector field**; that is, $\boldsymbol{F} = \nabla f$ for some f. Then

$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{16.29}$$

Problem 16.47. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a conservative vector field, where P, Q, R have continuous first-order partial derivatives. Then,

$$P_y = Q_x, \quad P_z = R_x, \quad Q_z = R_y. \tag{16.30}$$

Solution. ***Hint***: Use Clairout's theorem.

□

Problem 16.48. Show that $\int_C y\,dx + x\,dy + yz\,dz$ is not independent of path.

Solution. ***Hint***: Use (16.30) to check if it is conservative.

Exercises 16.3

1. The figure shows a curve C and a contour map of a function f whose gradient is continuous. Find $\int_C \nabla f \cdot d\mathbf{r}$.

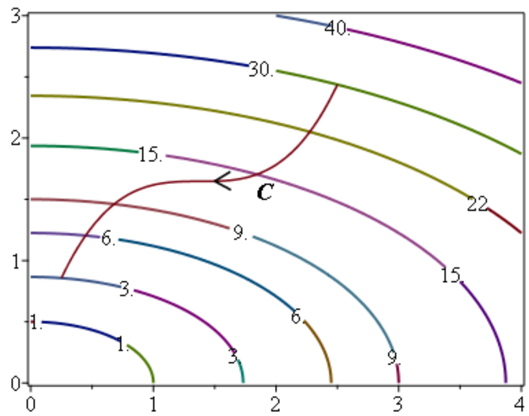


Figure 16.12

2. Determine whether the vector field $\boldsymbol{F}$ is conservative or not. If it is, find its potential.

   (a) $\boldsymbol{F}(x, y) = \langle x + y, x - y \rangle$
   (b) $\boldsymbol{F}(x, y) = \langle 2xy, x^2 + 2xy \rangle$
   (c) $\boldsymbol{F}(x, y) = \langle 2xy^4, x^2y^3 \rangle$
   (d) $\boldsymbol{F}(x, y) = \langle ye^x, e^x - 2y \rangle$

3. (i) Find the potential of $\boldsymbol{F}$ and (ii) use part (i) to evaluate $\int_C \boldsymbol{F} \cdot d\mathbf{r}$ along the given curve C.

   (a) $\boldsymbol{F}(x, y) = \langle e^y,\ xe^y + \sin y \rangle$, $C : \mathbf{r}(t) = \langle -\cos t, e^t \sin t \rangle,\ \ 0 \le t \le \pi$
   (b) $\boldsymbol{F}(x, y, z) = \langle 2y + z, 2x + z, x + y \rangle$, C is the line segment from $(1, 0, 0)$ to $(2, 2, 2)$
   (c) $\boldsymbol{F}(x, y, z) = \langle \sin z,\ -\sin y,\ x \cos z \rangle$, $C : \mathbf{r}(t) = \langle \cos t, \sin t, t \rangle,\ \ 0 \le t \le \pi/2$

   Ans: (a) 2; (b) 16; (c) $\cos(1) - 1$

4. Show that the line integral is independent of path and evaluate the integral.

   (a) $\displaystyle\int_C x\,dx - y\,dy$, C is any path from $(0, 1)$ to $(3, 0)$
   (b) $\displaystyle\int_C (\sin y - ye^{-x})\,dx + (e^{-x} + x \cos y)\,dy$, C is any path from $(1, 0)$ to $(0, \pi)$

   Ans: (a) 5; (b) π

5. The figure below depicts two vector fields, one of which is conservative. Which one is it? Why is the other one not conservative?

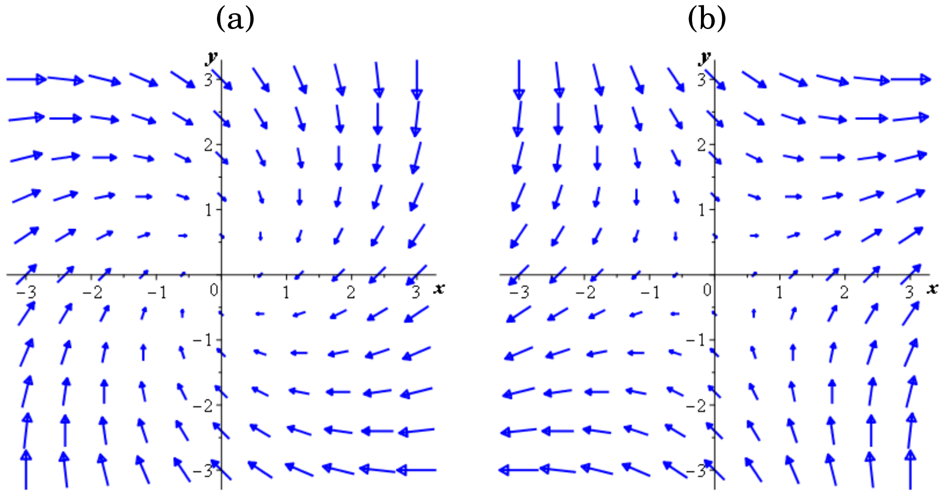


Figure 16.13: Two vector fields, one of which is conservative.

16.4. Green's Theorem

Green's Theorem gives the **relationship** between a **line integral** around a simple closed curve C and a **double integral** over the plane region D bounded by C.

Definition 16.49. *The* ***positive orientation*** *of a simple closed curve C refers to a single counterclockwise traversal of C (with keeping the domain on the left). The other directional orientation is called the* ***negative orientation****.*

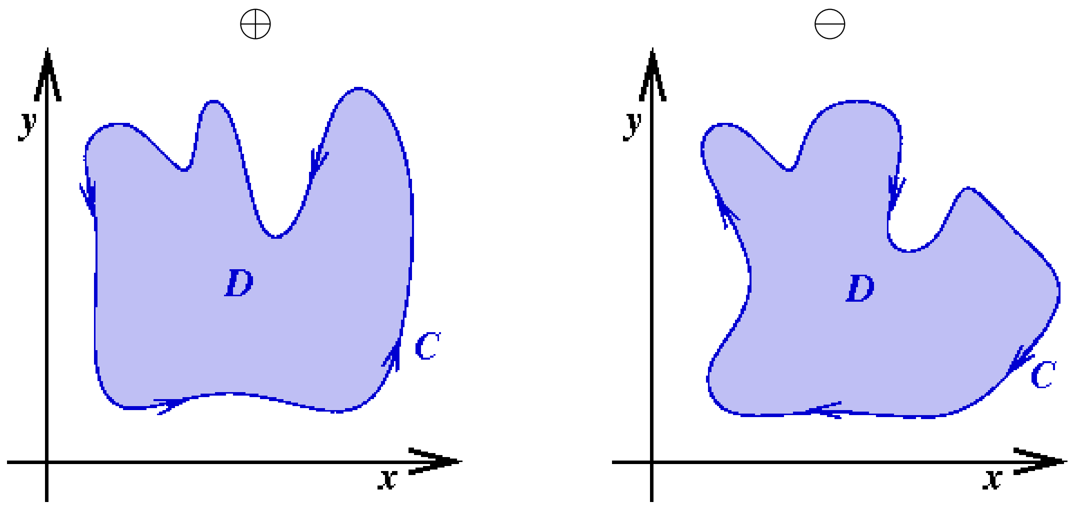


Figure 16.14: $\oplus$-orientation and $\ominus$-orientation of a simple closed curve C.

Theorem 16.50. ***(Green's Theorem)****. Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ has* ***continuous partial derivatives*** *on an open region including D, then*

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{16.31}$$

Note: The proof of Green's Theorem on simple regions is based on the following identities

$$\oint_C P dx = -\iint_D \frac{\partial P}{\partial y}\, dA, \quad \oint_C Q dy = \iint_D \frac{\partial Q}{\partial x}\, dA. \tag{16.32}$$

Notation 16.51. We denote the line integral calculated by using the **positive orientation** of the closed curve C by

$$\oint_C P\,dx + Q\,dy, \quad \oint_C P\,dx + Q\,dy, \text{ or } \oint_C P\,dx + Q\,dy.$$

We denote line integrals calculated by using the **negative orientation** of the closed curve C by

$$\oint_C P\,dx + Q\,dy.$$

Problem 16.52. Evaluate $\oint_C x^4 dx + xy dy$, where C is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$.

Solution. Although the given line integral could be evaluated by the methods of Section 6.2, we would use Green's Theorem.

Ans: $\frac{1}{6}$

Problem 16.53. Evaluate $\oint_C \boldsymbol{F} \cdot d\mathbf{r}$, where $\boldsymbol{F} = \langle y - \cos y, x \sin y \rangle$ and C is the circle $(x-3)^2 + (y+4)^2 = 4$ oriented clockwise.

Solution. ***Hint***: Check the orientation of the curve.

Ans: 4π

16.4.1. Application to Area Computation

Recall

$$A(D) = \iint_D 1\, dA.$$

If we choose P and Q such that

$$\partial Q/\partial x - \partial P/\partial y = 1, \tag{16.33}$$

then the area of D can be computed as

$$A(D) = \iint_D 1\, dA = \oint_C P dx + Q dy. \tag{16.34}$$

The following choices are common:

$$\begin{cases} P(x,y) = 0 \\ Q(x,y) = x \end{cases} \quad \begin{cases} P(x,y) = -y \\ Q(x,y) = 0 \end{cases} \quad \begin{cases} P(x,y) = -\frac{y}{2} \\ Q(x,y) = \frac{x}{2} \end{cases} \tag{16.35}$$

Then, Green's Theorem give the following formulas for the area of D:

$$A(D) \;=\; \oint_C x\, dy \;=\; -\oint_C y\, dx \;=\; \frac{1}{2}\oint_C x\, dy - y\, dx \tag{16.36}$$

Problem 16.54. Find the area enclosed by $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$, an ellipse.

Solution. ***Clue***: The ellipse has parametric equations $x = a\cos t$ and $y = b\sin t$, $0 \le t \le 2\pi$. ***Hint***: You may use $\sin^2 x = \frac{1-\cos 2x}{2}$ or $\cos^2 x = \frac{1+\cos 2x}{2}$.

Ans: $ab\pi$

Problem 16.55. Use a formula in (16.36) to find the area of the shaded region in Figure 16.15.

Solution. ***Hint***: For the slanted edge $(C_3) : x = t,\ y = 3 - t,\ 1 \le t \le 3$.

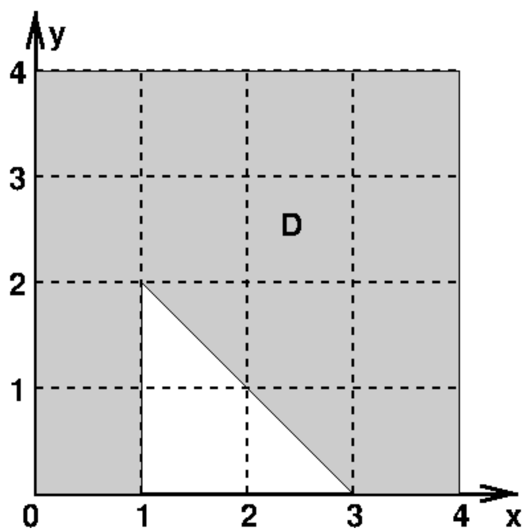


Figure 16.15

Ans: 14

Problem 16.56. Using the identity (an application of Green's Theorem)

$$A(D) = \iint\limits_{D} dA = \oint_{\partial D} x\,dy,$$

we will try to show that the area of D (the shaded region) is 6.

- First, observe that the line integrals on vertical and horizontal line segments of the figure are all zero.

- Thus the area must be the same as the line integral on the slant side, the line segment from $P(4,0)$ to $Q(2,2)$, which we denote by C_2.

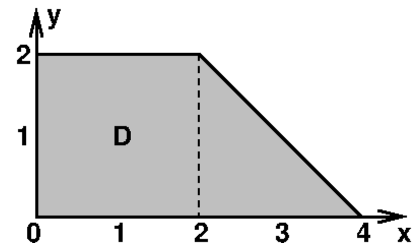


Figure 16.16

(a) Evaluate $\displaystyle\int_{C_2} x\,dy$, where C_2 is parametrized by

$$\mathbf{r}(t) = (1-t)P + t\,Q, \;\; 0 \le t \le 1.$$

(b) Evaluate $\displaystyle\int_{C_2} x\,dy$, where C_2 is parametrized by

$$\mathbf{r}(t) = \langle t,\, 4-t\rangle, \text{ with } t \text{ moving } 4 \searrow 2.$$

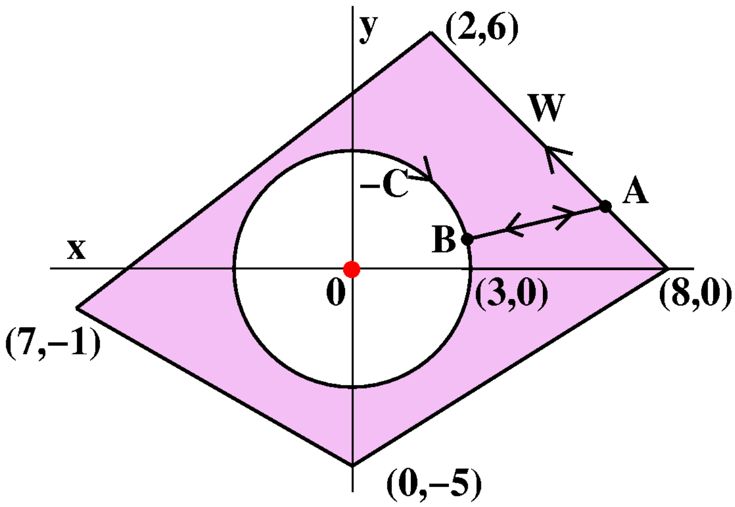
(c) Find "**the mid value of x**" and "**the change in y**", on C_2. Multiply the results to see if it is the same as the output in (a) and (b).[1]

Solution.

[1]The method introduces an effective algorithm for the computation of area. See P.4.5, p.713.

16.4.2. Generalization of Green's Theorem

Green's Theorem is proved for vector fields $\boldsymbol{F} = \langle P, Q\rangle$ defined in *simple regions* D; we can extend it to the cases where D is either **a finite union of simple regions** or **of holes**.



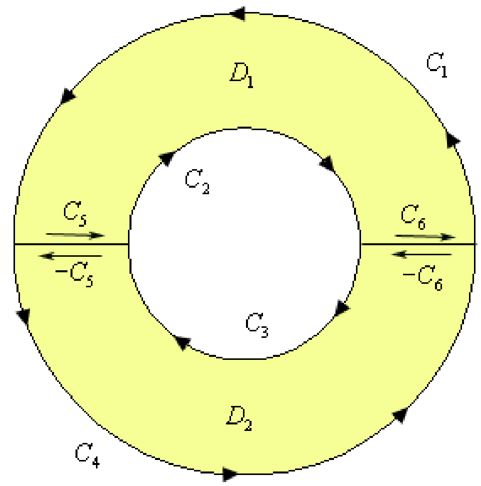


Figure 16.17: Regions having holes.

For example: For the right figure above,

$$\begin{aligned}\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA &= \iint_{D_1} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA + \iint_{D_2} \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \\ &= \oint_{\partial D_1} P\,dx + Q\,dy + \oint_{\partial D_2} P\,dx + Q\,dy.\end{aligned} \tag{16.37}$$

Along the common boundary, the opposite directional line integral will be canceled. Thus

$$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \oint_{\partial D} P\,dx + Q\,dy, \tag{16.38}$$

where ∂D is the collection of all boundaries of D.

Theorem 16.57. ***(Generalized Green's Theorem)***. *Let D be either a finite union of simply-connected regions or of holes, of which* ***the boundary is finite and oriented****. If $\mathbf{F} = \langle P, Q\rangle$ has* ***continuous partial derivatives*** *on an open region including D, then*

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA, \tag{16.39}$$

where ∂D is the boundary of D ***positively oriented****.*

Problem 16.58. Evaluate $\oint_C (1-y^3)dx + (x^3 + e^{y^2})dy$, where C is the boundary of the region between the circles $x^2+y^2=4$ and $x^2+y^2=9$, having the positive orientation.

Solution.

Ans: $\frac{195\pi}{2}$

Problem 16.59. Let $\boldsymbol{F}(x,y) = \langle P, Q\rangle = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$. Then

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = \frac{y^2-x^2}{(x^2+y^2)^2} - \frac{y^2-x^2}{(x^2+y^2)^2} = 0, \quad \text{for } (\mathbf{x},\mathbf{y}) \neq (\mathbf{0},\mathbf{0}). \tag{16.40}$$

Show that $\oint_C \boldsymbol{F}\cdot d\mathbf{r} = 0$ for any simple closed path C that does not pass through or enclose the origin.

Solution. You may use Green's Theorem.

Example 16.60. *Let $\boldsymbol{F}(x,y) = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2}\rangle$, the same as in the above example. Show that $\oint_C \boldsymbol{F}\cdot d\mathbf{r} = 2\pi$ for any positively oriented simple closed curve C that encloses the origin.*

Warning: You CANNOT use Green's Theorem for this problem. Why?

Solution. ***Clue***: Choose $C' : x^2+y^2=a^2$, for small a. Then,

$$\oint_{\partial D} \boldsymbol{F}\cdot d\mathbf{r} = \oint_C \boldsymbol{F}\cdot d\mathbf{r} + \oint_{-C'} \boldsymbol{F}\cdot d\mathbf{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = 0,$$

where D is the region bounded by C and $-C'$. Thus we have

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} = \oint_{C'} \boldsymbol{F}\cdot d\mathbf{r} \tag{16.41}$$

By introducing parametric representation of $C' : \mathbf{r}(t) = \langle a\cos t, a\sin t\rangle, \; 0 \le t \le 2\pi$, we can conclude

$$\oint_{C'} \boldsymbol{F}\cdot d\mathbf{r} = \int_0^{2\pi} \boldsymbol{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_0^{2\pi} \frac{a^2\sin^2 t + a^2\cos^2 t}{a^2}\,dt = 2\pi. \quad \square$$

Let's try to solve another problem before closing the section.

Problem 16.61. Evaluate $\oint_c y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannual region D in the upper half-plane between the circles $x^2+y^2=1$ and $x^2+y^2=4$.

Solution.

Ans: $14/3$

Summary 16.62. *Green's Theorem can be summarized as follows.*

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA \tag{16.42}$$

is applicable when

1. *The boundary of D is* ***finite and oriented****.*
2. *The vector field $\boldsymbol{F} = \langle P, Q\rangle$ has* ***continuous partial derivatives*** *over the whole region D. (It is about* ***quality*** *of the vector field.)*

Exercises 16.4

1. Evaluate the line integral $\oint_C y^2\,dx + 3xy\,dy$, where C is the triangle with vertices $(0,0)$, $(2,0)$, and $(2,2)$:

   (a) directly (b) using Green's theorem

   Hint: For (a), you should parametrize each of three line segments.
   For example: $C_3 : \mathbf{r}(t) = \langle t, t\rangle,\ t = 2 \searrow 0$.

   Ans: $4/3$

2. Use Green's Theorem to evaluate the line integral along the given positively oriented curve.

   (a) $\int_C (2y + \ln(1+x^2))dx + (6x + y^2)dy$, where C is the triangle with vertices $(0,0)$, $(3,0)$, and $(1,1)$

   (b) $\int_C (x^2 - y^3 + y)dx + (x^3 + x - y^2)dy$, where C is the circle $x^2 + y^2 = 4$

   Ans: (b) 24π

3. Use Green's Theorem to evaluate $\int_C \boldsymbol{F}\cdot d\mathbf{r}$. (Check the orientation of the curve before applying the theorem.)

   (a) $\boldsymbol{F}(x,y) =< y^3\cos x, x + 3y^2\sin x >$, C is the triangle from $(0,0)$ to $(8,0)$ to $(4,4)$ to $(0,0)$

   (b) $\boldsymbol{F}(x,y) =< 5y - 2030x^2 + \sin y,\ y^2 + x\cos y >$, C consists of the three line segments: from the origin to $(0,2)$, then to $(2,0)$, and then back down to the origin

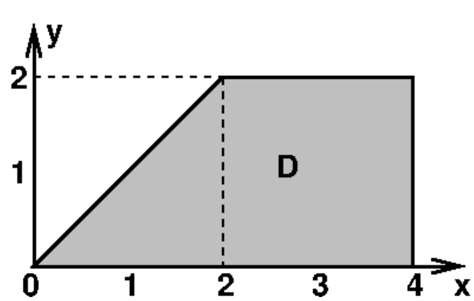
   (c) $\boldsymbol{F}(x,y) =< y + y^2 - \cos y, x\sin y >$, C is the circle $x^2 + y^2 = 4$ oriented clockwise

   Ans: (a) 16; (c) 4π

4. Use the identity (an application of Green's Theorem)

$$A(D) = \iint_D dA = \int_{\partial D} x\,dy$$

   to show that the area of D (the shaded region) is 6. You should compute the line integral for each line segment of the boundary, first introducing an appropriate parametrization.



16.5. Curl and Divergence

16.5.1. Curl

We now define the curl of a vector field, which helps us represent rotations of different sorts in physics and such fields. It can be used, for instance, to represent the velocity field in fluid flow.

Definition 16.63. *Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist. Then the* ***curl*** *of $\boldsymbol{F}$ is the vector field on $\mathbb{R}^3$ defined by*

$$\textbf{\textit{curl F}} = \left\langle \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z},\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x},\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\rangle. \tag{16.43}$$

Definition 16.64. *Define the* ***vector differential operator*** ∇ (*"**del**"*) *as*

$$\nabla = \left\langle \frac{\partial}{\partial x},\ \frac{\partial}{\partial y},\ \frac{\partial}{\partial z} \right\rangle = \langle \partial_x, \partial_y, \partial_z\rangle.$$

Then

$$\begin{aligned} \nabla \times \boldsymbol{F} &= \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ P & Q & R \end{bmatrix}\right) \\ &= \langle R_y - Q_z,\ P_z - R_x,\ Q_x - P_y \rangle \\ &= \textbf{\textit{curl F}} \end{aligned} \tag{16.44}$$

So, the easiest way to remember Definition 16.63 is

$$\textbf{\textit{curl F}} = \nabla \times \boldsymbol{F}. \tag{16.45}$$

Note: If $\boldsymbol{F}$ represents the velocity field in fluid flow, then the particles in the fluid tend to rotate about the axis that points in the direction of $\nabla \times \boldsymbol{F}$; the magnitude $|\nabla \times \boldsymbol{F}|$ measures how quickly the fluid rotates.

Quesiton. Why do **tornado** evolve? What is ***the change in the air*** after a tornado? *Answer*: Energy consumption

Remark 16.65. *If* $\boldsymbol{F}$ *is* ***conservative*** *and has* ***continuous partial derivatives****, then*

$$\textbf{\textit{curl}}\ \boldsymbol{F} = 0. \tag{16.46}$$

(See also Problem 16.47 on p.557.)

Theorem 16.66. *If* f *is a function of three variables that has continuous second-order partial derivatives, then*

$$\textbf{\textit{curl}}\,(\nabla f) = \nabla \times (\nabla f) = 0. \tag{16.47}$$

Proof. Use Clairout's Theorem.

Problem 16.67. Show that the vector field $\boldsymbol{F} = \langle xz, xyz, -y^2 \rangle$ is not conservative.

Solution. ***Clue***: Check if $\textbf{\textit{curl}}\,\boldsymbol{F} \neq 0$.

Theorem 16.68. *If $\boldsymbol{F}$ is a vector field whose component functions have* ***continuous partial derivatives*** *on a* ***simply-connected*** *domain and* ***curl*** $\boldsymbol{F} = 0$*, then $\boldsymbol{F}$ is* ***conservative****.*

Note: The above theorem is a 3D version of Theorem 16.42, p. 553.

Problem 16.69. Let $\boldsymbol{F} = \langle y^2 z^3,\, 2xyz^3,\, 3xy^2 z^2 \rangle$.

(a) Show that $\boldsymbol{F}$ is conservative.

(b) Find f such that $\boldsymbol{F} = \nabla f$.

Solution.

Ans: (a) ***curl*** $\boldsymbol{F} = 0$, (b) $f(x, y) = xy^2 z^3 + K$

16.5.2. Divergence

Definition 16.70. *Let $\boldsymbol{F} = \langle P, Q, R \rangle$ be a vector field on $\mathbb{R}^3$ and its partial derivatives exist. Then the* **divergence** *of $\boldsymbol{F}$ is defined as*

$$\mathbf{div}\,\boldsymbol{F} \equiv \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \boldsymbol{F}.$$

Theorem 16.71. *Let $\boldsymbol{F} = \langle P, Q, R \rangle$ whose components have continuous second-order partial derivatives. Then*

$$\nabla \cdot (\nabla \times \boldsymbol{F}) = 0. \tag{16.48}$$

Note: The above theorem is analogous to $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = 0$ for all $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$.

Problem 16.72. Show that $\boldsymbol{F} = \langle xz, xyz, -y^2 \rangle$ cannot be the curl of another vector field.

Solution. ***Clue***: Check if $\nabla \cdot \boldsymbol{F} = 0$

Remark 16.73. The reason for the name **divergence** can be understood in the context of fluid flow. If $\boldsymbol{F}$ is the velocity of a fluid, the $\boldsymbol{div}\,\boldsymbol{F}$ represents the **net change rate of the mass per unit volume**. Thus, if $\boldsymbol{div}\,\boldsymbol{F} = 0$, then $\boldsymbol{F}$ is said to be **incompressible**. Another differential operator occurs when we compute the divergence of a gradient vector field ∇f:

$$\boldsymbol{div}\,(\nabla f) = \nabla\cdot(\nabla f) = \nabla^2 f = \Delta f.$$

The operator $\nabla^2 = \nabla\cdot\nabla = \Delta$ is called the **Laplace operator**, which is also applicable to vector fields like

$$\Delta\boldsymbol{F} = \Delta\langle P,\, Q,\, R\rangle = \langle \Delta P,\, \Delta Q,\, \Delta R\rangle.$$

Vector Forms of Green's Theorem

Recall: Green's Theorem (p. 560): Let $\boldsymbol{F} = \langle P,\, Q\rangle$. Then

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \equiv \oint_C P dx + Q dy = \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\, dA. \tag{16.49}$$

Now, regard $\boldsymbol{F}$ as a vector field in $\mathbb{R}^3$ with the 3rd component 0. Then

$$\nabla\times\boldsymbol{F} = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ P & Q & 0 \end{bmatrix}\right) = \langle 0,\, 0,\, Q_x - P_y\rangle.$$

So we can rewrite the equation in Green's Theorem as

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \equiv \oint_C \boldsymbol{F}\cdot\boldsymbol{T} ds = \iint_D (\nabla\times\boldsymbol{F})\cdot\mathbf{k}\, dA, \tag{16.50}$$

which expresses the line integral of the **tangential component** of $\boldsymbol{F}$ along C as the double integral of the **vertical component** of $\boldsymbol{curl}\,\boldsymbol{F}$ over the region D enclosed by C.

Line integral of the normal component of $\boldsymbol{F}$

Example 16.74. Let $\boldsymbol{F} = \langle P,\, Q\rangle$. What is $\oint_C \boldsymbol{F}\cdot\mathbf{n}\,ds$?

Solution. Let $\mathbf{r} = \langle x(t),\, y(t)\rangle$ define the curve C. Then

$$\boldsymbol{T} = \frac{\mathbf{r}'}{|\mathbf{r}'|} = \frac{\langle x', y'\rangle}{|\mathbf{r}'|} \quad \text{and} \quad \mathbf{n} = \frac{\langle y', -x'\rangle}{|\mathbf{r}'|}, \tag{16.51}$$

where $\mathbf{n}$ is the **outward unit normal vector**, $90°$ clockwise rotation of $\boldsymbol{T}$. Thus we have

$$\begin{aligned} \boldsymbol{F}\cdot\mathbf{n}\,ds &= \langle P,\, Q\rangle \cdot \frac{\langle y', -x'\rangle}{|\mathbf{r}'|}\,|\mathbf{r}'|\,dt \\ &= (P\,y' - Q\,x')\,dt \\ &= -Q\,dx + P\,dy. \end{aligned}$$

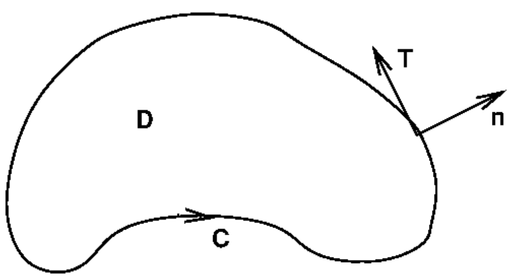


Figure 16.18

It follows from Green's Theorem that

$$\begin{aligned} \oint_C \boldsymbol{F}\cdot\mathbf{n}\,ds &= \oint_C -Q\,dx + P\,dy = \iint_D \big(P_x - (-Q)_y\big)\,dA \\ &= \iint_D \big(P_x + Q_y\big)\,dA = \iint_D \nabla\cdot\boldsymbol{F}\,dA. \end{aligned} \tag{16.52}$$

when P and Q have continuous partial derivatives over D. □

Exercises 16.5

1. Find (i) the curl and (ii) the divergence of the vector field.

   (a) $\boldsymbol{F}(x,y,z) = x^2yz\,\mathbf{j} + y^2z^2\,\mathbf{k}$ (b) $\boldsymbol{F}(x,y,z) = \langle x\sin y,\ y\sin z,\ z\sin x\rangle$

   Ans: (b) $\nabla\times\boldsymbol{F} = -\langle y\cos z, z\cos x, x\cos y\rangle$, $\nabla\cdot\boldsymbol{F} = \sin x + \sin y + \sin z$

2. The vector field $\boldsymbol{F}$ is shown in the xy-plane and looks the same in all other horizontal planes. (That is, $\boldsymbol{F}$ is independent of z and its third component is 0.)

   (a) Is ***div*** $\boldsymbol{F}$ positive, negative, or zero? Explain.
   (b) Determine whether ***curl*** $\boldsymbol{F} = \mathbf{0}$. If not, in which direction does it point?
   (c) Use Theorem 16.68 to conclude if $\boldsymbol{F}$ is conservative.

   Hint: **The vector field in (I)**: You may express it as $\boldsymbol{F} = \langle P(x), 0, 0\rangle$, where P is a decreasing function of x only. Thus ***div*** $\boldsymbol{F} < 0$. **The vector field in (II)**: Let $\boldsymbol{F} = \langle P(x,y), Q(x,y), 0\rangle$. Then ***div*** $\boldsymbol{F} = P_x + Q_y$ and ***curl*** $\boldsymbol{F} = \langle 0, 0, Q_x - P_y\rangle$. For example, $P_y < 0$ in (II), because the horizontal components of the arrows (P) become smaller as y increases. What can you say about P_x, Q_y, and Q_x?

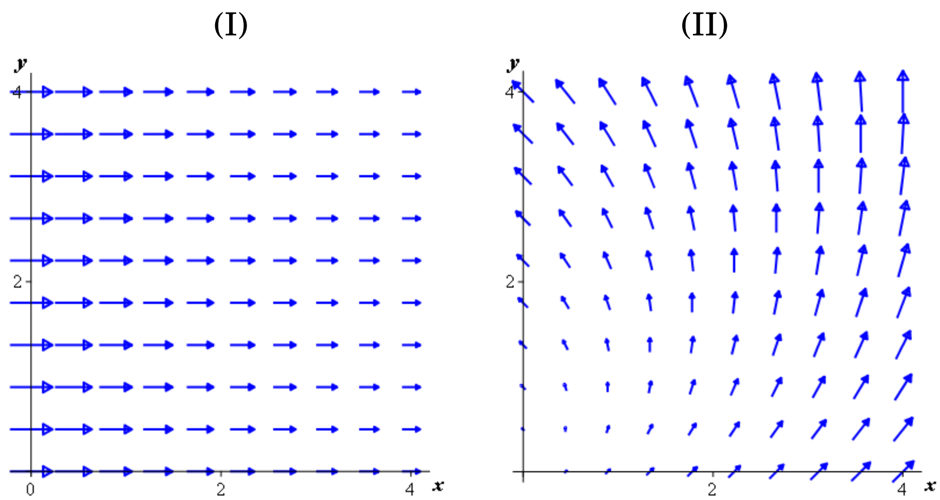


Figure 16.19

3. Determine whether or not $\boldsymbol{F}$ is conservative. If it is conservative, find its potential.

   (a) $\boldsymbol{F} = \langle yz^4, xz^4 + 2y, 4xyz^3\rangle$
   (b) $\boldsymbol{F} = \langle \sin z, 1, x\cos z\rangle$

   Ans: (b) $f = y + x\sin z + K$

16.6. Parametric Surfaces and Their Areas

16.6.1. Parametric Surfaces

Goal: This section will aim to describe surfaces by a function $\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle$, in a similar fashion that we described vector functions by $\mathbf{r}(t)$ earlier.

Definition 16.75. *A* ***parametric surface*** *is the set of points* $\{(x,y,z)\}$ *in* $\mathbb{R}^3$ *such that the components are expressed by a vector function of the form*

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle, \quad (u,v) \in D \subset \mathbb{R}^2.$$

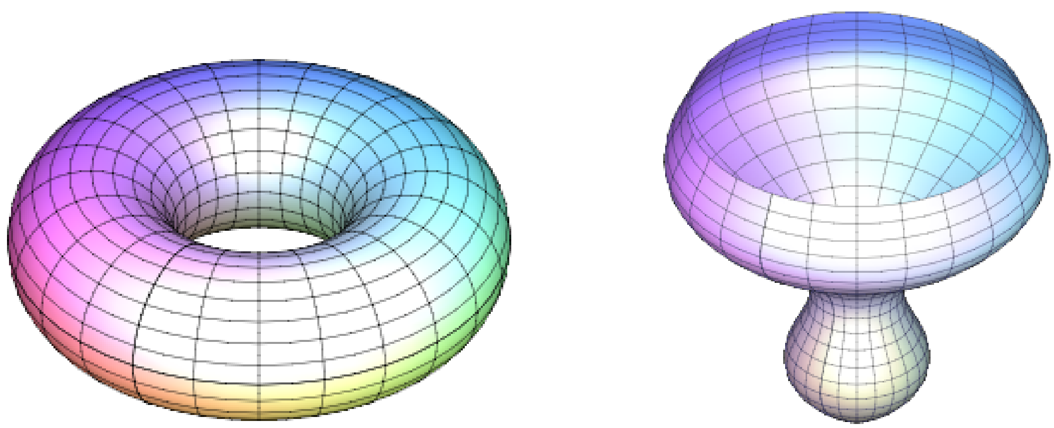


Figure 16.20: Examples of parametric surfaces.

Maple Script

```
with(plots): with(plottools):

plot3d([(4+2*cos(p))*cos(t), (4+2*cos(p))*sin(t), 2*sin(p)], p = 0..2*Pi, t = 0..2*Pi,
    axes = none, lightmodel = light1, scaling = constrained, orientation = [30,55]);

r := z/2+sin(z):
plot3d([r, t, z], t = 0..2*Pi, z =0..10, coords = cylindrical,
    axes = none, lightmodel = light1, scaling = constrained, orientation = [30,55]);
```

Problem 16.76. Identify and sketch $\mathbf{r}(u,v) = \langle 2\cos u,\ v,\ 2\sin u\rangle$, when $(u,v) \in D \equiv [0,2\pi] \times [0,5]$.

Solution. ***Clue***: $x^2 + z^2 = 4$.

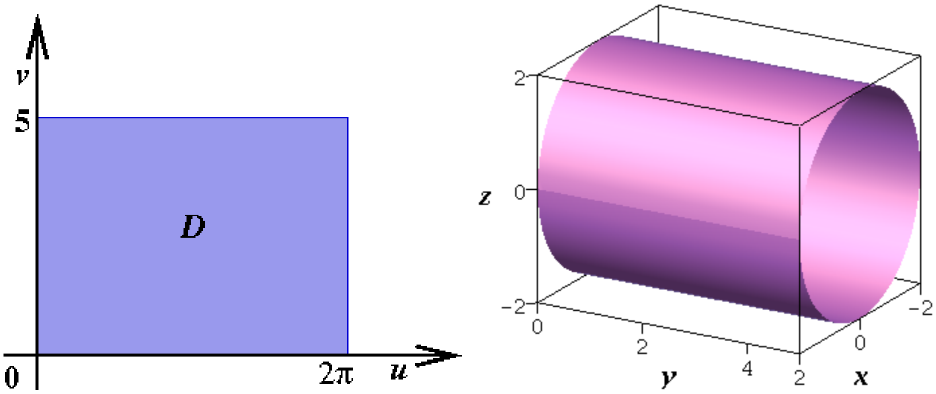


Figure 16.21

Self-study 16.77. Sketch $\mathbf{r}(s,t) = \langle s\cos 3t,\ s\sin 3t,\ s^2\rangle$, when $(s,t) \in [0,2] \times [0,2\pi]$. Discuss what the effect of the "3" is.

Quesiton. Given a surface, what is a parametric representation of it?

Problem 16.78. Find a parametric representation of the plane which passes $P_0(1,1,1)$ and contains $\mathbf{a} = \langle 1,2,0\rangle$ and $\mathbf{b} = \langle 2,0,-3\rangle$.

Solution. ***Clue***: $\mathbf{r}(u,v) = P_0 + u\,\mathbf{a} + v\,\mathbf{b}$.

Problem 16.79. Find a parametric representation of $x^2+y^2+z^2=a^2$.

Solution. ***Clue***: Use the spherical coordinates; the parameters are $(\theta,\,\phi)$.

Ans: $\mathbf{r}(\theta,\phi) = \langle a\sin\phi\cos\theta, a\sin\phi\sin\theta, a\cos\phi\rangle$, D?

Problem 16.80. Find a parametric representation of the cylinder $x^2+y^2=4$, $0 \le z \le 1$.

Solution. ***Hint***: Use cylindrical coordinates $(r=2,\theta,z)$.

Problem 16.81. Find a vector representation of the elliptic paraboloid $z=x^2+2y^2$.

Solution. ***Hint***: Let x, y be parameters.

Ans: $\mathbf{r}(x,y)=\langle x,y,x^2+2y^2\rangle$

In general, for $z = f(x, y)$,

$$\mathbf{r}(x, y) = \langle x,\ y,\ f(x, y)\rangle \tag{16.53}$$

is considered as a parametric representation of the surface.

Note: Parametric representations are not unique.

Problem 16.82. Find a parametric representation of $z = 2\sqrt{x^2 + y^2}$.

Clue: A representation is as in (16.53), while another one can be formulated using (r, θ) as with polar coordinates. Also, recall that when polar coordinates are considered, $x = r\cos\theta$, $y = r\sin\theta$.

Solution. ①

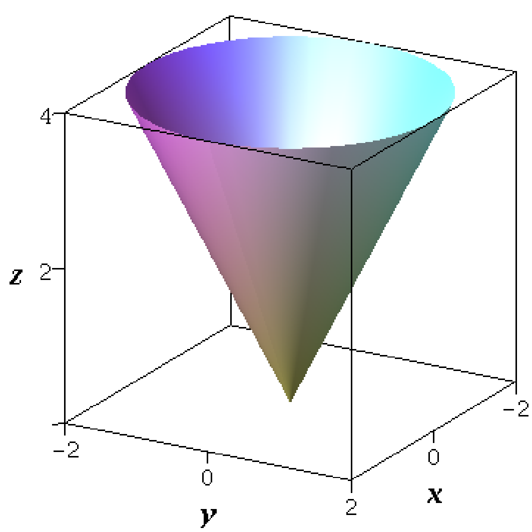


Figure 16.22

②

Surfaces of Revolution

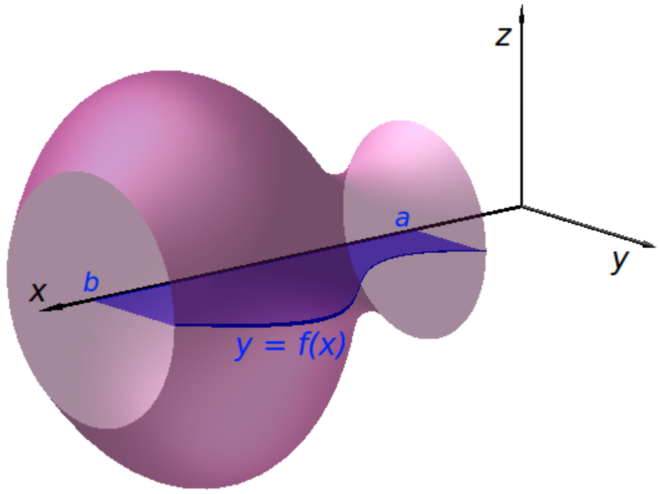


Figure 16.23: Surface of revolution

Let S be the surface obtained by rotating

$$y = f(x), \quad a \le x \le b,$$

about the x-axis (where $f(x) \ge 0$). Then, S can be represented as

$$\begin{aligned} \mathbf{r}(x,\theta) &= \langle x,\ f(x)\cos\theta,\ f(x)\sin\theta\rangle, \\ &\quad (x,\theta) \in [a,b] \times [0,2\pi]. \end{aligned} \tag{16.54}$$

Problem 16.83. Find parametric equations for the surface generated by rotating the curve $y = \sin(x)$, $0 \le x \le 2\pi$, about the x-axis.

Solution.

16.6.2. Tangent Planes and Surface Area

Recall: The **plane** passing $\mathbf{x}_0 = (x_0, y_0, z_0)$ and having a normal vector $\mathbf{v} = \langle a, b, c\rangle$ can be formulated as

$$\mathbf{v} \cdot (\mathbf{x} - \mathbf{x}_0) = 0,$$

or equivalently

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0. \tag{16.55}$$

Now, we will find the tangent plane to a parametric surface S traced out by

$$\mathbf{r}(u, v) = \langle x(u, v),\ y(u, v),\ z(u, v)\rangle$$

at a point P_0 with position vector $\mathbf{r}(u_0, v_0)$.

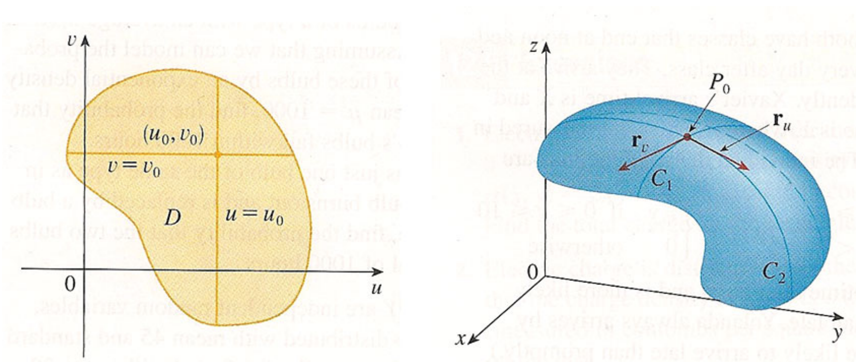


Figure 16.24

What we need: a normal vector, which can be determined by

$$\mathbf{r}_u \times \mathbf{r}_v.$$

Definition 16.84.

1. *A surface S represented by $\mathbf{r}$ is **smooth** if $\mathbf{r}_u \times \mathbf{r}_v \neq 0$ over the whole domain.*
2. *A **tangent plane** is the plane containing $\mathbf{r}_u$ and $\mathbf{r}_v$ and having a normal vector $\mathbf{r}_u \times \mathbf{r}_v$.*

Problem 16.85. Find the tangent plane to

$$S : x = u^2,\ y = v^2,\ z = u + 2v; \quad \textbf{at } (1, 1, 3)$$

Solution.

Ans: $-2(x-1) - 4(y-1) + 4(z-3) = 0$

Surface Area

Let $\mathbf{r} : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$. Then the **surface area** of S is

$$A(S) = \iint_S dS. \tag{16.56}$$

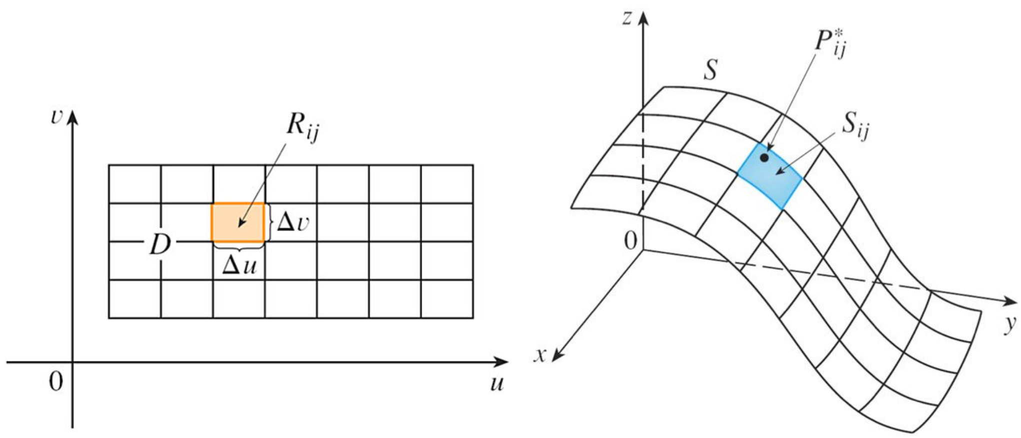


Figure 16.25: $\mathbf{r} : R_{ij} \mapsto S_{ij}$.

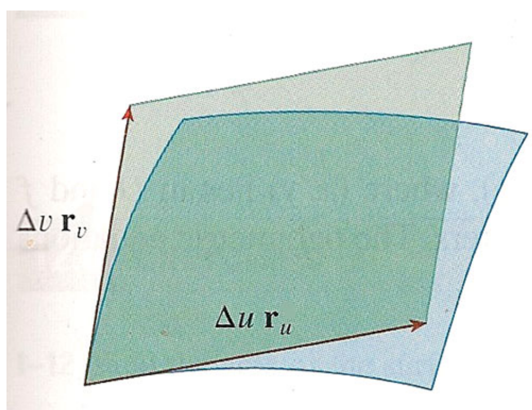


Figure 16.26: Approximating a patch by a parallelogram.

The area of the patch S_{ij} can be approximated by

$$\begin{aligned} \Delta S_{ij} &\approx A(\text{parallelogram}) \\ &= |(\Delta u\, \mathbf{r}_u) \times (\Delta v\, \mathbf{r}_v)| = |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v \end{aligned} \tag{16.57}$$

Definition 16.86. *If a smooth surface S is represented by*

$$\mathbf{r}(u,v) = \langle x(u,v),\, y(u,v),\, z(u,v)\rangle, \quad (u,v) \in D,$$

and S is covered just once as (u,v) ranges throughout the parameter domain D, then the ***surface area*** *of S is*

$$A(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA. \tag{16.58}$$

That is, $dS = |\mathbf{r}_u \times \mathbf{r}_v| \, dA$.

Problem 16.87. Find the area of the surface given by parametric equations $x = u^2$, $y = uv$, $z = \frac{1}{2}v^2$, $0 \le u \le 1$, $0 \le v \le 1$.

Solution.

Ans: 1

Change of Variables vs. $\boxed{\Delta S \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v}$

Recall: (**Summary 15.70 in § 15.9, p. 518**). For a differentiable transformation $T : Q \subset \mathbb{R}^2 \to R \subset \mathbb{R}^2$ given by $\mathbf{r}(u,v) = \langle x(u,v), y(u,v)\rangle$,

$$\Delta A \approx \left|\frac{\partial(x,y)}{\partial(u,v)}\right| \Delta u\, \Delta v, \tag{16.59}$$

where $\partial(x,y)/\partial(u,v)$ is the **Jacobian** of T defined as

$$\frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u\, y_v - x_v\, y_u. \tag{16.60}$$

Now, consider R as a flat region embedded in $\mathbb{R}^3$. Define

$$\widetilde{R} = R \times \{0\} \subset \mathbb{R}^3.$$

Then, $\widetilde{T} : Q \to \widetilde{R}$ is represented by $\widetilde{\mathbf{r}}(u,v) = \langle x(u,v), y(u,v), 0\rangle$;

$$\widetilde{\mathbf{r}}_u \times \widetilde{\mathbf{r}}_v = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ x_u & y_u & 0 \\ x_v & y_v & 0 \end{bmatrix}\right) = \langle 0, 0, x_u\, y_v - x_v\, y_u\rangle\,. \tag{16.61}$$

Therefore

$$|\widetilde{\mathbf{r}}_u \times \widetilde{\mathbf{r}}_v| = \left|\frac{\partial(x,y)}{\partial(u,v)}\right|. \tag{16.62}$$

Eqn. (16.59) is a special case of $\boxed{\Delta S \approx |\mathbf{r}_u \times \mathbf{r}_v|\, \Delta u\, \Delta v}$.

Summary 16.88. *Let $\mathbf{r} : D \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$ be a* ***parametric representation*** *of the surface S. Then*

1. *The map $\mathbf{r}$ can be viewed as a* ***change of variables****.*
2. *The quantity $|\mathbf{r}_u \times \mathbf{r}_v|$ is simply the* ***scaling factor*** *for $\mathbf{r}$.*

Surface Area of the Graph of a Function

As a special case, consider the surface S made by the graph of

$$z = g(x, y), \ \ (x, y) \in D.$$

Then the surface S can be represented by

$$\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle.$$

Since

$$\mathbf{r}_x = \langle 1, 0, g_x\rangle \ \text{ and } \ \mathbf{r}_y = \langle 0, 1, g_y\rangle,$$

we obtain

$$\mathbf{r}_x \times \mathbf{r}_y = \det\left(\begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{bmatrix}\right) = \langle -g_x,\, -g_y,\, 1\rangle \tag{16.63}$$

Thus we conclude the following.

Let S be made by the graph of $z = g(x, y), \ (x, y) \in D$. Then the surface area of S is

$$A(S) = \iint_D \sqrt{g_x^2 + g_y^2 + 1}\, dA. \tag{16.64}$$

Problem 16.89. Find the area of the part of paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.

Solution. (See Problem 15.42 on p. 497.)

Ans: $\frac{\pi}{6}(37\sqrt{37} - 1)$

Exercises 16.6

1. Identify the surface with the vector equation.

   (a) $\mathbf{r}(u, v) = \langle u - 3, u + v, 4u + 3v - 2\rangle$ (b) $\mathbf{r}(s, t) = \langle 2\cos t, s, 2\sin t\rangle, \quad 0 \le t \le \pi$

2. Match the parametric equations with the graphs labeled (I)–(III) and give reasons for your choices. Determine which families of grid curves on the surface have u constant and which have v constant.

   (a) $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$
   (b) $\mathbf{r}(u, v) = \langle v, 2\cos u, 2\sin u\rangle$
   (c) $\mathbf{r}(u, v) = \langle v\sin u, v\cos u, \cos v \sin v\rangle$

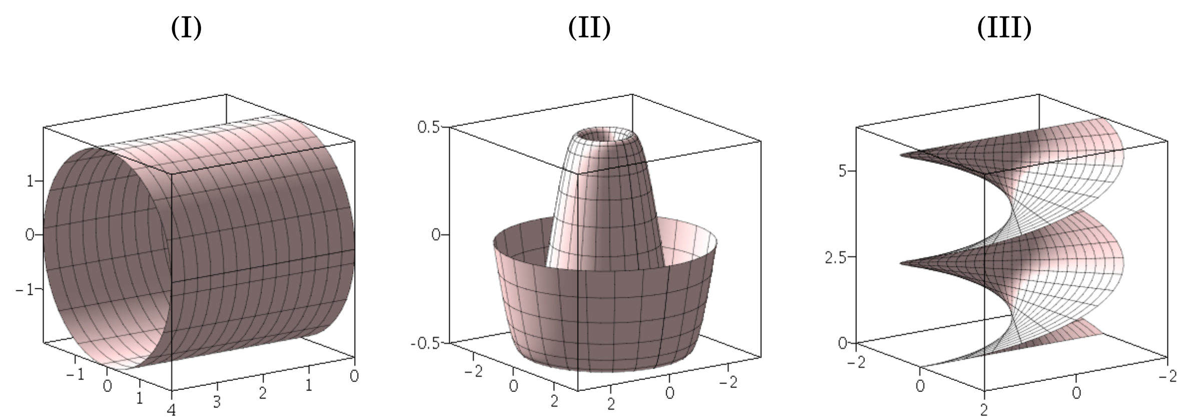


Figure 16.27

3. Find the parametric representation for the surface.

   (a) The part of the sphere $x^2 + y^2 + z^2 = 4$ that lies above the plane $z = 1$.
   (b) The part of the plane $y + z = 1$ that lies inside the cylinder $x^2 + z^2 = 1$. (See Figure 16.28.)

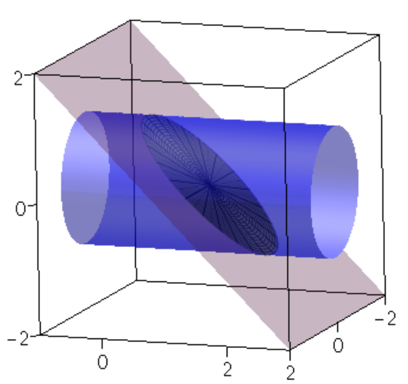


Figure 16.28

Hint: For (a), use the spherical coordinates (with $\rho = 2$) to specify the values of ϕ appropriately. Of course, $0 \le \theta \le 2\pi$. For (b), use the polar coordinates for the region in the xz-plane; that is, $x = r\cos\theta,\ z = r\sin\theta$. Then, you may set $y = 1 - z$. You have to specify the domain, values of r and θ, appropriately.

4. Find an equation of the tangent plane to the given surface at the specific point.

   (a) $\mathbf{r}(x, y) = \langle x, y, x^2 - y^2 \rangle$, $\quad (2, 1, 3)$
   (b) $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v \rangle$, $\quad (u, v) = (1, \pi/2)$

   Ans: (b) $\mathbf{r}_u \times \mathbf{r}_v(1, \pi/2) = \langle 1, 0, 1 \rangle \Rightarrow 1 \cdot (x - 0) + 0 \cdot (y - 1) + 1 \cdot (z - \pi/2) = x + z - \pi/2 = 0$

5. Find the area of the surface.

   (a) The part of the paraboloid $y = x^2 + z^2$ cut off by the plane $y = 6$
   (b) The surface parametrized by $\mathbf{r}(u, v) = \langle u^2, uv, \dfrac{v^2}{2} \rangle$, defined on $\{(u, v) \mid u^2 + v^2 \leq 1\}$

   Ans: (a) $\frac{62\pi}{3}$; (b) $3\pi/4$

16.7. Surface Integrals

This section deals with surface integrals of the form

$$\iint_S f(x,y,z)\,dS \quad \textbf{or} \quad \iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$$

16.7.1. Surface Integrals of Scalar Functions

Suppose that the surface S has a parametric representation

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v)\in D.$$

Then, from the previous section, we have

$$dS = |r_u \times r_v|\,dA$$

Thus we can reach at the formula

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA. \tag{16.65}$$

$\boxed{\textbf{Remark}}$ **16.90.**

- *When* $z = g(x,y)$, $\boxed{\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle}$. *Thus the formula (16.65) reads*

$$\boxed{\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA.} \tag{16.66}$$

- **Similarity**: *For line integrals,* $\displaystyle\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$

Problem 16.91. Compute the surface integral $\iint_S xy\,dS$, where S is the triangular region with vertices $(1,0,0)$, $(0,2,0)$, and $(0,0,2)$.

Solution. ***Clue***: The surface S (triangular region) can be expressed by $\frac{x}{1}+\frac{y}{2}+\frac{z}{2}=1.$ Thus $z=2-2x-y.$ Now, what is D?

Ans: $\frac{1}{\sqrt{6}}$

$\boxed{\textbf{Problem}}$ **16.92.** Evaluate $\displaystyle\iint_S z\,dS$, where S is the surface whose side S_1 is given by the cylinder $x^2+y^2=1$, whose bottom S_2 is the disk $x^2+y^2\le 1$ in the plane $z=0$, and whose top S_3 is the disk $x^2+y^2\le 1$ in the plane $z=1$.

Solution. ***Clue***: $S_1 : x = \cos\theta,\ y = \sin\theta,\ z = z;\ (\theta, z) \in D \equiv [0, 2\pi] \times [0, 1]$. Then $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$.

Ans: $\pi + 0 + \pi = 2\pi$

16.7.2. Surface Integrals of Vector Fields

Oriented Surfaces

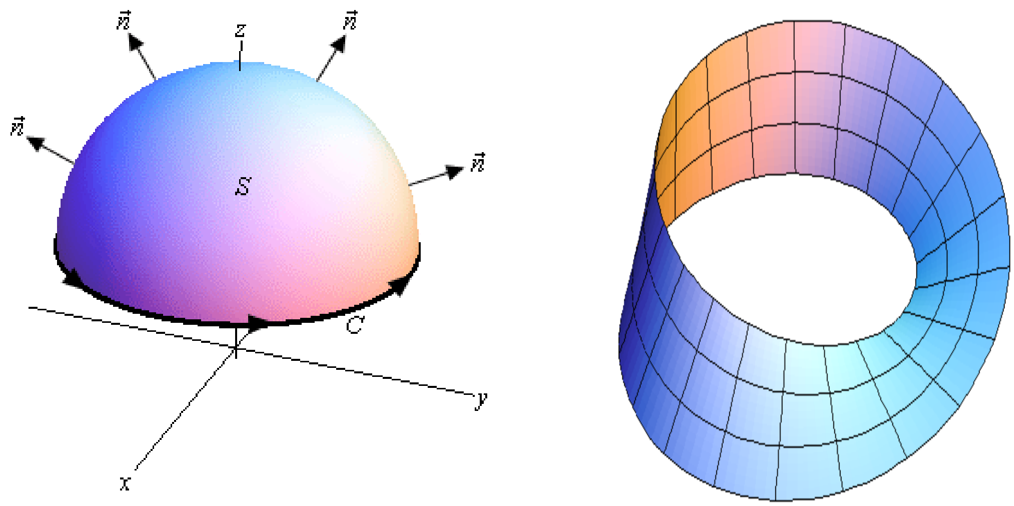


Figure 16.29: Oriented surface and Möbius strip.

Definition 16.93. *Let the surface S have a vector representation r.*

- *A unit normal vector n is defined as*

$$\mathrm{n} = \frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}. \tag{16.67}$$

- *The surface S is called an* ***oriented surface*** *if the (chosen) unit normal vector n* ***varies continuously*** *over S.*
  (A counter example: Möbius strip.)
- *For closed surfaces, the* ***positive orientation*** *is the one outward.*

Is it confusing? Then, consider this:

Definition 16.94. *A surface S is called* ***orientable*** *if it has* ***two separate sides****.*

A Historic View, for Surface Integrals of Vector Fields

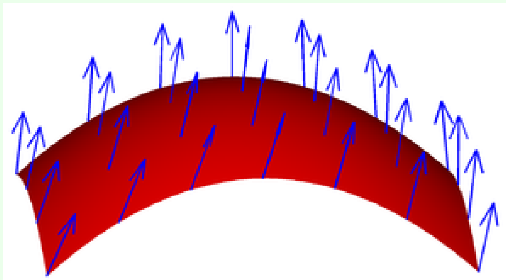


Figure 16.30: A vector field on a surface.

Suppose that S is an oriented surface. Imagine we have a fluid flowing through S, such that $\mathbf{v}(\mathbf{x})$ determines the velocity of the fluid at $\mathbf{x}$. The **flux** is defined as the quantity of fluid flowing through S per unit time.

The illustration implies that if the vector field is tangent to S at each point, then the flux is zero because the fluid just flows in parallel to S, and neither in nor out.

Thus, if $\mathbf{v}$ has both a tangential and a normal component, then **only the normal component contributes to the flux**. Based on this reasoning, to find the flux, we need to take the dot product of $\mathbf{v}$ with the unit surface normal $\mathbf{n}$ to S, which will give us a scalar field to be integrated over S appropriately.

Definition 16.95. *Let* $\boldsymbol{F}$ *be a continuous vector field defined on an oriented surface* S *with unit normal vector* $\mathbf{n}$*. The* ***surface integral of*** $\boldsymbol{F}$ ***over*** S *is*

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \stackrel{\text{def}}{=\!=} \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS. \tag{16.68}$$

This integral is also called the ***flux*** *of* $\boldsymbol{F}$ *across* S*.*

For the computation of the flux, the right side of (16.68), you may utilize

$$\mathbf{n} = \frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|} \quad\text{and}\quad dS = |\mathbf{r}_u\times\mathbf{r}_v|\,dA, \tag{16.69}$$

when S is parametrized by $\mathbf{r}: D\to S$.

Surface Integrals of Vector Fields. Let r be a parametric representation of S, from $D \subset \mathbb{R}^2$. The flux **across the surface** S can be measured by

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} &\overset{\text{def}}{=} \iint_S \boldsymbol{F}\cdot \mathbf{n}\, dS \\ &= \iint_D \boldsymbol{F}(\mathbf{r})\cdot\left(\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right)|\mathbf{r}_u\times\mathbf{r}_v|\,dA \\ &= \iint_D \boldsymbol{F}(\mathbf{r})\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.\end{aligned} \tag{16.70}$$

Note that $\boldsymbol{F}\cdot\mathbf{n}$ and $\boldsymbol{F}(\mathbf{r})\cdot(\mathbf{r}_u\times\mathbf{r}_v)$ are scalar functions.

Remark 16.96. Line integrals of vector fields is defined to measure quantities **along the curve**. That is,

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\mathbf{r} &\overset{\text{def}}{=} \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds \\ &= \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,|\mathbf{r}'(t)|\,dt = \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt,\end{aligned} \tag{16.71}$$

where C is parametrized by $\mathbf{r} : [a,b] \to C$.

Problem 16.97. Find the flux of $\boldsymbol{F} = \langle x, y, 1\rangle$ across a upward helicoid: $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 2$, $0 \le v \le \pi$.

Solution. ***Hint***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.

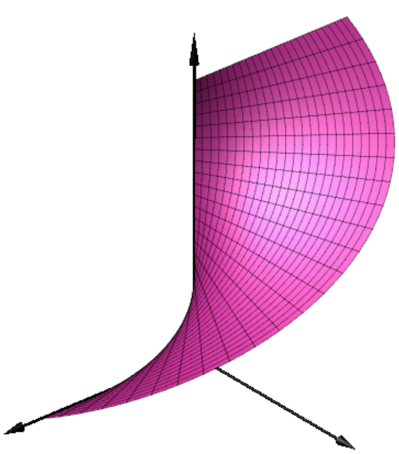


Figure 16.31

Ans: 2π

Example 16.98. *Find the flux of* $\boldsymbol{F} = \langle z, y, x\rangle$ *across the unit sphere* $x^2 + y^2 + z^2 = 1$.

Solution. First, consider a **vector representation** of the surface:

$$\mathbf{r}(\phi, \theta) = \langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi\rangle,\ \ 0 \le \phi \le \pi,\ 0 \le \theta \le 2\pi.$$

Then,

$$\begin{aligned}\boldsymbol{F}(\mathbf{r}) \quad &= \langle \cos\phi, \sin\phi\sin\theta, \sin\phi\cos\theta\rangle,\\ \mathbf{r}_\phi \times \mathbf{r}_\theta &= \langle \sin^2\phi\cos\theta, \sin^2\phi\sin\theta, \sin\phi\cos\theta\rangle,\end{aligned}$$

from which we have

$$\boldsymbol{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta) = 2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta.$$

Thus

$$\begin{aligned}\text{Flux} &= \iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}(\mathbf{r}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, dA\\ &= \int_0^{2\pi}\int_0^{\pi} (2\sin^2\phi\cos\phi\cos\theta + \sin^3\phi\sin^2\theta)\, d\phi d\theta\\ &= \int_0^{2\pi}\int_0^{\pi} (\sin^3\phi\sin^2\theta)\, d\phi d\theta\\ &= \int_0^{2\pi} \sin^2\theta d\theta \int_0^{\pi} \sin^3\phi\, d\phi = \pi \cdot \frac{4}{3}. \qquad \square\end{aligned}$$

Note: The answer of the previous example is actually the volume of the unit sphere. In Section 16.9, we will study the so-called **Divergence Theorem** (formulated for closed surfaces)

$$\iint_{\partial E} \boldsymbol{F} \cdot d\boldsymbol{S} = \iiint_E \nabla \cdot \boldsymbol{F}\, dV$$

The above example can be solved easily using the Divergence Theorem; see Problem 16.106, p. 605.

Surfaces defined by $z = g(x, y)$:

- A vector representation: $\mathbf{r}(x,y) = \langle x, y, g(x,y)\rangle$.
- Normal vector: $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$.
- Thus, when $\boldsymbol{F} = \langle P,\, Q,\, R\rangle$,

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \;=\; \iint_D \boldsymbol{F}\cdot(\mathbf{r}_x \times \mathbf{r}_y)\, dA = \iint_D (-P\, g_x - Q\, g_y + R)\, dA. \quad (16.72)$$

Problem 16.99. Evaluate $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle y, x, z\rangle$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution. ***Hint***: For S_1 (the upper part), use the formula in (16.72). For S_2 (the bottom: $z = 0$), you may try to get $\boldsymbol{F}\cdot\mathbf{n}$, where $\mathbf{n} = -\mathbf{k}$.

Ans: $\frac{\pi}{2} + 0 = \frac{\pi}{2}$

Formula **16.100.** *Let* $\boldsymbol{F} =< P, Q, R >$.

- $\displaystyle\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA$

  $\displaystyle\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA$, *when* S *is given by* $z = g(x,y)$
- $\displaystyle\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_S \boldsymbol{F}\cdot \mathbf{n}\,dS = \iint_D \boldsymbol{F}\cdot(\mathbf{r}_u \times \mathbf{r}_v)\,dA$

  $\displaystyle\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D (-Pg_x - Qg_y + R)\,dA$, *when* S *is given by* $z = g(x,y)$
- **Note**: *When* S *is given by* $z = g(x,y)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$

Exercises 16.7

1. Evaluate the surface integral $\displaystyle\iint_S f(x,y,z)\,dS$.

   (a) $f(x,y,z) = x$, S is the helicoid given by the vector equation $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 1$, $0 \le v \le \pi/2$ (***Hint***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.)

   (b) $f(x,y,z) = (x^2+y^2)z$, S is the hemisphere $x^2+y^2+z^2 = 1$, $z \ge 0$

   Ans: (a) $(2\sqrt{2}-1)/3$; (b) $\pi/2$

2. Evaluate the surface integral $\displaystyle\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$.

   (a) $\boldsymbol{F}(x,y,z) = x\mathbf{i} + y\mathbf{j} + 2z\mathbf{k}$, S is the part of the paraboloid $z = x^2+y^2$, $z \le 1$

   (b) $\boldsymbol{F}(x,y,z) = \langle z, x-z, y\rangle$, S is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$, oriented downward

   (c) $\boldsymbol{F} = \langle y, -x, z\rangle$, S is the upward helicoid parametrized by $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 2$, $0 \le v \le \pi$ (***Hint***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.)

   Ans: (a) 0; (b) $-1/3$; (c) $2\pi + \pi^2$

3. CAS Use a CAS to find the integral, either $\iint_S f(x,y,z)\,dS$ or $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$. First try to find the exact value; if the CAS does not work properly for the exact value, then try to estimate the integral correct four decimal places.

   (a) $f(x,y,z) = 2x^2+2y^2+z^2$, S is the surface $z = x\cos y$, $0 \le x \le 1$, $0 \le y \le 1$

   (b) $\boldsymbol{F}(x,y,z) = \langle x^2+y^2, y^2+z^2, x^2\rangle$, S is the part of the cylinder $x^2+z^2 = 1$ that lies above the xy-plane and between the planes $y = 0$ and $y = 1$, with upward orientation ***Hint***: You may use $\mathbf{r}(\theta, y) = \langle \cos\theta, y, \sin\theta\rangle$, for a representation of S.

   Ans: (b) $2/3$

16.8. Stokes's Theorem

Stokes' Theorem is a high-dimensional version of Green's Theorem studied in § 16.4.

Recall: **(Green's Theorem, p. 560).** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ have **continuous partial derivatives** on an open region including D, then

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \stackrel{\text{def}}{=\!=} \oint_C P\,dx + Q\,dy \;=\; \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot \mathbf{k}\,dA. \tag{16.73}$$

(For the last equality, see (16.50) on p.574.)

Theorem 16.101. (Stokes's Theorem) *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\boldsymbol{F} = \langle P,\, Q,\, R\rangle$ be a vector field whose components have **continuous partial derivatives** on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \tag{16.74}$$

Remark 16.102.

- *See Figure 16.29(left) on p. 594, for an oriented surface of which the boundary has positive orientation.*
- ***Computation** of the surface integral: for $\mathbf{r}: D \to S$,*

$$\iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \stackrel{\text{def}}{=\!=} \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot \mathbf{n}\,dS = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot (\mathbf{r}_u\times\mathbf{r}_v)\,dA. \tag{16.75}$$

- *Green's Theorem is a special case in which S is flat and lies on the xy-plane ($\mathbf{n} = \mathbf{k}$). Compare the last terms in (16.73) and (16.75).*

Problem 16.103. Evaluate $\int_C \boldsymbol{F} \cdot d\mathrm{r}$, where $\boldsymbol{F} = \langle -y^2, x, z^2 \rangle$ and C is the curve of intersection of the plane $y + z = 2$ and the cylinder $x^2 + y^2 = 1$.

Solution. ***Clue***: You may start with the computation of $\nabla \times \boldsymbol{F}$ and consider a vector representation for S: $z = g(x, y) = 2 - y$. Then use the formula (16.75).

Ans: π

Problem 16.104. Use Stokes's Theorem to compute the surface integral $\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle xz, yz, xy \rangle$ and S is the part of sphere $x^2+y^2+z^2 = 4$ that lies inside the cylinder $x^2+y^2 = 1$ and above the xy-plane.

Solution. ***Hint***: $\displaystyle\iint_S (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{S} = \oint_C \boldsymbol{F} \cdot d\mathbf{r} = \int_a^b \boldsymbol{F} \cdot \mathbf{r}'(t)\, dt.$ A vector representation of C is $\mathbf{r}(t) = \langle \cos t, \sin t, \sqrt{3} \rangle, \;\; 0 \le t \le 2\pi.$

Ans: 0

Exercises 16.8

1. A hemisphere H and a part P of a paraboloid are shown in the figure below. Let $\boldsymbol{F}$ be ba vector field on $\mathbb{R}^3$ whose components have continuous partial derivatives. Which of the following is true? Give reasons for your choice.

   A. $\displaystyle\iint_H (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S} < \iint_P (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S}$

   B. $\displaystyle\iint_H (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S} = \iint_P (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S}$

   C. $\displaystyle\iint_H (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S} > \iint_P (\textbf{\textit{curl}}\,\boldsymbol{F}) \cdot d\boldsymbol{S}$

   D. cannot compare

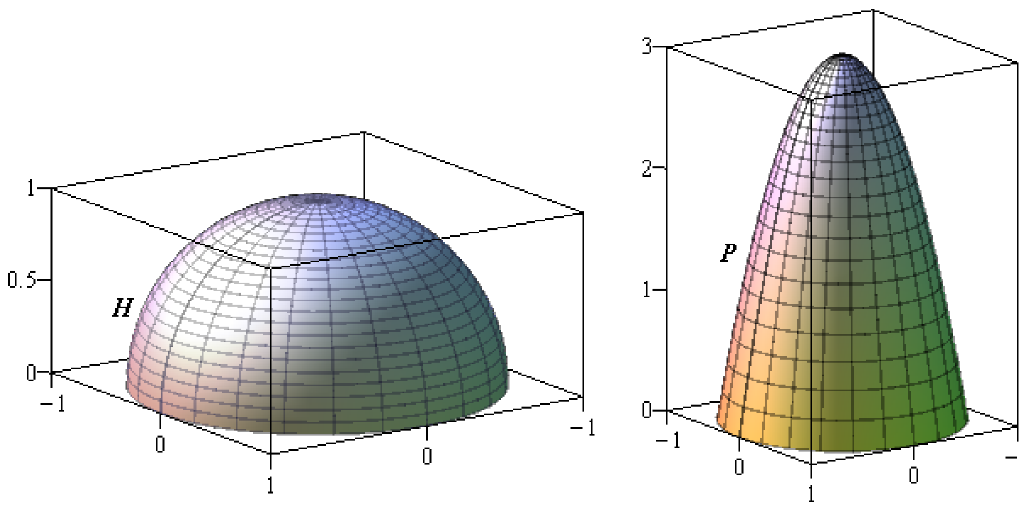


Figure 16.32

2. Use Stokes's Theorem to evaluate $\displaystyle\iint_S \textbf{\textit{curl}}\,\boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x,y) =< -y, x, x^2+y^2 >$ and S is the part of the sphere $x^2+y^2+z^2 = 8$ that lies inside the cone $z = \sqrt{x^2+y^2}$, oriented upward. (***Clue***: The boundary of S can be parametrized as $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 2\rangle$, $0 \le t \le 2\pi$.) ***Hint***: Use the formula given in the hint of Problem 16.104.

   Ans: 16π

3. Use Stokes's Theorem to evaluate $\displaystyle\int_C \boldsymbol{F} \cdot d\mathbf{r}$. For each case, let C be oriented counter-clockwise when viewed from above.

   (a) $\boldsymbol{F}(x,y,z) =< z^2+x, x^2+y, y^2+z >$, C is the triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$

   (b) $\boldsymbol{F}(x,y,z) =< x, y, z-x >$, C is the curve of intersection of the plane $2y+z=2$ and the cylinder $x^2+y^2=1$

Hint: $\boxed{\int_C \boldsymbol{F} \cdot d\mathbf{r} = \iint_S \textbf{\textit{curl}}\,\boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \textbf{\textit{curl}}\,\boldsymbol{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)dS}$. (a) $\textbf{\textit{curl}}\,\boldsymbol{F} = \langle 2y, 2z, 2x\rangle$ and $\mathbf{r}_x \times \mathbf{r}_y = \langle 1,1,1\rangle$. Figure out yourself what S, D, and $\mathbf{r}$ are.

Ans: (a) 1; (b) 2π

16.9. The Divergence Theorem

Recall: Let $\boldsymbol{F} = \langle P, Q\rangle$. In § 16.5.2, we considered vector forms of Green's Theorem including

$$\oint_C \boldsymbol{F}\cdot\mathbf{n}\,ds = \iint_D \nabla\cdot\boldsymbol{F}\,dA. \tag{16.76}$$

(See (16.52), p. 575.)

The Divergence Theorem is a generalization of the above.

Theorem 16.105. (Divergence Theorem) *Let E be a simple solid region and S be the boundary surface of E, given with positive (outward) orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ have **continuous partial derivatives** on an open region that contains E. Then*

$$\oiint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV. \tag{16.77}$$

Note: Let a surface S is parametrized by r. Then, from § 16.7.2 (p. 594), we know

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} \overset{\text{def}}{=\!=} \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS = \iint_D \boldsymbol{F}\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA, \tag{16.78}$$

whether or not S is closed. □

Note: The Divergence Theorem is developed mainly for closed surfaces; however, it can be applied for **unclosed surfaces**. We will consider problems in Chapter Review.

Problem 16.106. (Revisit of Example 16.98, p. 597) Find the flux of $\boldsymbol{F} = \langle z, y, x\rangle$ over the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution.

Ans: $\frac{4}{3}\pi$, the volume of the unit sphere

Problem 16.107. Find the flux of $\boldsymbol{F}$ across S, where

$$\boldsymbol{F}(x, y, z) = (\cos z + xy^2)\,\mathbf{i} + xe^{-z}\,\mathbf{j} + (\sin y + x^2 z)\,\mathbf{k}$$

and S is the surface of the solid bounded by the paraboloid $z = x^2 + y^2$ and the plane $z = 4$.

Solution.

Ans: $\frac{32}{3}\pi$

Problem 16.108. Use the Divergence Theorem to evaluate $\iint_S (x^2 + 2y^2 + z\,e^x)\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 1$.

Solution. ***Hint***: Find n and express the integrand as $\boldsymbol{F} \cdot$ n; then try to use the Divergence Theorem.

Ans: 4π

Problem 16.109. Assume that S and E satisfy the conditions of the Divergence Theorem and functions have all required continuous partial derivatives, first or second-order. Prove the following.

1. $\iint_S \mathbf{a} \cdot \mathbf{n}\,dS = 0$, where a is a constant vector.

2. $V(E) = \dfrac{1}{3} \iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F}(x, y, z) = \langle x, y, z \rangle$.

3. $\iint_S \boldsymbol{curl}\, \boldsymbol{F} \cdot d\boldsymbol{S} = 0$.

Exercises 16.9

1. Verify the Divergence Theorem is true for the vector field $\boldsymbol{F}$ defined on the region E.

   $\boldsymbol{F}(x,y,z) = \langle 2x, yz, xy\rangle$, $E = [0,1]\times[0,1]\times[0,1]$, the unit cube
   Clue: For the computation of $\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$, you should evaluate it on each of the six sides.

2. Use the Divergence Theorem to evaluate the total flux $\displaystyle\iint_S \boldsymbol{F}\cdot d\boldsymbol{S}$.

   (a) $\boldsymbol{F}(x,y,z) = y\,\mathbf{i} + x\,\mathbf{j} + 2z\,\mathbf{k}$, S is the boundary of of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$
   (b) $\boldsymbol{F}(x,y,z) = (x + y^2 + \cos z)\,\mathbf{i} + [\sin(\pi z) + xe^{-z}]\,\mathbf{j} + z\,\mathbf{k}$, S is a part of the cylinder $x^2 + y^2 = 1$ that lies between $z = 0$ and $z = 1$
   (c) $\boldsymbol{F}(x,y,z) = \langle x^2y^2, xye^z, xy^2z - xe^z\rangle$, S is the boundary of the box bounded by the coordinate planes and the planes $x = 1$, $y = 3$, and $z = 4$

   Ans: (b) 2π; (c) 54

3. As a variant of Problem 16.108, let's consider the following problem:
   Evaluate $\displaystyle\iint_S (x^2 + 2y^2 + 3z^2 + z\,e^x)\,dS$, where S is the unit sphere $x^2 + y^2 + z^2 = 4$.

   Ans: 128π

F.1. Formulas for Chapter 16

Line Integrals

Formula 16.110. **(16.17)** *If f is defined on a smooth curve C given by a vector equation $\mathbf{r}(t)=\langle x(t),y(t),z(t)\rangle$, $a\le t\le b$, then **line integral of** f **along** C is*

$$\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt = \int_a^b f(x(t),y(t),z(t))\,\sqrt{(x')^2+(y')^2+(z')^2}\,dt. \quad (16.79)$$

Formula 16.111. **(16.22)** *Let $\boldsymbol{F}$ is a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t)$, $a\le t\le b$. Then the line integral of $\boldsymbol{F}$ along C is*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds = \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt. \quad (16.80)$$

The Fundamental Theorem for Line Integrals

Formula 16.112. **(16.25)** *Suppose that $\boldsymbol{F}$ is continuous, and is a **conservative** vector field; that is, $\boldsymbol{F}=\nabla f$ for some scalar-valued function f. Then*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} = \int_C \nabla f\cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \quad (16.81)$$

Note: If $\boldsymbol{F}=\langle P,Q\rangle$ satisfies $P_y=Q_x$ over an open simply-connected domain, then $\boldsymbol{F}$ is conservative.

Green's Theorem

Formula 16.113. **(16.31)** *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F}=\langle P,Q\rangle$ have **continuous partial derivatives** on an open region including D, then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \oint_C Pdx+Qdy = \iint_D \left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA. \quad (16.82)$$

Surface Integrals

Formula 16.114. **(16.65)** *Suppose the surface S is defined by a vector function $\mathbf{r}(u,v)=\langle x(u,v),y(u,v),z(u,v)\rangle,\ \ (u,v)\in D$. Then*

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u\times r_v|\,dA. \quad (16.83)$$

Formula 16.115. **(16.66)** *When $z=g(x,y)$, $\mathbf{r}_{\boldsymbol{x}}\times\mathbf{r}_{\boldsymbol{y}}=\langle -\boldsymbol{g_x},-\boldsymbol{g_y},1\rangle$. Thus the formula (16.83) reads*

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2+g_y^2+1}\,dA. \quad (16.84)$$

Surface Integrals of Vector Fields

Formula 16.116. (16.71) *Let $\boldsymbol{F}$ be a continuous vector field defined on an oriented surface S with unit normal vector n. The* ***surface integral of*** $\boldsymbol{F} = \langle P, Q, R\rangle$ ***over*** S *is*

$$\begin{aligned}\int_C \boldsymbol{F}\cdot d\mathbf{r} &\stackrel{\text{def}}{=} \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds \\ &= \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\,|\mathbf{r}'(t)|\,dt = \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt,\end{aligned} \tag{16.85}$$

Formula 16.117. (16.72) *When the surface S is defined by $z = g(x, y)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$ and*

$$\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_D \boldsymbol{F}\cdot(\mathbf{r}_x\times\mathbf{r}_y)\,dA = \iint_D(-P\,g_x - Q\,g_y + R)\,dA. \tag{16.86}$$

Stokes' Theorem

Formula 16.118. (16.75) *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a vector field whose components have* ***continuous partial derivatives****. Then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S(\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \stackrel{\text{def}}{=} \iint_S(\textbf{curl}\,\boldsymbol{F})\cdot\mathrm{n}\,dS = \iint_D(\textbf{curl}\,\boldsymbol{F})\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA. \tag{16.87}$$

The Divergence Theorem

Formula 16.119. (16.77) *Let E be a simple solid region and S be the boundary surface of E, given with positive (outward) orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ have* ***continuous partial derivatives*** *on an open region that contains E. Then*

$$\oiint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV. \tag{16.88}$$

APPENDIX **C**

Chapter Reviews

Contents of Review

C.1. Functions

§1.3. Trigonometric Functions

Definition 1.26: An **angle** is the figure formed by two rays sharing a common endpoint, called the **vertex of the angle**.

The **angle** can be defined with the *unit circle*, the circle of radius 1.

"The angle is θ (radian), when the corresponding arc length is θ."

- The angle of the whole circle is 2π (radian).
- $2\pi = 360°$
- $\pi = 180° \quad \Rightarrow \quad \boxed{\circ = \dfrac{\pi}{180}}$

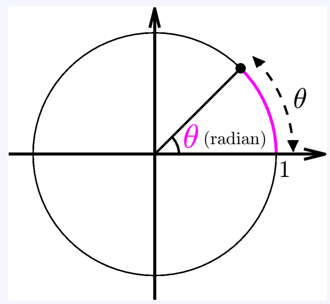


Figure C.1: Geometric definition of the angle.

Geometric interpretation of trigonometric functions

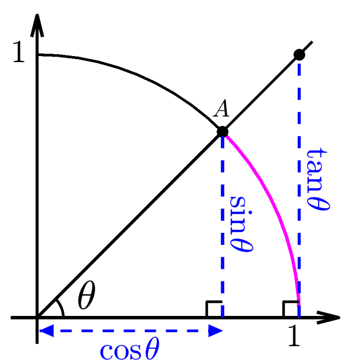


Figure C.2: Geometric interpretation of trigonometric functions.

Formula 1.35. Frequently Used Trigonometric Formulas:
For all angle x,

$$\sin^2 x + \cos^2 x = 1 \qquad \tan x = \frac{\sin x}{\cos x} = \text{slope}$$

Example C.1. Identify the **amplitude** and the **period** for the function and sketch its graph.

$$y = -2\cos\left(\frac{x}{2} + \pi\right) + 1$$

Solution.

Example C.2. Find all possible values of a for the following equation to **admit a unique solution** θ in $[0, \pi]$.

$$\sin^2\theta \;=\; a\cos\theta.$$

Solution.

Ans: $a \neq 0$

§1.6. Inverse Functions and Logarithms

Definition 1.52. Let f be a one-to-one function with domain X and range Y. Then its **inverse function** f^{-1} has domain Y and range X and is defined by

$$f^{-1}(y) = x \iff f(x) = y, \tag{C.1.1}$$

for any $y \in Y$.

Strategy 1.57. How to Find the Inverse Function of a One-to-One Function f: Write $y = f(x)$.

Step 1: Solve this equation for x in terms of y (if possible).

Step 2: Interchange x and y; the resulting equation is $y = f^{-1}(x)$.

Example 1.59. Find the inverse of the function $f(x) = x^3 + 2$, expressed as a function of x.

Solution. Write $y = x^3 + 2$.

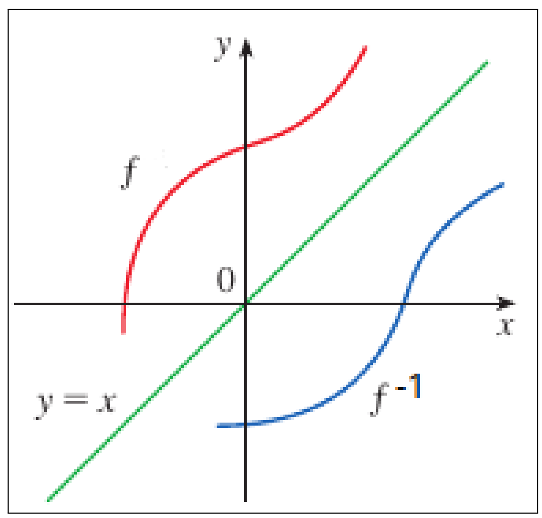
Step 1: Solve it for x:

$$x^3 = y - 2 \;\;\Rightarrow\;\; \boldsymbol{x = \sqrt[3]{y-2}}.$$

Step 2: Exchange x and y:

$$\boldsymbol{y = \sqrt[3]{x-2}}.$$

Therefore the inverse function is
$f^{-1}(x) = \sqrt[3]{x-2}$.



Observation 1.60. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

Definition 1.62. The **logarithmic function with base a**, written $y = \log_a x$, is the inverse of $y = a^x$ ($a > 0,\ a \neq 1$). That is,

$$\log_a x = y \iff a^y = x. \tag{C.1.2}$$

Algebraic Properties of Logarithms: for $(a > 0,\ a \neq 1)$

$$\begin{array}{lrcl} \text{Product Rule:} & \log_a xy & = & \log_a x + \log_a y \\ \text{Quotient Rule:} & \log_a \dfrac{x}{y} & = & \log_a x - \log_a y \\ \text{Power Rule:} & \log_a x^{\alpha} & = & \alpha \log_a x \\ \text{Reciprocal Rule:} & \log_a \dfrac{1}{x} & = & -\log_a x \end{array} \tag{C.1.3}$$

Claim 1.69.

(a) Every exponential function is a power of the natural exponential function.

$$a^x = e^{x \ln a}. \tag{C.1.4}$$

(b) Every logarithmic function is a constant multiple of the natural logarithm.

$$\log_a x = \frac{\ln x}{\ln a}, \quad (a > 0,\ a \neq 1) \tag{C.1.5}$$

which is called the **Change-of-Base Formula**.

Example 1.68. Solve for x.

(a) $e^{5-3x} = 3$.

(b) $\log_3 x + \log_3(x-2) = 1$

(c) $\ln(\ln x) = 0$

Solution.

Example 1.72. Is it correct? If not, why?

(a) $\arcsin\left(\sin\frac{9\pi}{4}\right) = \frac{9\pi}{4}$

(b) $\cos(\arccos 2) = 2$

Solution.

Self-study 1.76. Use a **geometric manipulation** to simplify the expression $\sin(\tan^{-1} x)$.

Solution.

C.2. Limits and Continuity

§2.3. The Precise Definition of a Limit

Definition 2.22. Let $f(x)$ be defined on an open interval about a, except possibly at a itself. We say that

the limit of $f(x)$ is L as x approaches a,

and write

$$\lim_{x \to a} f(x) = L, \tag{C.2.1}$$

if, for every number $\varepsilon > 0$, there exists a corresponding number $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta. \tag{C.2.2}$$

Strategy 2.26. The process of finding a $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x - a| < \delta$$

can be accomplished in two steps.

1. Solve the inequality $|f(x) - L| < \varepsilon$ to find an open interval (c, d) containing a.
2. Find a value of a $\delta > 0$ that places the open interval $(a - \delta, a + \delta)$ **inside the interval (c, d)**.

Example C.3. For the limit $\lim_{x\to 4} \sqrt{2x+1} = 3$, find a δ that works for $\varepsilon = 1$. That is, find a δ such that

$$|\sqrt{2x+1} - 3| < 1 \text{ whenever } 0 < |x - 4| < \delta.$$

Solution.

Ans: $\delta = 2.5$.

§2.5. Continuity

Definition 2.36. A function f is **continuous** at a if

$$\lim_{x \to a} f(x) = f(a). \tag{C.2.3}$$

Definition 2.40. **One-Sided Continuity**

- A function f is **right-continuous** at a (or **continuous from the right**) if
$$\lim_{x \to a^+} f(x) = f(a). \tag{C.2.4}$$
- A function f is **left-continuous** at a (or **continuous from the left**) if
$$\lim_{x \to a^-} f(x) = f(a). \tag{C.2.5}$$

Example 2.47. For what value of the constant c is the function f continuous everywhere?

$$f(x) = \begin{cases} cx^2 + 2x, & \text{if } x < 2 \\ x^3 - cx & \text{if } x \geq 2 \end{cases}$$

Solution.

Theorem 2.50. If $\lim_{x\to a} f(x) = b$ and g is continuous at b, then

$$\lim_{x\to a} g(f(x)) = g\left(\lim_{x\to a} f(x)\right) = g(b) \tag{C.2.6}$$

Note: **Continuity and limit are commutative**, when the limit exists.

Theorem 2.52. **Intermediate Value Theorem (IVT)**
Suppose that f is **continuous** on a **closed interval** $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c \in (a, b)$ such that $f(c) = N$.

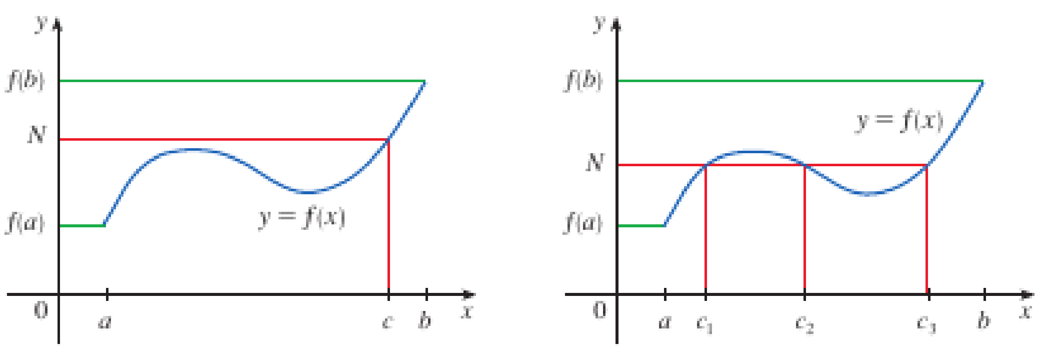


Figure C.3: There is *at least* one such c that $f(c) = N$.

Remark 2.53. **Consequences of the IVT**

- **Connectedness of the Graph**: The IVT implies that the graph of a continuous function cannot have any breaks over the interval. It will be **connected** – a single, unbroken curve.
- **Root Finding**: We call a solution of the equation $f(x) = 0$ a **root of the equation** or a **zero of the function** f. The IVT tells us that if f is continuous, then any interval on which f changes sign contains a zero of the function.

Example C.4. Use the Intermediate Value Theorem to show that there is a root of the given equation in the specified interval.

$$x^3 + 3x^2 = x + 2, \quad [0, 1].$$

Solution.

Continuous Extension to a Point

Example C.5. Define $f(3)$ in a way that extends $f(x) = \dfrac{x^2 - 9}{x^2 + x - 12}$ to be continuous at $x = 3$.

Solution.

§2.6. Limits Involving Infinity; Asymptotes

Definition 2.59. The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x\to\infty} f(x) = L \quad \text{or} \quad \lim_{x\to-\infty} f(x) = L. \tag{C.2.7}$$

Definition 2.73 A line $x = a$ is a **vertical asymptote** of the graph of a function $y = f(x)$ if either

$$\lim_{x\to a^+} f(x) = \pm\infty \qquad \text{or} \qquad \lim_{x\to a^-} f(x) = \pm\infty. \tag{C.2.8}$$

Example C.6. Find the **vertical, horizontal, and oblique asymptotes** of each curve, if any.

(a) $f(x) = \dfrac{x-1}{x^2-2x}$

(b) $g(x) = \dfrac{x^2-2x}{x-1}$

C.3. Derivatives

§3.2. The Derivative as a Function

Definition 3.13. The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \tag{C.3.1}$$

provided the limit exists.

Remark 3.14. The domain of f' may be either **the same as** or **smaller than** the domain of f.

Theorem 3.21. **Differentiability Implies Continuity**:
If f has a derivative at $x = c$, then f is continuous at $x = c$.

Example C.7. Let $f(x) = 1/x$.

(a) Use the definition of derivative to find f'.

(b) Find the tangent line to the curve $y = f(x)$ at $x = 1$.

Solution.

§3.6. The Chain Rule

Theorem 3.45. **The Chain Rule**: If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x) \; = \; \frac{d\, f(g(x))}{d\, g(x)} \cdot \frac{dg(x)}{dx}. \tag{C.3.2}$$

Letting $y = f(u)$ and $u = g(x)$, a simpler form of **Leibniz's notation** reads

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}, \tag{C.3.3}$$

where dy/du is evaluated at $u = g(x)$.

Example 3.50. Find the value of $(f \circ g)'$ at the given value of x. Use this to find an equation of the tangent line to the curve at the given point.

$$f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$$

Solution.

§3.8. Derivatives of Inverse Functions and Logarithms

Let f be differentiable and have inverse f^{-1}.

- Then
$$f(f^{-1}(x)) = x.$$
- Applying Chain Rule results in
$$f'(f^{-1}(x)) \cdot [f^{-1}]'(x) = 1 \quad \text{(C.3.4)}$$
and therefore
$$\boxed{[f^{-1}]'(x) = \frac{1}{f'(f^{-1}(x))},} \quad \text{(C.3.5)}$$
which can be interpreted geometrically as in the figure.

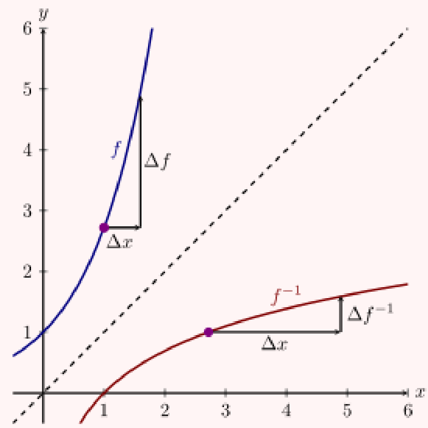


Figure C.4: The derivative of the inverse function.

Example C.8. Let $f(x) = x^2 - 2x - 1$, $x < 1$. Find $\dfrac{df^{-1}}{dx}(7)$.

Solution.

Derivatives of Logarithmic Functions

Let $f(x) = \log_a x$.

- It follows from the definition of $\log$ that

$$y = \log_a x \iff x = a^y = e^{\ln a^y} = e^{y \ln a}. \tag{C.3.6}$$

- Apply implicit differentiation to the right side of (3.36) to get

$$1 = e^{y \ln a} \cdot y' \ln a \quad \Longrightarrow \quad y' = \frac{1}{e^{y \ln a} \cdot \ln a} = \frac{1}{x \ln a}. \tag{C.3.7}$$

Summary 3.59. We may apply above arguments and Chain Rule to get the following formulas.

$$\begin{array}{lclclcl} \dfrac{d}{dx} \log_a x & = & \dfrac{1}{x \ln a} & \quad & \dfrac{d}{dx} \ln x & = & \dfrac{1}{x} \\ \dfrac{d}{dx} a^x & = & a^x \ln a & & \dfrac{d}{dx} e^x & = & e^x \\ \dfrac{d}{dx} \ln f(x) & = & \dfrac{f'(x)}{f(x)} & & \dfrac{d}{dx} \ln |x| & = & \dfrac{1}{x} \end{array} \tag{C.3.8}$$

Logarithmic Differentiation

Algorithm 3.63. **Logarithmic Differentiation**

1. **Take natural logarithms** of both sides of an equation $y = f(x)$ and use the Laws of Logarithms to conveniently reform the right side.
2. **Differentiate implicitly** with respect to x.
3. **Solve the resulting equation for y'.**
4. **Replace y with $f(x)$.**

Example C.9. Use logarithmic differentiation to find dy/dx.

(a) $y = x^{\ln x}$ (b) $y^x = x^y$

Solution.

Example C.10. (Exercise 3, §3.8).
Suppose that the function f ans its derivative have the following values at $x = 0, 1, 2, 3, 4$.

x	0	1	2	3	4
$f(x)$	-4	3	-1	2	1
$f'(x)$	3	2	$5/4$	$2/3$	$1/5$

Assuming the inverse function f^{-1} is differentiable, find $[f^{-1}]'(x)$ at

(a) $x = 1$ (b) $x = 2$ (c) $x = 3$

Solution.

Ans: (c) $1/2$

§3.11. Linearization and Differentials

Definition 3.78. If f is differentiable at $x = a$, then the approximating function

$$L(x) := f(a) + f'(a)(x - a) \tag{C.3.9}$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x) \tag{C.3.10}$$

of f by L is the **linear approximation** (or, **tangent line approximation**) of f at a. The point $x = a$ is the **center** of the approximation.

Definition 3.83. Let $y = f(x)$ be a differentiable function. The **differential** dx is an independent variable. The **differential** dy is a dependent variable, defined as

$$dy = f'(x)dx. \tag{C.3.11}$$

Remark 3.84. **What is dy?**
Often the variable dx is chosen to be Δx, the change in x. Then the differential dy is the change in the linearization of f at $x = a$, $\boldsymbol{\Delta L}$.

Example C.11. Let $f(x) = 2x^4 + x$.

(a) Find the linearization $L(x)$ of $f(x)$ at $x = 1$.

(b) Use $L(x)$ to approximate $f(1.1)$. (The exact value is 4.0282.)

(c) Find the value of dy when $x = 1$ and $dx = 0.1$.

C.4. Applications of Derivatives

§4.4. Concavity and Curve Sketching

Definition 4.22. The graph of a differentiable function $y = f(x)$ is

(a) **concave up (convex)** on an open interval I if f' is increasing on I;

(b) **concave down** on an open interval I if f' is decreasing on I.

The Second Derivative Test for Concavity
Let $y = f(x)$ be twice-differentiable on an interval I.

(a) If $f'' > 0$ on I, the graph of f over I is concave up.

(b) If $f'' < 0$ on I, the graph of f over I is concave down.

Definition 4.23. A point $(c, f(c))$ where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

Theorem 4.27. **Second Derivative Test for Local Extrema**
Suppose f'' is continuous on an open interval that contains $x = c$.

(a) If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.

(b) If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.

(c) If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.

Strategy 4.29. Procedure for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find the derivatives f' and f''.
3. Find the **critical points** of f, if any, and identify the function's behavior at each one.
4. Find where the curve is **increasing** and where it is **decreasing**.
5. Find the **points of inflection**, if any occur, and determine the **concavity** of the curve.
6. **Plot key points**, such as the intercepts and the points found in Steps 3–5, and **sketch the curve** together **with any asymptotes** that exist.

Example 4.28. Sketch a graph of the function $f(x) = x^4 - 4x^3 + 10$. Identify the coordinates of any local extreme points, inflection points, and concavity.

Solution.

x		0		2		3	
$f'(x)$		0				0	
$f''(x)$		0		0			
$f(x)$							
Behavior of f							

§4.5. Indeterminate Forms and L'Hôpital's Rule

Theorem 4.33. L'Hôpital's Rule (Bernoulli's Rule)
Assume that f and g are differentiable and $g'(x) \neq 0$ on an open interval I containing a (except possibly at a). Suppose that

$$\lim_{x\to a} f(x) = 0 \quad \text{and} \quad \lim_{x\to a} g(x) = 0 \tag{C.4.1}$$

or that

$$\lim_{x\to a} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to a} g(x) = \pm\infty. \tag{C.4.2}$$

(In other words, we have an indeterminate form of type $\dfrac{0}{0}$ or $\dfrac{\infty}{\infty}$.) Then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}. \tag{C.4.3}$$

Example C.12. Use L'Hôpital's Rule to evaluate the limit.

(a) $\displaystyle\lim_{x\to 0} \frac{(x^5+1)(e^x-1)}{2^x-1}$

(b) $\displaystyle\lim_{\theta\to 0} \frac{\cos^2\theta - 1}{e^\theta - \theta - 1}$

Other Indeterminate Forms

Example C.13. Use L'Hôpital's Rule to find the limit.

(a) $\lim_{x\to 0^+} x \ln x^2 \quad (0 \cdot \infty)$

(b) $\lim_{x\to\infty} [\ln 2x - \ln(x+1)] \quad (\infty - \infty)$

(c) $\lim_{x\to 0^+} (1+x)^{3/x} \quad (1^\infty)$

(d) $\lim_{x\to\infty} x^{1/\sqrt{x}} \quad (\infty^0)$

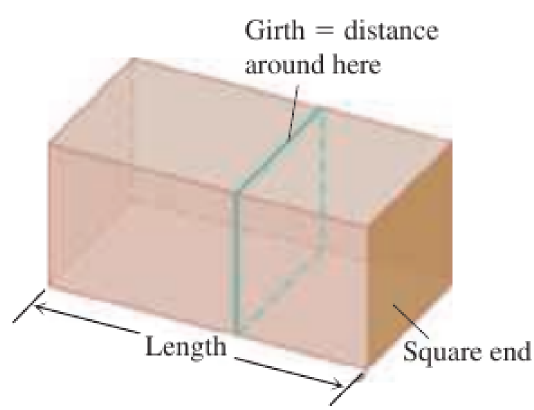
§4.6. Applied Optimization

Strategy 4.41. **Solving Applied Optimization Problems**

1. *Read the problem*. Read the problem until you ***understand*** it. What is **given**? What is the **unknown quantity to be optimized**?
2. *Introduce variables*. List every relevant relation in the problem as an equation. In most problem it is helpful to draw a picture.
3. *Write an equation for the unknown quantity*.
   Express **the quantity to be optimized** as a function of **a single variable**. This may require considerable manipulation.
4. *Test the critical points and endpoints in the domain of the function found in the previous step*. Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Note: Mostly, the optimization problem will be formulated with **two quantities and two variables**; **the quantity to be optimized** can be expressed as a function of **a single variable.**

Example C.14. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 *in*. What dimensions will give a box with a square end the largest possible volume?



Solution.

Ans: Length = 36 in, Side = 18 in ($18^2 \cdot 36 = 11664\,\text{in}^3$)

§4.7. Newton's Method

- Given an **initial approximation x_0**, the point-slope equation for the tangent to the curve at $(x_0, (x_0))$ is

$$y = f'(x_0)(x - x_0) + f(x_0). \tag{C.4.4}$$

- We can find where it crosses the x-axis by setting $y = 0$:

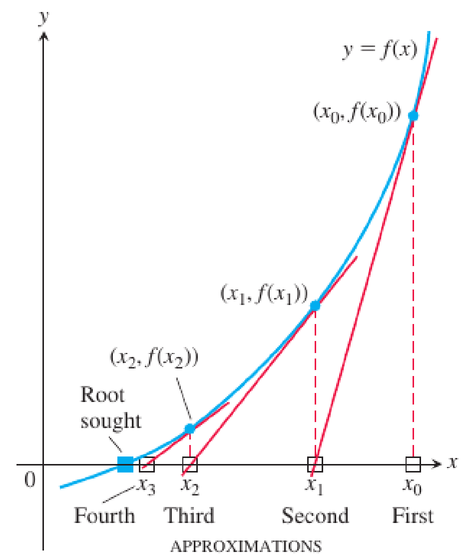
$$f'(x_0)(x - x_0) = -f(x_0),$$

which implies

$$x = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{C.4.5}$$

when $f'(x_0) \neq 0$.

- This value of x is the **next approximation x_1**.
- Repeat the steps to find new approximations.



Algorithm 4.48. Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad f'(x_n) \neq 0. \tag{C.4.6}$$

Example C.15. Let us try to find $5^{1/3}$ applying the Newton's method. ($\sqrt[3]{5} \approx 1.7099759466766968$.)

(a) Use the *Intermediate Value Theorem* to show that there is a zero of $f(x) = x^3 - 5$ in $[1, 2]$.

(b) Run two iterations of the Newton's method to find x_2 when $x_0 = 1.5$.

Solution.

Ans: $x_1 = \dfrac{47}{27} \approx 1.7407407407, \quad x_2 = 1.7105164618$

C.5. Integrals

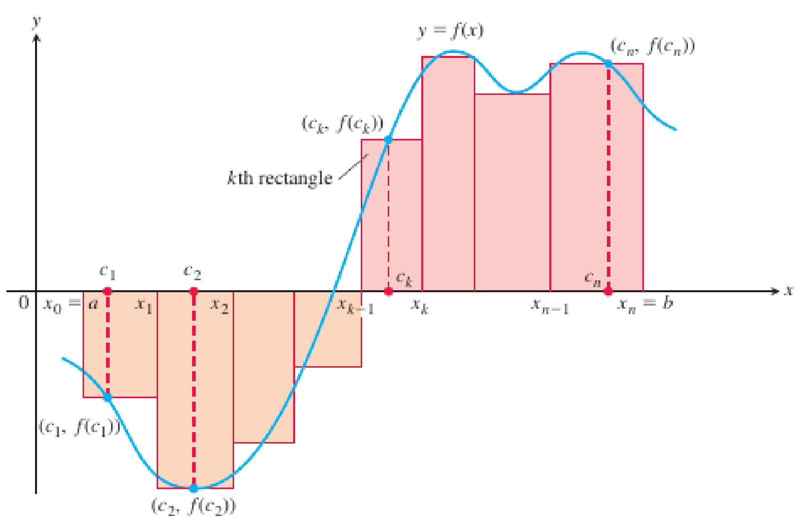
§5.3. The Definite Integral

Definition 5.16. Let $f(x)$ be a function defined on a closed interval $[a, b]$. We say that a number J is the **definite integral** of f over $[a, b]$ and that J is **the limit of the Riemann sums**:

$$J = \lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k \tag{C.5.1}$$

if the following condition is satisfied:

> Given any number $\varepsilon > 0$ there is a corresponding number $\delta > 0$ such that for **every partition** $P = \{x_0, x_1, \cdots, x_n\}$ with $\|P\| < \delta$ and **any choice** of $c_k \in [x_{k-1}, x_k]$, we have
>
> $$\left| \sum_{k=1}^{n} f(c_k)\Delta x_k - J \right| < \varepsilon. \tag{C.5.2}$$



The Definite Integral as the Limit of Riemann Sums

If the definite integral exists, then instead of writing J we write

$$\int_a^b f(x)\,dx = \lim_{||P||\to 0} \sum_{k=1}^{n} f(c_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_k)\Delta x_k \tag{C.5.3}$$

Definition 5.17. When the definite integral exists, we say that the Riemann sums of f on $[a, b]$ **converge** to the definite integral $J = \int_a^b f(x)dx$ and that f is **integrable** over $[a, b]$.

Theorem 5.18. **Integrability of Continuous Functions**:
If a function f is continuous over the interval $[a, b]$, or if f has at most finitely many jump discontinuities there, then the definite integral $\int_a^b f(x)dx$ exists and f is integrable over $[a, b]$.

Example 5.20. Express the limit as a definite integral.

$$\lim_{||P||\to 0} \sum_{k=1}^{n} (c_k^2 - 3c_k)\Delta x_k,$$

where P is a partition of $[-7, 5]$.
Solution.

Recall: **Definite Integral as the Limit of Riemann Sums** (C.5.3):

$$\int_a^b f(\mathbf{x})\,\mathbf{dx} = \lim_{||P||\to 0} \sum_{k=1}^{n} f(\mathbf{c_k})\mathbf{\Delta x_k} = \lim_{n\to\infty} \sum_{k=1}^{n} f(\mathbf{c_k})\mathbf{\Delta x_k}$$

Equal-Width Subintervals

Example 5.22. Express the limit as a definite integral.

$$\lim_{n\to\infty} \sum_{k=1}^{n} \left[\left(3 + \frac{2k}{n}\right)^2 + 4\left(3 + \frac{2k}{n}\right) \right] \left(\frac{2}{n}\right)$$

Note that the expression is not unique!
Solution.

Ans: $\displaystyle\int_0^2 [(3+x)^2 + 4(3+x)]\, dx = \int_3^5 (x^2 + 4x)\, dx.$

Properties of Definite Integrals

Example 5.24. Suppose f and g are integrable and that

$$\int_1^9 f(x)\, dx = -1, \qquad \int_7^9 f(x)\, dx = 5, \qquad \int_7^9 g(x)\, dx = -4$$

Find

(a) $\displaystyle\int_7^9 [2f(x) - 3g(x)]\, dx$

(b) $\displaystyle\int_1^7 f(x)\, dx$

(c) $\displaystyle\int_9^7 [g(x) - f(x)]\, dx$

Area under the Graph of a Nonnegative Function

Example C.16. Evaluate the integral by interpreting it in terms of areas.

(a) $\displaystyle\int_{-1}^{3} (|2x| + 1)\, dx$

(b) $\displaystyle\int_{-3}^{3} \sqrt{9 - x^2}\, dx$

Average Value of a Continuous Function: Revisited

Definition 5.28. If f is integrable on $[a, b]$, then its **average value** on $[a, b]$, which is also called its **mean**, is

$$\mathrm{av}(f) = \frac{1}{b-a} \int_a^b f(x)\, dx. \tag{C.5.4}$$

Example C.17. Find the average value of $f(x) = 1 + \sqrt{4 - x^2}$ on $[-2, 2]$.

Solution.

§5.4. The Fundamental Theorem of Calculus

Theorem 5.30. **(FTC1)** If f is continuous on $[a, b]$, then

$$F(x) = \int_a^x f(t)\,dt \tag{C.5.5}$$

is continuous on $[a, b]$ and differentiable on (a, b) and its derivative is $f(x)$:

$$F'(x) = \frac{d}{dx}\int_a^x f(t)\,dt = f(x). \tag{C.5.6}$$

That is, F is an **antiderivative** of f.

Theorem 5.32. **(FTC2)** If f is continuous over $[a, b]$ and F is any antiderivative of f on $[a, b]$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{C.5.7}$$

Note:

$$\begin{aligned} \frac{d}{dx}\int_{h(x)}^{g(x)} f(t)\,dt &= \frac{d}{dx}\left[F(t)\Big|_{h(x)}^{g(x)} \right] = \frac{d}{dx}\left[F(g(x)) - F(h(x))\right] \\ &= f(g(x))\cdot g'(x) - f(h(x))\cdot h'(x). \end{aligned} \tag{C.5.8}$$

Example C.18. Find the derivative of the function.

(a) $f(x) = \displaystyle\int_0^x \cos(1+t^4)\,dt$

(b) $g(x) = \displaystyle\int_2^{x^3} e^{-t^2}\,dt$

(c) $y = \displaystyle\int_{\sin x}^1 \frac{t^2}{\sqrt{1+t^2}}\,dt$

Mean Value Theorem for Definite Integrals

Theorem 5.36. The Mean Value Theorem for Definite Integrals.
If f is continuous on $[a,b]$, then at some point $c \in [a,b]$,

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx. \tag{C.5.9}$$

Position, Velocity, Displacement, and Distance

The **position** of an object moving along a line at time t, denoted $s(t)$, is the location of the object relative to the origin.

(a) The **velocity** of an object at time t is $v(t) = s'(t)$.

(b) The **Net Change Theorem** says that

$$s(b) - s(a) \;=\; \int_a^b v(t)\,dt, \tag{C.5.10}$$

so the integral of velocity is the **displacement** of the object over the time interval $[a,b]$.

(c) The **distance traveled** over the time interval $[a,b]$ is

$$\text{Distance traveled} = \int_a^b |v(t)|\,dt, \tag{C.5.11}$$

where $|v(t)|$ is the **speed** of the object at time t.

Example C.19. The velocity function (in meters per second) is given for a particle moving along a line, as

$$v(t) = t^2 - 4t + 3, \qquad t \in [2,5].$$

(a) Find the displacement.

(b) Find the distance traveled.

Solution.

§5.6. Definite Integral Substitutions and the Area between Curves

The Substitution Rule (§5.5)

Theorem 5.48. If $u = g(x)$ is a differentiable function whose range is an interval I, and f is continuous on I, then

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du, \tag{C.5.12}$$

where $du = g'(x)\, dx$.

Substitution in Definite Integrals

Theorem 5.55. If $g'(x)$ is continuous on the interval $[a, b]$ and $f(x)$ is continuous on the range of $g(x) = u$, then

$$\int_a^b f(g(x)) \cdot g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du. \tag{C.5.13}$$

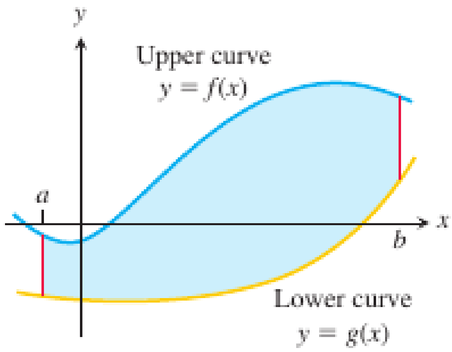
Example C.20. Evaluate definite integrals.

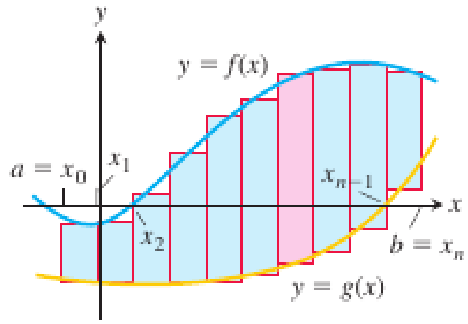
(a) $\displaystyle\int_0^1 (6x^2 + 4x)\sqrt{x^3 + x^2}\, dx.$ (b) $\displaystyle\int_0^{\pi/2} \sin^2 x \cos x\, dx.$

Solution.

Ans: (a) $8\sqrt{2}/3$. (b) $1/3$.

Areas between Curves





Definition 5.61. If f and g are continuous with $f(x) \geq g(x)$ throughout $[a, b]$, then the area of the region between the curves $y = f(x)$ and $y = g(x)$ from over $[a, b]$ is the integral of $(f - g)$ from a to b:

$$A = \lim_{||P|| \to 0} \sum_{k=1}^{n} [f(c_k) - g(c_k)] \Delta x_k = \int_a^b [f(x) - g(x)] dx. \tag{C.5.14}$$

Example C.21. Find the *total area* of the region between the curves $y = x^2 - 4$ and $y = x - 2$, between $x = 0$ and $x = 3$.

Solution.

C.6. Applications of Definite Integrals

§6.1. Volume Using Cross-Sections

Proposition 6.1. Suppose that we want to find the volume of a solid S like the one pictured in Figure C.5 (a).

- At each point $x \in [a, b]$, we form a **cross-section** $S(x)$ by intersecting S with a plane perpendicular to the x-axis through the point x, which gives a planar region whose area is $A(x)$.

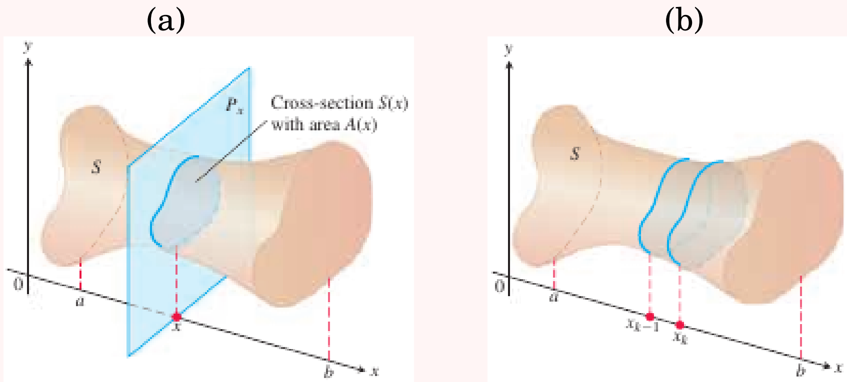


Figure C.5: (a) A cross-section $S(x)$ of the solid S. (b) A thin slab in the solid S.

- **Riemann sum**:
  - Partition $[a, b]$ into subintervals
  - Approximate the **thin slab** by a **cylindrical solid**.
  - Then a **Riemann sum** for the function $A(x)$ on $[a, b]$ reads

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\Delta x_k, \tag{C.6.1}$$

where $c_k = x_k$.

This method is known as the **method of slicing** for computing volumes.

The Method of Slicing

Definition 6.2. The **volume** of a solid of integrable cross-sectional area $A(x)$ for $x \in [a,b]$ is the integral of A over $[a,b]$

$$V = \lim_{n\to\infty} \sum_{k=1}^{n} A(x_k)\Delta x_k = \int_a^b A(x)\,dx. \tag{C.6.2}$$

Strategy 6.3. **Calculating the Volume of a Solid**

(a) **Sketch the solid** to understand.

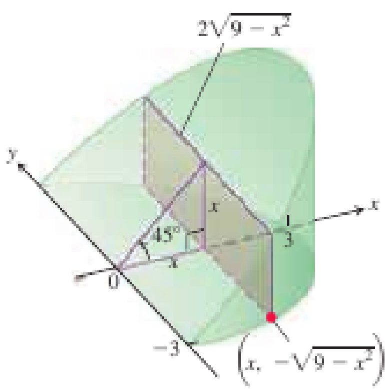
(b) Find **the limits of integration, $[a,b]$.**

(c) For each $x \in [a,b]$, find a **formula for $A(x)$**,
the area of a typical cross-section.

(d) **Integrate $A(x)$** to find the volume.

Example 6.5. A curved wedge is cut from a circular cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a $45°$ angle at the center of the cylinder. Find the volume of the wedge.

Solution. Note that $A(x) = x \cdot 2\sqrt{9-x^2}$.



Ans: 18.

The Disk Method

Volume by Disks for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx, \tag{C.6.3}$$

where $R(x)$ is the radius.

Example C.22. Find the volume of the solid obtained by rotating the region between $x = y^2 + 1$ and $x = 5$ about the line $x = 5$.
Solution.

Ans: $\frac{32}{3}\pi$.

The Washer Method

Volume by Washers for Rotation About the x-Axis

$$V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx, \tag{C.6.4}$$

where $R(x)$ is the outer radius and $r(x)$ is the inner radius.

Example C.23. The region bounded by the parabola $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.
Solution.

Ans: $\frac{117}{5}\pi$.

§6.2. Volumes Using Cylindrical Shells

The Shell Method

- Consider the region bounded by the graph of a function $y = f(x)$ and the x-axis over the closed interval $[a, b]$; see Figure 6.3(a).
- We assume $L \leq a$. We generate a solid S by rotating the region about the **vertical line $x = L$**.

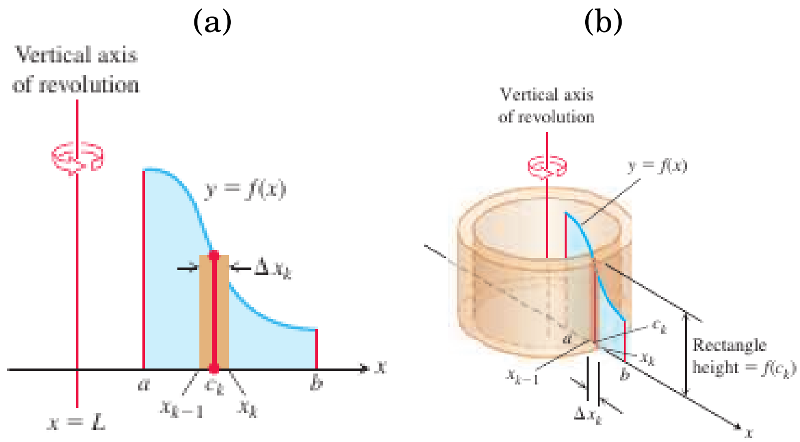


Figure C.6: A region is resolved about the vertical line $y = L$.

- **Partitioning**. Let P be a partition of the interval $[a, b]$ by the points

$$a = x_0 < x_1 < \cdots < x_n = b.$$

- **Approximation**. **We approximate the region in Figure 6.3(a) with the collection of rectangles based on this partition.**
- **Rectangles**. As usual, choose a point $c_k \in [x_{k-1}, x_k]$, e.g., the midpoint of the subinterval. **A typical approximating rectangle** has

$$\text{height} = f(c_k) \ \text{ and } \ \text{width} = \Delta x_k = x_k - x_{k-1}.$$

- **Rotation: Cylindrical shells**. If **such a rectangle** is rotated about the vertical line $y = L$, then **a shell** is swept out, as in Figure 6.3(b).
- **Riemann sum**. We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles:

$$V \approx \sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \qquad \text{(C.6.5)}$$

Formula 6.11. Shell Formula for Revolution About a Vertical Line. The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0,\ \ L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b \underbrace{2\pi(\text{shell-radius}) \cdot (\text{shell-height})}_{\text{Area of the thin shell}}\, dx. \tag{C.6.6}$$

Example C.24. The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Use the shell method to find the volume of the solid.

Solution.

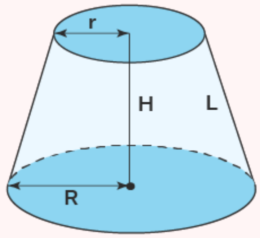
Ans: $\frac{128}{5}\pi$.

§6.4. Areas of Surfaces of Revolution

Note: From Section 6.3, the **differential of arc length** reads

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy. \tag{C.6.7}$$

Formula 6.25. Curved Surface Area of a Frustum



$$\textbf{Frustum Surface Area} = \boxed{2\pi \cdot \frac{R+r}{2} \cdot L} \tag{C.6.8}$$

Surface Area for Revolution about the x-axis

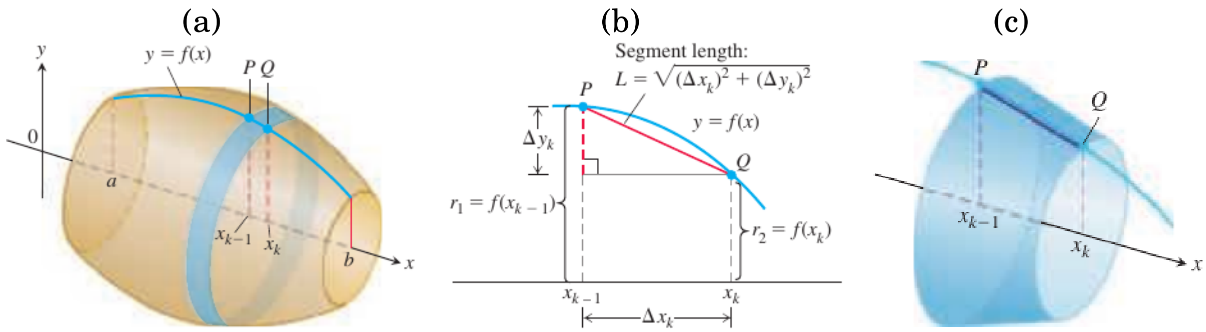


Figure C.7: Surface area for revolution about the x-axis.

We will **find the area of the surface** generated by revolving the graph of $y = f(x)$, $a \le x \le b$, about the x-axis, as in Figure C.7(a).

- **Partitioning** $\Rightarrow$ **Approximation** $\Rightarrow$ **Riemann sum**

$$\begin{aligned} S &\approx \sum_{k=1}^{n} 2\pi \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{1 + [f'(c_k)]^2}\Delta x_k \\ &\approx \sum_{k=1}^{n} 2\pi f(c_k) \cdot \sqrt{1 + [f'(c_k)]^2}\Delta x_k. \end{aligned} \tag{C.6.9}$$

- See details on page 282.

Definition 6.26. **Surface Area for Revolution about the x-axis**
If the function $y = f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area of the surface** generated by revolving the graph of $y = f(x)$ **about the x-axis** is

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}\, dx. \tag{C.6.10}$$

Definition 6.27. **Surface Area for Revolution about the y-axis**
If the function $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the **area of the surface** generated by revolving the graph of $x = g(y)$ **about the y-axis** is

$$S = \int_c^d 2\pi x \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy. \tag{C.6.11}$$

Claims 6.31 and 6.33 . Curves, Revolved About the Other Axis

- Let $y = f(x)$ be a monotone function defined on $[a, b]$, $a \geq 0$, such that

$$[a, b] \;\underset{g = f^{-1}}{\overset{f}{\rightleftarrows}}\; [c, d]. \tag{C.6.12}$$

- Then, for example, **the area of the surface** obtained by rotating the curve $y = f(x)$ about the **y-axis** reads, with **(radius = x)**,

$$S = \int 2\pi r\, ds = \int_a^b 2\pi x\sqrt{1 + [f'(x)]^2}\, dx = \int_c^d 2\pi g(y)\sqrt{1 + [g'(y)]^2}\, dy \tag{C.6.13}$$

Example C.25. Find the area of the surface obtained by rotating the curve $y = x^2,\ 0 \le x \le 2$, about the ***y*-axis**.

Solution.

(a) ***x*-integration**:

(b) ***y*-integration**:

Ans: $\frac{\pi}{6}(17\sqrt{17} - 1)$

Example C.26. (Exercise 3, § 6.4) Find the area of the surface obtained by rotating the curve about the ***y*-axis**.

$$y = \frac{x^4}{4} + \frac{1}{8x^2}, \quad 1 \le x \le 2$$

Hint: Use the formula in (6.26).

Solution.

Ans: $253\pi/20$.

C.7. Integrals and Transcendental Functions

§7.2. Exponential Change and Separable Differential Equations

In many real-world situations, **the rate of change of a quantity y** is proportional to **its size** at a given time t.

$$\frac{dy}{dt} \sim y \implies \frac{dy}{dt} = ky. \tag{C.7.1}$$

- Examples of such quantities include the size of a population, the amount of a decaying radioactive material, and the temperature difference between a hot object and its surrounding medium.
- Such quantities are said to undergo **exponential change**.

Example C.27. Solve the differential equation (C.7.1).

Solution. Divide (C.7.1) by y to get

$$\begin{aligned} &\frac{1}{y}\cdot\frac{dy}{dt} = k \ \Rightarrow\ \int \frac{\mathbf{1}}{\mathbf{y}}\cdot\frac{\mathbf{dy}}{\mathbf{dt}}\,\mathbf{dt} = \int k\,\mathbf{dt} \\ &\Rightarrow\ \ln|y| = kt + C \quad\Rightarrow\ |y| = e^{kt+C} \\ &\Rightarrow\ y = \pm e^{C}\cdot e^{kt} = Ae^{kt} \end{aligned} \tag{C.7.2}$$

The solution of the **initial value problem**

$$\frac{dy}{dt} = ky, \quad y(0) = y_0, \tag{C.7.3}$$

is

$$y = y_0\, e^{kt}. \tag{C.7.4}$$

Note: $\int \frac{1}{y}\cdot\frac{\mathrm{dy}}{\mathrm{dt}}\,\mathrm{dt} = \int \frac{1}{y}\,\mathrm{dy}$, the integration of $\frac{1}{y}$ with respect to y. That is, **the first line in (C.7.2) and (C.7.3)** can be written as

$$\boldsymbol{\frac{1}{y}\,dy = k\,dt} \tag{C.7.5}$$

Separable Differential Equations

- More general differential equations are of the form

$$\frac{dy}{dx} = f(x,y) = f(x, y(x)). \tag{C.7.6}$$

- The differential equation (C.7.6) is **separable** if f can be expressed as a product of a function of x and a function of y:

$$\frac{dy}{dx} = g(x)\, h(y). \tag{C.7.7}$$

- We can solve (C.7.7), using the same arguments introduced in Example C.27 (***separate the variables and integrate***):

$$\frac{1}{h(y)}\, dy = g(x)\, dx \quad \Rightarrow \quad \int \frac{1}{h(y)}\, dy = \int g(x)\, dx. \tag{C.7.8}$$

Population Growth

Example C.28. The biomass of a yeast culture in an experiment is initially 20 grams. After 30 minutes the mass is 40 grams. Assuming that the equation for unlimited population growth gives a good model for the growth of the yeast ($y' = ky$) when the mass is below 100 grams, how long will it take for the mass to **triple** from its initial value?

Solution. Begin with $y = y_0\, e^{kt}$.

Ans: $k = \frac{1}{30}\ln(\frac{40}{20}) \approx 0.0231$, $t = \frac{\ln 3}{k} \approx 47.55$.

Radioactivity

- **Some atoms are unstable** and can spontaneously emit mass or radiation. This process is called **radioactive decay**.
- **Experiments** have shown that at any given time the rate at which a radioactive element decays is approximately proportional to the number of radioactive nuclei present: proportional to the number of radioactive nuclei present.
- **Modeling**: Thus the decay of a radioactive element is described by the equation
$$\frac{dy}{dt} = -ky, \quad k > 0, \tag{C.7.9}$$
of which the solution reads
$$y = y_0\, e^{-kt}. \tag{C.7.10}$$
- The **half-life** of a radioactive element is the time expected to pass until half of the radioactive nuclei present in a sample decay:
$$\frac{1}{2}y_0 = y_0\, e^{-kt}.$$
Thus the half-life reads
$$\text{Half-life} = \frac{\ln 2}{k}. \tag{C.7.11}$$

Example 7.10. (**Plutonium-239**) The half-life of the plutonium isotope is 24,360 years. If 10 g of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?
Solution.

Ans: $24360 \cdot \frac{\ln 5}{\ln 2} \approx 56562$.

§7.4. Relative Rates of Growth and Convergence: Big-oh and Little-oh

Relative Rates of Growth

Definition 7.28. Let $f(x)$ and $g(x)$ be positive for x sufficiently large.

(a) f **grows faster than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \infty,$$

or, equivalently, if

$$\lim_{x\to\infty} \frac{g(x)}{f(x)} = 0.$$

We also say that g **grows slower than** f as $x \to \infty$.

(b) f and g **grow at the same rate** as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L,$$

where L is finite and positive.

Big-oh and Little-oh

Definition 7.31. A function f is **of smaller order than** g as $x \to \infty$ if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = 0. \tag{C.7.12}$$

We indicate this by writing $f = o(g)$ ("f is **little-oh** of g").

Definition 7.32. Let $f(x)$ and $g(x)$ be positive for x sufficiently large. Then f is **of at most the order of** g as $x \to \infty$ if there is a positive number M such that

$$\frac{f(x)}{g(x)} \le M, \quad \text{for } x \text{ sufficiently large.} \tag{C.7.13}$$

We indicate this by writing $f = \mathcal{O}(g)$ ("f is **big-oh** of g").

Relative Rates of Convergence

Definition 7.34. Suppose $\lim_{h\to 0} G(h) = 0$.

(a) A quantity $F(h)$ is said to be **in little-oh** of $G(h)$ as $h \to 0$, if

$$\lim_{h\to 0} \frac{|F(h)|}{|G(h)|} = 0. \tag{C.7.14}$$

In this case, we denote $F(h) \in o(G(h))$ or $F(h) = o(G(h))$.

(b) A quantity $F(h)$ is said to be **in big-oh** of $G(h)$ as $h \to 0$, if there is a positive number K such that

$$\frac{|F(h)|}{|G(h)|} \le K, \text{ for } h \text{ sufficiently small.} \tag{C.7.15}$$

In this case, we denote $F(h) \in \mathcal{O}(G(h))$ or $F(h) = \mathcal{O}(G(h))$.

Example C.29. Answer the assertions by "Yes" or "No" for each of $\{f, g\}$-pairs under the given conditions

$\{f, g\}$	$f = o(g)$	$f = \mathcal{O}(g)$
$\{f(x) = \tan^{-1} x,\ g(x) = x^2 - 2x\}$, as $x \to 0$		
$\{f(x) = x^3 - \sin^3 x,\ g(x) = x^3\}$, as $x \to \infty$		
$\{f(h) = 1 - e^h,\ g(h) = h\}$, as $h \to 0$		
$\{f(n) = n^2 - n,\ g(n) = n^3\}$, as $n \to \infty$		

C.8. Techniques of Integration

§8.2. Integration by Parts

Formula 8.8. Integration by Parts:

$$\int u(x)v'(x)\,dx \;=\; u(x)v(x) - \int u'(x)v(x)\,dx, \tag{C.8.1}$$

whose differential version reads

$$\int u\,dv = uv - \int v\,du. \tag{C.8.2}$$

Remark 8.9. Integration by Parts: Alternative Form
Let v_1 is the antiderivative of v with $C = 0$. Then

$$\int u(x)v(x)\,dx \;=\; u(x)v_1(x) - \int u'(x)v_1(x)\,dx. \tag{C.8.3}$$

Remark 8.14. Tabular Integration by Parts:
While the aforementioned **recursive definition** is correct, it is often tedious to remember and implement. A much easier visual representation of this process is often taught to students and is called the **tabular method** or the **tic-tac-toe method**.

- **Let v_{k+1} be the antiderivative of v_k with $C = 0$**, where $v = v_0$. Then

$$\begin{aligned}
\int uv \;&=\; \mathbf{uv_1} - \int \mathbf{u'v_1} \;=\; uv_1 - \Big(u'v_2 - \int u''v_2\Big)\\
&=\; \mathbf{uv_1 - u'v_2} + \int \mathbf{u''v_2}\\
&=\; \mathbf{uv_1 - u'v_2 + u''v_3} - \int \mathbf{u'''v_3}\\
&=\; \mathbf{uv_1 - u'v_2 + u''v_3 - u'''v_4} + \int \mathbf{u''''v_4} \;=\; \cdots
\end{aligned} \tag{C.8.4}$$

Note: u is involved in the form of $(-1)^k \boldsymbol{u}^{(k)}$.

Example C.30. Evaluate the integrals.

(a) $\displaystyle\int x(\ln x)^2\,dx$ (b) $\displaystyle\int \cos\sqrt{x}\,dx$

Solution.

Example C.31. Find the area of the region bounded by the curve $y = \sqrt{x}\ln x$ and the x-axis from $x = 1$ to $x = 4$.

Solution.

Ans: $\displaystyle\frac{1}{3}u^3\ln u - \frac{1}{9}u^3\Big|_1^2$.

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int y f'(y)\,dy \qquad y = f^{-1}(x),\ x = f(y),\ dx = f'(y)dy \\ &= y f(y) - \int f(y)\,dy \\ &= x f^{-1}(x) - \int f(y)\,dy\end{aligned} \tag{C.8.5}$$

Note that $y = f^{-1}(x)$ (on the first line) can be viewed as a substitution.

Formula 8.18. Integrating Inverses of Functions:

$$\int f^{-1}(x)\,dx = x f^{-1}(x) - \int f(y)\,dy. \qquad y = f^{-1}(x) \tag{C.8.6}$$

Example C.32. Evaluate the integrals, using the formula in (C.8.6).

(a) $\displaystyle\int \arcsin x\,dx$ (b) $\displaystyle\int_1^e \ln x\,dx$

Solution.

Ans: (a) $x \arcsin x + \cos(\arcsin x) + C$. (b) 1.

§8.4. Trigonometric Substitutions

Table C.1: Trigonometric substitutions.

Expression	Substitution	Identity
$\sqrt{a^2+x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1+\tan^2\theta = \sec^2\theta$
$\sqrt{a^2-x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$	$1-\sin^2\theta = \cos^2\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta,\quad 0 \le \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta \le \pi$	$\sec^2\theta - 1 = \tan^2\theta$

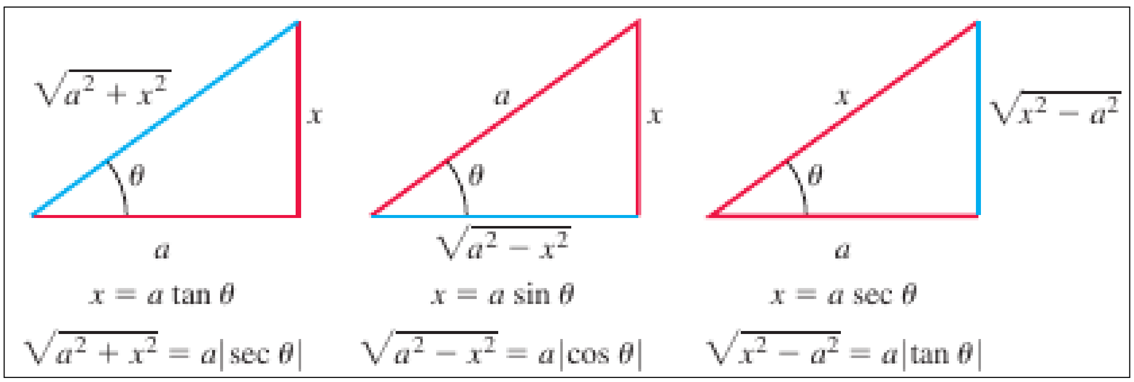


Figure C.8: Reference triangles for the three basic trigonometric substitutions.

Remark 8.29. With the substitution $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta. \tag{C.8.7}$$

The substitution requires

$$\theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \frac{\pi}{2}, & \text{if } \frac{x}{a} \ge 1, \\ \frac{\pi}{2} < \theta \le \pi, & \text{if } \frac{x}{a} \le -1. \end{cases} \tag{C.8.8}$$

For $\sec^{-1}$, see Figure 1.25, p. 54.

Example C.33. Evaluate the integrals.
You may have to use the formula $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$.

(a) $\displaystyle\int_0^2 \frac{x^2}{(x^2+4)^{3/2}}\, dx$

Ans: $\ln|\sec\theta + \tan\theta| - \sin\theta\Big|_0^{\pi/4}$

(b) $\displaystyle\int_0^{\sqrt{3}/2} \frac{x^3}{(1-x^2)^{5/2}}\, dx$

Ans: $4/3$.

§8.5. Integration of Rational Functions by Partial Fractions

Strategy 8.36. Method of Partial Fractions
When $f(x)/g(x)$ is Proper

1. **Linear factors of g**:
   Let $(x-r)$ be a factor of $g(x)$ with $(x-r)^m$ being its highest power. Then, to this factor, assign the sum of the m partial fractions:
   $$\frac{A_1}{x-r} + \frac{A_2}{(x-r)^2} + \cdots + \frac{A_m}{(x-r)^m}. \tag{C.8.9}$$

2. **Quadratic factors of g**:
   Let (x^2+px+q) be a factor of $g(x)$ with $(x^2+px+q)^n$ being its highest power. Then, to this factor, assign the sum of the n partial fractions:
   $$\frac{B_1x+C_1}{x^2+px+q} + \frac{B_2x+C_2}{(x^2+px+q)^2} + \cdots + \frac{B_nx+C_n}{(x^2+px+q)^n}. \tag{C.8.10}$$

3. **Combine all the partial fractions**:
   Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions.

4. **Determine the coefficients**:
   Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- **The degree of $f(x)$ must be less than the degree of $g(x)$.**
  - That is, the fraction must be **proper**.
  - If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
  $$\frac{f(x)}{g(x)} = Q(x) + \frac{r(x)}{g(x)}. \tag{C.8.11}$$
- **We must know the factors of $g(x)$.**

Example 8.38. Evaluate the integrals.

(a) $\displaystyle\int \frac{x-1}{(x+1)^3}\,dx$ (b) $\displaystyle\int \frac{10}{(x-1)(x^2+9)}\,dx$

Example 8.39. Evaluate the integrals.

(a) $\displaystyle\int \frac{x^2+2x+1}{(x^2+1)^2}\,dx$ (b) $\displaystyle\int \frac{\sqrt{x+1}}{x}\,dx$ (*Hint*: Let $u=\sqrt{x+1}$)

Ans: (a) $\ln(x^2+1)-\frac{1}{x^2+1}+C$. (b) $2[\sqrt{x+1}-\frac{1}{2}\ln(\sqrt{x+1}+1)+\frac{1}{2}\ln|\sqrt{x+1}-1|+C$.

§8.8. Improper Integrals

Expansion 8.49. Definite Integrals

- Up to now, we have required **definite integrals** to satisfy two properties:
  1. The **domain of integration** $[a, b]$ must be **finite**.
  2. The **range of the integrand** must be **finite** on this domain.
- In practice, we may encounter problems that fail to meet one or both of these conditions.
- In either case, the integrals are said to be **improper** and are calculated as limits.

Definition 8.51. Integrals with infinite intervals are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx. \tag{C.8.12}$$
2. If $f(x)$ is continuous on $(-\infty, b]$, then
$$\int_{-\infty}^b f(x)\,dx = \lim_{a\to-\infty} \int_a^b f(x)\,dx. \tag{C.8.13}$$
3. If $f(x)$ is continuous on $(-\infty, \infty)$, then
$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^c f(x)\,dx + \int_c^\infty f(x)\,dx, \tag{C.8.14}$$
where c is any real number.

In each case, if the limit exists and is finite, we say that the improper integral **converges** and that the limit is the value of the improper integral. If the limit fails to exist, the improper integral **diverges**.

The Integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$

Example 8.54. For what values of p does the integral $\displaystyle\int_1^\infty \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

Solution. Consider cases: ① $p = 1$, ② $p \neq 1$

Ans: **When $p > 1$, the improper integral converges to $1/(p-1)$.**

Example C.34. Determine whether the integral converges or diverges. Evaluate if it converges.

$$\int_1^\infty \frac{2}{x^2 + 2x}\, dx$$

Solution.

Type II: Discontinuous Integrands

Definition 8.59. Integrals of functions that **become infinite at a point** within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and discontinuous at a, then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx. \tag{C.8.15}$$
2. If $f(x)$ is continuous on $[a, b)$ and discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx. \tag{C.8.16}$$
3. If $f(x)$ is continuous on $[a, b]$ except at c, $a < c < b$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx. \tag{C.8.17}$$

In each case, if the limit exists and is finite, we say the improper integral **converges** and that the limit is the value of the improper integral. If the limit does not exist, the integral **diverges**.

The Integral $\displaystyle\int_0^1 \frac{dx}{x^p}$

Example C.35. For what values of p does the integral $\displaystyle\int_0^1 \frac{dx}{x^p}$ converge? When the integral does converge, what is its value?

Solution.

Ans: **When $p < 1$, the improper integral converges to $1/(1-p)$.**

Example C.36. Determine whether the integral converges or diverges. Evaluate if it converges.

(a) $\displaystyle\int_0^1 \frac{1+x}{\sqrt{x^2+2x}}\,dx$ (b) $\displaystyle\int_0^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$

Solution.

Ans: (a) $\sqrt{3}$. (b) π.

Example C.37. Determine whether the integral converges or diverges. Evaluate if it converges.

$$\int_0^4 \frac{1}{(x-2)^{2/3}}\,dx$$

Solution.

Tests for Convergence and Divergence

Remark 8.63. When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges.

- If the integral diverges, that's the end of the story.
- If it converges, we can use **numerical methods** to approximate its value.
- The principal tests for convergence or divergence are the **Direct Comparison Test** and the **Limit Comparison Test**.

Theorem 8.64. Direct Comparison Test
Let f and g be continuous on $[a, \infty)$ with $0 \le f(x) \le g(x)$ for all $x \ge a$. Then

1. If $\displaystyle\int_a^\infty g(x)\,dx$ converges, then $\displaystyle\int_a^\infty f(x)\,dx$ converges.
2. If $\displaystyle\int_a^\infty f(x)\,dx$ diverges, then $\displaystyle\int_a^\infty g(x)\,dx$ diverges.

Theorem 8.66. Limit Comparison Test
If the positive functions f and g are continuous on $[a, \infty)$, and if

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = L, \quad 0 < L < \infty, \tag{C.8.18}$$

then

$$\int_a^\infty f(x)\,dx \quad \text{and} \quad \int_a^\infty g(x)\,dx$$

either *both converge* or *both diverge*.

$\boxed{\textbf{Example}}$ **C.38.** Show that $\displaystyle\int_0^\infty \frac{1}{\sqrt{x+x^3}}\,dx$ converges.

Solution.

$\boxed{\textbf{Example}}$ **C.39.** Use integration, the Direct Comparison Test, or the Limit Comparison Test to investigate the convergence of

(a) $\displaystyle\int_1^2 \frac{1}{x\ln x}\,dx$ (b) $\displaystyle\int_e^\infty \frac{1}{\ln x}\,dx$ (c) $\displaystyle\int_4^\infty \frac{1}{x^{3/2}-x}\,dx$

Solution.

Ans: (b) diverge. (c) converge.

C.14. Partial Derivatives

§14.4. Tangent Planes & Linear Approximations

Definition C.40. *Given $z = f(x, y)$, the **linear (tangent plane) approximation** of f near (a, b) is*

$$L(x, y) \equiv z_0 + f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b), \tag{C.14.1}$$

where $z_0 = f(a, b)$.

Note: The equation of the **tangent plane** is

$$z - z_0 = f_x(a, b)\,(x - a) + f_y(a, b)\,(y - b),$$

or equivalently

$$-f_x(a, b)\,(x - a) - f_y(a, b)\,(y - b) + (z - z_0) = 0. \tag{C.14.2}$$

A level surface form of $z = f(x, y)$ can be rewritten as

$$F(x, y, z) = z - f(x, y) = 0;$$

its gradient becomes

$$\nabla F = \langle -f_x,\ -f_y,\ 1\rangle. \tag{C.14.3}$$

Preveal C.41. (§ 16.6. Parametric Surfaces and Their Areas): Let a surface S be formed by the graph of $z = f(x, y)$ and parametrized by $\mathbf{r}(x, y) = \langle x,\ y,\ f(x, y)\rangle$. Then, as in (16.63) on p. 588,

$$\mathbf{r}_x \times \mathbf{r}_y = \langle -f_x,\ -f_y,\ 1\rangle. \tag{C.14.4}$$

Theorem C.42. *If f_x and f_y **exist** near (a, b) and **continuous** at (a, b), then f is **differentiable** at (a, b).*

Definition C.43. *For a differentiable function $z = f(x, y)$, the **(total) differential** is*

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy, \tag{C.14.5}$$

where dx and dy represent the change in the x and y directions, respectively.

Example C.44. Find an equation for the tangent plane to the elliptic paraboloid $z = x^2 + 4y^2$ at the point $(1, 1, 5)$.
Solution.

Ans: $z - 5 = 2 \cdot (x - 1) + 8 \cdot (y - 1) \Leftrightarrow z = 2x + 8y - 5$.

Example C.45. Let $f(x, y) = \ln(x+1) + \cos(x/y)$. Explain why the function is differentiable at $(0, 2)$.

Example C.46. Use a linear approximation to estimate $f(2.2, 4.9)$, provided that $f(2, 5) = 6$, $f_x(2, 5) = 1$, and $f_y(2, 5) = -1$.
Solution.

Ans: 6.3

§14.6. Directional Derivatives and the Gradient Vector

Claim C.47. *For a unit vector* $\mathbf{u}$*, the* ***directional derivative*** *for a differential function* f *is*

$$D_{\mathbf{u}}f(x,y) = \nabla f(x,y)\cdot\mathbf{u}.$$

Theorem C.48. *Let* f *be differentiable. Then,*

$$\max_{\mathbf{u}} D_{\mathbf{u}}f = |\nabla f| \tag{C.14.6}$$

Note: The gradient vector ∇f is directing the fastest increasing direction.

Tangent Plane and Normal Line to a Level Surface

Suppose S is a surface given as $F(x,y,z) = k$ and $\mathbf{x}_0 = (x_0, y_0, z_0)$ is on S. Then the **tangent plane** to S at $\mathbf{x}_0$ is

$$\nabla F(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0) = F_x(\mathbf{x}_0)(x-x_0) + F_y(\mathbf{x}_0)(y-y_0) + F_z(\mathbf{x}_0)(z-z_0) = 0. \tag{C.14.7}$$

The **normal line** to S at $\mathbf{x}_0$ is

$$\frac{x-x_0}{F_x(\mathbf{x}_0)} = \frac{y-y_0}{F_y(\mathbf{x}_0)} = \frac{z-z_0}{F_z(\mathbf{x}_0)}. \tag{C.14.8}$$

Example C.49. Let $f(x, y) = x + \sin(xy)$.

1. Find the directional derivative of f at the point $(1, 0)$ in the direction given by the angle $\theta = \pi/3$.
2. In what direction does f have the maximum rate of change? What is the maximum rate of change?

Solution.

Ans: (a) $(1+\sqrt{3})/2$ (b) $\sqrt{2}$

Example C.50. Find the equations of the tangent plane and the normal line at $P(0, 0, 1)$ to $x + y + z = e^{xyz}$.

Solution.

Ans: (a) $x + y + z = 1$ (b) $x = y = z - 1$

§14.8. Method of Lagrange Multipliers

Consider the optimization problem

$$\left[\begin{array}{l}\max / \min f(\mathbf{x}) \\ \quad \text{subject to } \underline{g(\mathbf{x}) = c}\end{array}\right.$$

Strategy C.51. **(Method of Lagrange multipliers)**. For the max/min values of the optimization problem,

(a) Find all values of x, y, z, and λ such that

$$\boxed{\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \;\text{ and }\; g(x,y,z) = c.}$$

(b) Evaluate f at all these points, to find the maximum and minimum.

Example C.52. Use Lagrange Multipliers to prove that the rectangle with maximum area that has a give perimeter p is a square.

Example C.53. Find the maximum and minimum values of $f(x,y) = 2x^2 + (y-1)^2$ on the **circle** $x^2 + y^2 = 4$.

Solution. $\nabla f = \lambda \nabla g \Longrightarrow \begin{bmatrix} 4x \\ 2(y-1) \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. Therefore, $\begin{cases} 2x = \lambda\, x & ① \\ y - 1 = \lambda\, y & ② \\ x^2 + y^2 = 4 & ③ \end{cases}$

From ①, $x = 0$ or $\lambda = 2$.

Ans: min: $f(0,2) = 1$; max: $f(\pm\sqrt{3}, -1) = 10$

Example C.54. Find the maximum and minimum values of $f(x,y) = 2x^2 + (y-1)^2$ on the **disk** $x^2 + y^2 \leq 4$.

Solution. ***Hint***: You should check values at **critical points** as well.

Ans: min: $f(0,1) = 0$; $f(\pm\sqrt{3}, -1) = 10$

C.15. Multiple Integrals

§15.2. Double Integrals over General Regions

Multiple integrals can be computed with **iterated integral** where the given domain must be covered ***once-and-only-once***, without missing and without overlap. Furthermore, you should be able to change the order of integration properly.

Example C.55. (Problem 15.16). Find the volume of the solid that lies under the plane $z = 1 + 2y$ and above the region D in the xy-plane bounded by the line $y = 2x$ and the parabola $y = x^2$.

Solution. Try for both orders.

Ans: $28/5$

Example **C.56.** Evaluate the integral by reversing the order of integration:

$$\int_0^4 \int_{\sqrt{y}}^2 e^{x^3}\, dx\, dy$$

Solution.

Ans: $\frac{1}{3}(e^8 - 1)$

Self-study **C.57.** Sketch the region of integration and change the order of integration.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}} f(x,y)\, dx\, dy \qquad\qquad \int_0^{\ln 2}\int_{e^y}^{2} f(x,y)\, dx\, dy$$

Solution.

§15.7. Triple Integrals in Cylindrical Coordinates

Definition C.58. **(Definition 15.54).** *The conversion between the **Cylindrical Coordinates** and the Rectangular Coordinate system gives*

$(x, y, z)_R \leftarrow (r, \theta, z)_C$	$(r, \theta, z)_C \leftarrow (x, y, z)_R$
$x = r\cos\theta$	$r^2 = x^2 + y^2$
$y = r\sin\theta$	$\tan\theta = \dfrac{y}{x}$
$z = z$	$z = z$

(C.15.1)

Note: The triple integral with a Cylindrical Domain E can be carried out by first separating the domain like

$$E = D \times [u_1(x, y), u_2(x, y)], \quad \text{where } D \text{ is a polar region.}$$

Example C.59. Evaluate $\iiint_E y\, dV$, where E is the solid that lies between the cylinders $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$, above the xy-plane, and below the plane $z = y + 3$.

Solution.

Ans: 20π

$\boxed{\textbf{Self-study}}$ **C.60.** Use the cylindrical coordinates to find the volume of the solid E that is enclosed by the cone $z = \sqrt{x^2 + y^2}$ and the sphere $x^2+y^2+z^2 = 8$.

Solution.

Ans: $\frac{32}{3}\pi(\sqrt{2} - 1)$

§15.9. Change of Variables in Multiple Integrals

Definition C.61. *A change of variables is a* ***transformation*** $T : Q \to R$ *(from the* uv*-plane to the* xy*-plane),* $T(u,v) = (x,y)$*, where* x *and* y *are related to* u *and* v *by the equations*

$$x = g(u,v), \quad y = h(u,v). \qquad [\textbf{or}, \;\; \mathbf{r}(u,v) = \langle g(u,v), h(u,v)\rangle]$$

We usually take these transformations to be C^1***-Transformation****, meaning* g *and* h *have continuous first-order partial derivatives, and* ***one-to-one****.*

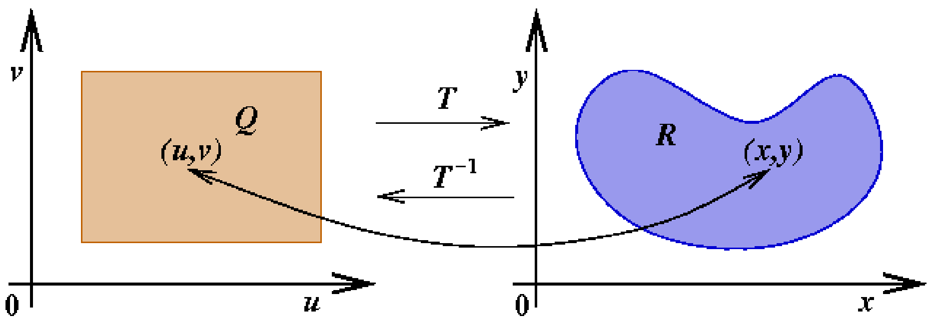


Figure C.9: Transformation: $R = T(Q)$, the **image** of T.

Definition C.62. *The* ***Jacobian of*** $T : x = g(u,v),\ y = h(u,v)$ *is*

$$\frac{\partial(x,y)}{\partial(u,v)} \overset{\text{def}}{=\!=} \det \begin{bmatrix} x_u & x_v \\ y_u & y_v \end{bmatrix} = x_u\, y_v - x_v\, y_u. \tag{C.15.2}$$

Claim C.63. *Suppose* $T : Q \to R$ *is an one-to-one* C^1 *transformation whose Jacobian is nonzero. Then*

$$\iint_R f(x,y)\, dA = \iint_Q f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv \tag{C.15.3}$$

Note: In **linear algebra**, an $n \times n$ matrix A is considered as a transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$. Furthermore its **determinant** can be viewed as a *volume scaling factor*. For details, see Section 3.1 of *Introduction to Linear Algebra*: https://skim.math.msstate.edu/LectureNotes/Linear_Algebra_LectureNote.pdf.

Example C.64. Make an appropriate change of variables to evaluate the integral

$$\iint\limits_R \sin(9x^2 + 4y^2)\, dA,$$

where R is the region in the **first quadrant** bounded by the ellipse $9x^2 + 4y^2 = 1$.

Solution.

Ans: $\frac{\pi(1-\cos 1)}{24}$

C.16. Integrals and Vector Fields

§16.2. Line Integrals

Definition C.65. *If f is defined on a smooth curve C given by*

$$\mathbf{r}(t) = \langle x(t),\, y(t)\rangle, \quad a \le t \le b, \tag{C.16.1}$$

*then **line integral of f along** C is*

$$\int_C f(x,y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i, \tag{C.16.2}$$

if this limit exists. Here $\Delta s_i = \sqrt{\Delta x_i^2 + \Delta y_i^2}$.

The line integral defined in (C.16.2) can be evaluated as

$$\begin{aligned} \textstyle\int_C f(x,y)\,ds &= \int_a^b f(x(t),y(t))\,\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\ &= \int_a^b f(x(t),y(t))\,|\mathbf{r}'(t)|\,dt. \end{aligned} \tag{C.16.3}$$

Definition C.66. *Let $\boldsymbol{F}$ be a continuous vector field defined on a smooth curve C given by $\mathbf{r}(t), a \le t \le b$. Then the **line integral of $\boldsymbol{F}$ along** C is*

$$\int_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \int_C \boldsymbol{F}\cdot\boldsymbol{T}\,ds \;=\; \int_a^b \boldsymbol{F}(\mathbf{r}(t))\cdot\mathbf{r}'(t)\,dt. \tag{C.16.4}$$

*We say that work is the line integral with respect to arc length of the **tangential component** of force.*

Example C.67. Evaluate the line integral $\displaystyle\int_C x\, e^{y^2-z^2}\, ds$, where C is the line segment from $(0,0,0)$ to $(2,-2,1)$.
Solution.

Ans: e^3-1

Example C.68. Find the work done by the vector field $\boldsymbol{F}(x,y)=\langle x, ye^x\rangle$ on the particle that moves along the parabola $x=y^2+1$ from $(1,0)$ to $(2,1)$.
Solution.

Ans: $\frac{3}{2}+\frac{e^2-e}{2}$

§16.3. The Fundamental Theorem for Line Integrals

Let C be a curve represented by

$$\mathbf{r}(t) = \langle x(t), y(t)\rangle \quad \textbf{or} \quad \mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle, \quad a \le t \le b.$$

Theorem C.69.

1. *Suppose that* $\boldsymbol{F}$ *is continuous, and is a* ***conservative vector field****; that is,* $\boldsymbol{F} = \nabla f$ *for some* f*. Then*
$$\int_C \boldsymbol{F} \cdot d\mathbf{r} = \int_C \nabla f \cdot d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)). \tag{C.16.5}$$
2. $\int_C \boldsymbol{F} \cdot d\mathbf{r}$ *is* ***independent of path*** *in* $D \iff \int_C \boldsymbol{F} \cdot d\mathbf{r} = 0$ *for every closed path in* D*.*
3. *Suppose* $\boldsymbol{F}$ *is a vector field that is continuous on an* ***open connected*** domain D*. If* $\int_C \boldsymbol{F} \cdot d\mathbf{r}$ *is* ***independent of path*** in D*, then* $\boldsymbol{F}$ is ***conservative*** *(i.e., there is* f *such that* $\boldsymbol{F} = \nabla f$*).*
4. *If* $\boldsymbol{F} = \langle P, Q\rangle$ *is conservative, where* P *and* Q *have continuous partial derivatives, then*
$$\frac{\partial Q}{\partial x} = \frac{\partial P}{\partial y}. \tag{C.16.6}$$
5. *When* D *is a simply-connected domain, the equality (C.16.6) implies conservativeness of* $\boldsymbol{F}$*.*

Roughly speaking: When $\boldsymbol{F} = \langle P, Q\rangle$ is smooth enough,

$$\boxed{\text{conservativeness}} \Leftrightarrow \boxed{\text{independence of path}} \Leftrightarrow \boxed{Q_x = P_y}$$

Example C.70. Find the work done by

$$\boldsymbol{F} = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\,\mathbf{j}$$

in moving an object from $A(1,1)$ to $B(2,4)$.
Solution. First, check if $\boldsymbol{F}$ is conservative: $Q_x = 3\sqrt{y},\ \ P_y = 2\cdot\frac{3}{2}y^{1/2} = 3\sqrt{y}.$

Ans: 30

Example C.71. Given $\boldsymbol{F}(x,y) = \langle e^y + y\cos x,\ xe^y + \sin x\rangle$,

(a) Find a potential.
(b) Evaluate $\int_C \boldsymbol{F}\cdot d\mathbf{r}$, where C is parameterized as

$$\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t\rangle,\ \ 0\le t\le \pi.$$

Solution.

Ans: (a) $f(x,y) = xe^y + y\sin x + K$ (b) $-e^{\pi} - 1$

§16.4. Green's Theorem

Theorem C.72. ***(Green's Theorem)***. *Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ has **continuous partial derivatives** on an open region including D, then*

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA. \tag{C.16.7}$$

The theorem gives the following formulas for the area of D:

$$A(D) = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx \tag{C.16.8}$$

Example C.73. Evaluate $\oint_C \boldsymbol{F}\cdot d\mathbf{r}$, where $\boldsymbol{F} = \langle e^{-x} + y^2, e^{-y} + x^2 + 2xy\rangle$ and C is the circle $x^2 + (y-1)^2 = 1$ oriented clockwise.

Solution. ***Hint***: Check the orientation of the curve.

Ans: 0

Example C.74. Use the identity (an example of Green's Theorem)

$$A(D) = \iint_D dA = \oint_{\partial D} x\,dy$$

to show that the area of D (the shaded region) is 6. You have to compute **the line integral for each of four line segments** of the boundary. For the slant line segment, in particular, you should introduce an appropriate parameterization for the line integral.

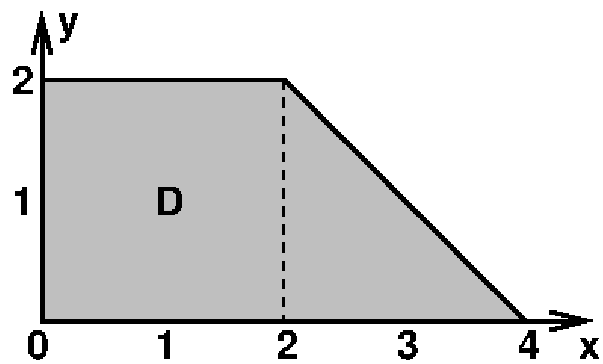


Figure C.10

§16.7. Surface Integrals

Suppose that the surface S has a parametric representation

$$\mathbf{r}(u,v) = \langle x(u,v), y(u,v), z(u,v)\rangle, \quad (u,v) \in D.$$

Then, surface integrals of **scalar functions** give

$$\iint_S f(x,y,z)\,dS = \iint_D f(\mathbf{r}(u,v))|r_u \times r_v|\,dA. \qquad \text{(C.16.9)}$$

Remark C.75.

- *$dS = |r_u \times r_v|\,dA$.*
- *For line integrals, $\int_C f(x,y,z)\,ds = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt$.*
- *When $z = g(x,y)$, $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$. Thus the formula (16.65) reads*

$$\iint_S f(x,y,z)\,dS = \iint_D f(x,y,g(x,y))\sqrt{g_x^2 + g_y^2 + 1}\,dA. \qquad \text{(C.16.10)}$$

Surface Integrals of Vector Fields. Let r be a parametric representation of S, from $D \subset \mathbb{R}^2$. The flux **across the surface S** can be measured by

$$\begin{aligned}\iint_S \boldsymbol{F}\cdot d\boldsymbol{S} &\overset{\text{def}}{=\!=} \iint_S \boldsymbol{F}\cdot\mathbf{n}\,dS \\ &= \iint_D \boldsymbol{F}(\mathbf{r})\cdot\left(\frac{\mathbf{r}_u\times\mathbf{r}_v}{|\mathbf{r}_u\times\mathbf{r}_v|}\right)|\mathbf{r}_u\times\mathbf{r}_v|\,dA \\ &= \iint_D \boldsymbol{F}(\mathbf{r})\cdot(\mathbf{r}_u\times\mathbf{r}_v)\,dA.\end{aligned} \qquad \text{(C.16.11)}$$

Note that $\boldsymbol{F}\cdot\mathrm{n}$ and $\boldsymbol{F}(\mathbf{r})\cdot(\mathbf{r}_u\times\mathbf{r}_v)$ are scalar functions.

Remark C.76. Line integrals of vector fields is defined to measure quantities **along the curve**. That is, for C parametrized by $\mathbf{r} : [a, b] \to C$,

$$\begin{aligned}\int_C \boldsymbol{F} \cdot d\mathbf{r} &\overset{\text{def}}{=\!=} \int_C \boldsymbol{F} \cdot \boldsymbol{T}\, ds \\ &= \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}\, |\mathbf{r}'(t)|\, dt = \int_a^b \boldsymbol{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.\end{aligned} \tag{C.16.12}$$

Surfaces defined by $z = g(x, y)$:

- A vector representation: $\mathbf{r}(x, y) = \langle x, y, g(x, y)\rangle$.
- Normal vector: $\mathbf{r}_x \times \mathbf{r}_y = \langle -g_x, -g_y, 1\rangle$.
- Thus, when $\boldsymbol{F} = \langle P,\, Q,\, R\rangle$,

$$\iint_S \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_D \boldsymbol{F} \cdot (\mathbf{r}_x \times \mathbf{r}_y)\, dA = \iint_D (-P\, g_x - Q\, g_y + R)\, dA. \tag{C.16.13}$$

Example C.77. Evaluate $\displaystyle\iint_S (x^2 + y^2 + z)\, dS$, where S is the surface whose side S_1 is given by the cylinder $x^2 + y^2 = 1$, whose bottom S_2 is the disk $x^2 + y^2 \le 1$ in the plane $z = 0$, and whose top S_3 is the disk $x^2 + y^2 \le 1$ in the plane $z = 1$.

Solution. ***Hint***: Use (C.58). ***Clue***: $S_1 : x = \cos\theta,\ y = \sin\theta,\ z = z;\ (\theta, z) \in D \equiv [0, 2\pi] \times [0, 1]$. Then $|\mathbf{r}_\theta \times \mathbf{r}_z| = 1$.

Ans: $3\pi + \frac{1}{2}\pi + \frac{3}{2}\pi = 5\pi$

Example C.78. Find the flux of $\boldsymbol{F} = \langle x, y, 1\rangle$ across a upward helicoid: $\mathbf{r}(u, v) = \langle u\cos v, u\sin v, v\rangle$, $0 \le u \le 2,\ \ 0 \le v \le \pi$.

Solution. ***Hint***: Use (C.16.11). ***Clue***: $\mathbf{r}_u \times \mathbf{r}_v = \langle \sin v, -\cos v, u\rangle$.

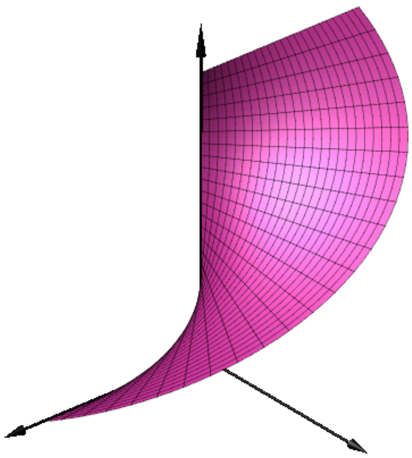


Figure C.11

Ans: 2π

Example C.79. Evaluate $\iint_S \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle x, y, z \rangle$ and S is the boundary of the solid region E enclosed by the paraboloid $z = 1 - x^2 - y^2$ and the plane $z = 0$.

Solution. ***Clue***: For S_1 (the upper part), use the formula in (C.16.13). For S_2 (the bottom: $z = 0$), you may try to get $\boldsymbol{F} \cdot \mathbf{n}$, where $\mathbf{n} = -\mathbf{k}$.

Ans: $\frac{3\pi}{2} + 0 = \frac{3\pi}{2}$

§16.8. Stokes's Theorem

Stokes's Theorem is a high-dimensional version of Green's Theorem studied in § 16.4.

Recall: **(Green's Theorem, p. 685).** Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q\rangle$ have **continuous partial derivatives** on an open region including D, then

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} \overset{\text{def}}{=\!=} \oint_C P\,dx + Q\,dy \;=\; \iint_D \Big(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\Big)\,dA = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot \mathbf{k}\,dA. \tag{C.16.14}$$

Theorem C.80. (Stokes's Theorem) *Let S be an oriented piecewise-smooth surface that is bounded by a simple, closed, piecewise-smooth curve C with positive orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ be a vector field whose components have **continuous partial derivatives** on an open region in $\mathbb{R}^3$ that contains S. Then*

$$\oint_C \boldsymbol{F}\cdot d\mathbf{r} = \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \tag{C.16.15}$$

Remark C.81.

- *See Figure 16.29(left) on p. 594, for an oriented surface of which the boundary has positive orientation.*
- ***Computation** of the surface integral: for $\mathbf{r} : D \to S$,*

$$\iint_S (\textbf{curl}\,\boldsymbol{F})\cdot d\boldsymbol{S} \overset{\text{def}}{=\!=} \iint_S (\textbf{curl}\,\boldsymbol{F})\cdot \mathbf{n}\,dS = \iint_D (\textbf{curl}\,\boldsymbol{F})\cdot (\mathbf{r}_u\times\mathbf{r}_v)\,dA. \tag{C.16.16}$$

- *Green's Theorem is a special case in which S is flat and lies on the xy-plane ($\mathbf{n} = \mathbf{k}$). Compare the last terms in (C.16.14) and (C.16.16).*

Try to solve problems in Section 16.8, once more.

§16.9. The Divergence Theorem

Theorem C.82. (Divergence Theorem) *Let E be a simple solid region and S be the boundary surface of E, given with positive (outward) orientation. Let $\boldsymbol{F} = \langle P, Q, R\rangle$ have* ***continuous partial derivatives*** *on an open region that contains E. Then*

$$\oiint_S \boldsymbol{F}\cdot d\boldsymbol{S} = \iiint_E \nabla\cdot\boldsymbol{F}\,dV. \tag{C.16.17}$$

Example C.83. Use the Divergence Theorem to evaluate the (total) flux $\iint_S \boldsymbol{F}\cdot dS$, where

$$\boldsymbol{F}(x,y,z) = (x+y^2+\cos z)\,\mathbf{i} + [\sin(\pi z) + xe^{-z}]\,\mathbf{j} + z\,\mathbf{k}$$

and S is a part of the cylinder $x^2+y^2=4$ that lies between $z=0$ and $z=1$.
Solution.

Ans: 8π

Example C.84. Use the Divergence Theorem to evaluate

$$\iint_S \left(x^2 + y \sin x + \frac{z^2}{2} \right) dS,$$

where S is the unit sphere $x^2 + y^2 + z^2 = 4$.

Solution. ***Clue***: What is n?

Ans: 32π

Appendix P
Projects

Finally we add projects.

Contents of Projects

P.1. Newton's Method for Finding Zeros and Extrema

In this project, you will implement Newton's method to find zeros and extrema of functions. Consider two functions defined on closed intervals.

$$\begin{aligned} f(x) &= 3x^4 - 4x^3 - 36x^2 - x + 20, \quad [-3, 4] \\ g(x) &= \frac{\sin(x^2 - \frac{\pi}{15})}{1 + x}, \quad [0, \pi] \end{aligned} \tag{P.1.1}$$

ID Function: Now, we will define the ID function so that each of you have your own function to work with.

Definition P.1. Let your ID be **123456789**. Then the **ID function** for you is defined as

$$I(x) = \mathbf{1} * x^4 - \mathbf{2} * x^3 + \mathbf{3} * x^2 - \mathbf{4} * x + \mathbf{5} * \sin(x) - \mathbf{6} * \cos\left(\frac{\mathbf{7} + \mathbf{8}}{2} * x\right) - \mathbf{9}. \tag{P.1.2}$$

For Matlab/Octave implementation, see **Example P.4** below.

What You Will Do

Problem P.2. Implement Newton's method

1. To find all zeros, for f, g, and I.
2. To find local/global extrema, for f and g.

Remark P.3. The **Newton's method** is introduced **to find zeros of functions**. Thus the method can be used **for finding extrema**: *Apply the Newton's method for* $f'(x) = 0$, which would result in **critical points**.

Details To Do

1. Implement Newton's method, Algorithm 4.48, p. 203.
(See `newton_diff_iteration.m` below.)
2. **Zeros**. For each of f, g, and I:
   (a) Plot the graph.
   (b) Guess an initial approximation for each zero.
   (c) Apply Newton's method to approximate the zero, in 10 decimal accuracy.
   (d) Add points to the plot, to see if the computation is correct.
3. **Local/Global Extrema**. For each of f and g:
   (a) Plot and recognize where local extrema are located.
   (b) Find the derivative of the function.
   (c) With an initial approximation, apply Newton's method to approximate the zero of the derivative, in 10 decimal accuracy.
   (d) Evaluate the function at the resulting point (a critical point).
   (e) Add points to the plot, to see if the computation is correct.
4. **Report your work and results, including code and plots.**

Example P.4. As an example, here we will consider **zero finding** for the ID function I. Let your ID be **674029185**.

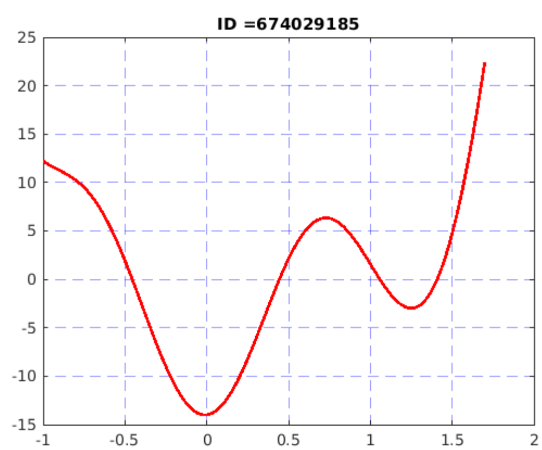


Figure P.1: Graph of the ID function $y = I(x)$, when ID = 674029185.

> You may have to implement a function for the Newton's method. The function `newton_diff_iteration.m` can be used for all other tasks.

project_ID_newton.m

```
ID = [6  7  4  0  2  9  1  8  5];  %if your ID = 674029185
I = @(x) (ID(1)*x.^4 -ID(2)*x.^3 +ID(3)*x.^2 -ID(4)*x ...
         +ID(5)*sin(x) -ID(6)*cos((ID(7)+ID(8))*x/2) - ID(9));

%% Here, you should plot to see where the zeros are

tol=1.e-10; itmax=100;
for x0 = [ -0.5 0.5 1 1.4]
    [x,it] = newton_diff_iteration(I,x0,tol,itmax);
    fprintf('x=%12.10f; it=%2d; I(x)=%g\n',x,it,I(x))
end
```

newton_diff_iteration.m

```
function [x,it] = newton_diff_iteration(f,x0,tol,itmax)

h = sqrt(tol);
x = x0;
for it = 1:itmax
    df = (f(x+h)-f(x-h))/(2*h);
    cor = f(x)/df;
    x = x - cor;
    if abs(cor)< tol, return; end
end
```

For the implementation:

Output

```
x=-0.4551039348; it= 4; I(x)=0
x= 0.4440727359; it= 5; I(x)=0
x= 1.0584112674; it= 4; I(x)= 2.66454e-15
x= 1.4090762364; it= 4; I(x)=-3.55271e-15
```

P.2. Numerical Integration

Note: **Definite Integrals and Numerical Integration**

- Often, it is **difficult (or, impossible) to find antiderivatives.**
  - When it is the case, we may approximate the **definite integral** using a computational method called numerical integration.
  - Numerical integration is a generalization of **Riemann sum**.
- **Numerical integration** can be performed as follows:

  (1) **Approximate the function f by a polynomial $p_n \in \mathbb{P}_n$**, and
  (2) **Integrate the polynomial over the prescribed interval**.
- That is, numerical integration is carried out as

$$\int_a^b f(x)\,dx \approx \int_a^b p_n(x)\,dx. \tag{P.2.1}$$

  - The **"polynomial approximation and integration"** can be applied for **each subinterval** or **each several subintervals.**

Theorem P.5. ***(Polynomial Interpolation Theorem):*** *If $x_0, x_1, x_2, \cdots, x_n$ are $(n+1)$ distinct real numbers, then for arbitrary values $y_0, y_1, y_2, \cdots, y_n$, there is a* **unique polynomial p_n** *of degree at most n such that*

$$p_n(x_i) = y_i \quad (0 \le i \le n). \tag{P.2.2}$$

P.2.1. Lagrange interpolating polynomials

Definition P.6. Let data points (x_i, y_i), $0 \le i \le n$ be given, where x_i are distinct. The nth-degree **Lagrange interpolating polynomial** p_n is a polynomial of the form

$$p_n(x) = y_0 L_{n,0}(x) + y_1 L_{n,1}(x) + \cdots + y_n L_{n,n}(x) = \sum_{i=0}^{n} y_k L_{n,i}(x), \tag{P.2.3}$$

where $L_{n,i}(x)$ are nth-degree polynomials such that

$$L_{n,i}(x_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases} \tag{P.2.4}$$

Polynomials $L_{n,i}(x)$ of such a property are called the **cardinal functions** and δ_{ij} is the **Kronecker delta**.

How to Construct the Basis Functions $L_{n,i}$

Example P.7. Find the Lagrange form of interpolating polynomial for the three-point table ($n = 2$)

x	x_0	x_1	x_2
y	y_0	y_1	y_2

Solution. We should find $L_{2,0}(x)$, $L_{2,1}(x)$, and $L_{2,2}(x)$ in $\mathbb{P}_2$.

① **Let's focus on $L_{2,0}(x) \in \mathbb{P}_2$**:

- It should satisfy $\boldsymbol{L_{2,0}(x_0) = 1}$ and $\boldsymbol{L_{2,0}(x_1) = L_{2,0}(x_2) = 0}$.
  - From $\boldsymbol{L_{2,0}(x_1) = L_{2,0}(x_2) = 0}$, the function $L_{2,0}$ must be of the form
$$L_{2,0}(x) = a\,(x - x_1)(x - x_2), \quad \text{for some } a. \tag{P.2.5}$$
  - From $\boldsymbol{L_{2,0}(x_0) = 1}$,
$$L_{2,0}(x_0) = a\,(x_0 - x_1)(x_0 - x_2) = 1 \;\Rightarrow\; a = \frac{1}{(x_0 - x_1)(x_0 - x_2)}. \tag{P.2.6}$$

- It follows from (P.2.5) and (P.2.6) that

$$L_{2,0}(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}. \tag{P.2.7}$$

② Similarly, we can formulate

$$L_{2,1}(x) = \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}, \quad L_{2,2}(x) = \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \tag{P.2.8}$$

Thus the Lagrange interpolating polynomial p_2 reads

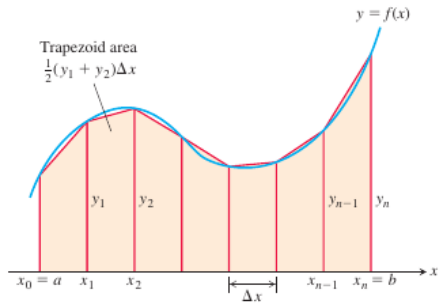
$$\begin{aligned} p_2(x) &= y_0 L_{2,0}(x) + y_1 L_{2,1}(x) + y_2 L_{2,2}(x) \\ &= y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ &\quad + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}. \end{aligned} \tag{P.2.9}$$

Note: In the following, we assume that the interval $[a,b]$ is partitioned into n uniform subintervals, **for simplicity**.

$$x_i = a + i \cdot h, \;\; i = 0, 1, 2, \cdots, n, \quad h = \frac{b-a}{n}. \tag{P.2.10}$$

P.2.2. Trapezoid Rule

The Trapezoid Rule approximates the function f with **a linear polynomial on each subinterval**.



Example **P.8.** Find $P_{0,1} \in \mathbb{P}_1$ which approximates f over $[x_0, x_1]$ and integrate it over the subinterval.

Solution. Let $f_i = f(x_i)$. The linear polynomial $P_{0,1} \in \mathbb{P}_1$ must read

$$P_{0,1}(x) = f_0 L_{1,0}(x) + f_1 L_{1,1}(x), \tag{P.2.11}$$

where

$$L_{1,0}(x) = \frac{x - x_1}{x_0 - x_1}, \quad L_{1,1}(x) = \frac{x - x_0}{x_1 - x_0}.$$

Integrating $P_{0,1}(x)$ over $[x_0, x_1]$ gives

$$\begin{aligned} \int_{x_0}^{x_1} P_{0,1}(x)\,dx &= f_0 \int_{x_0}^{x_1} L_{1,0}(x)\,dx + f_1 \int_{x_0}^{x_1} L_{1,1}(x)\,dx \\ &= f_0 \frac{x_1 - x_0}{2} + f_1 \frac{x_1 - x_0}{2} \\ &= (x_1 - x_0)\,\frac{f_0 + f_1}{2} = h\,\frac{f_0 + f_1}{2}. \quad \square \end{aligned} \tag{P.2.12}$$

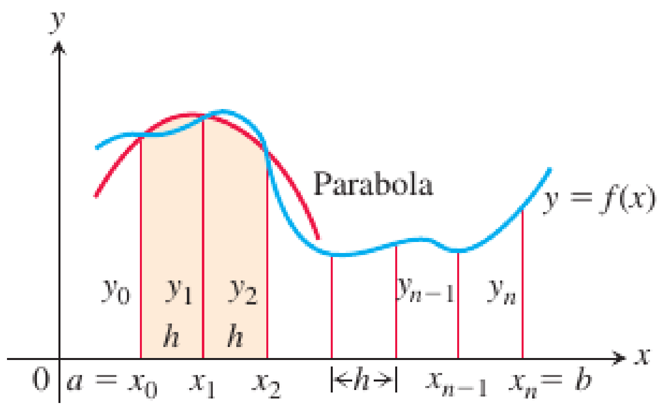
Algorithm **P.9.** **The Trapezoid Rule**

$$\begin{aligned} \int_a^b f(x)\,dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f(x)\,dx \\ &\approx \sum_{i=1}^n \int_{x_{i-1}}^{x_i} P_{i-1,i}(x)\,dx = \sum_{i=1}^n h \cdot \frac{f_{i-1} + f_i}{2} \\ &= h \cdot \left(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{n-1} + \frac{f_n}{2}\right), \end{aligned} \tag{P.2.13}$$

of which the error is known to be $\mathcal{O}(h^2)$.

P.2.3. Simpson's Rule

The Simpson's Rule approximates the function f with **a quadratic polynomial on each two subintervals**.



Example P.10. Find $P_{0,1,2} \in \mathbb{P}_2$ which approximates f over $[x_0, x_1] \cup [x_1, x_2]$ and integrate it over the subintervals.

Solution. It follows from (P.2.9) that

$$\begin{aligned} P_{0,1,2}(x) &= f_0 L_{2,0}(x) + f_1 L_{2,1}(x) + f_2 L_{2,2}(x) \\ &= f_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} \\ &\quad + f_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\ &= f_0 \frac{(x-x_1)(x-x_2)}{2h^2} + f_1 \frac{(x-x_0)(x-x_2)}{-h^2} \\ &\quad + f_2 \frac{(x-x_0)(x-x_1)}{2h^2}. \end{aligned} \tag{P.2.14}$$

Integrating $P_{0,1,2}(x)$ over $[x_0, x_2]$ gives

$$\int_{x_0}^{x_2} P_{0,1,2}(x)\,dx = \int_{x_0}^{x_0+2h} \sum_{i=0}^{2} f_i L_{2,i}(x)\,dx = \frac{2h}{6}(f_0 + 4f_1 + f_2). \tag{P.2.15}$$

Algorithm P.11. **The Simpson's Rule**. Let n be even.

$$\begin{aligned}\int_a^b f(x)\,dx &= \sum_{i=1}^{n/2}\int_{x_{2i-2}}^{x_{2i}} f(x)\,dx \approx \sum_{i=1}^{n/2}\int_{x_{2i-2}}^{x_{2i}} P_{2i-2,2i-1,2i}(x)\,dx \\ &= \frac{2h}{6}\cdot\Big[(f_0+4f_1+f_2)+(f_2+4f_3+f_4)+\cdots \\ &\qquad\qquad +(f_{n-2}+4f_{n-1}+f_n)\Big],\end{aligned} \tag{P.2.16}$$

of which the error is known to be $\mathcal{O}(h^4)$.

Matlab-code P.12. One of major techniques in computer programming is **looping**, which allows a set of selected operations to perform **repeatedly**.

- For example, let's use Simpson's Rule to approximate $\int_0^\pi \sin x\,dx$, with 10 and 20 subintervals ($n = 10, 20$). (The exact integral is **2**.)
- Below an example code is given.

simpson_rule.m

```
function simp = simpson_rule(f,a,b,n)
% function simp = simpson_rule(f,a,b,n)

%--------------------------
if mod(n,2)==1, error('n is odd'); end

h = (b-a)/n;
partition = linspace(a,b,n+1);
y = f(partition);

%--------------------------
simp = 0;
for i=2:2:n
   simp = simp + ( y(i-1)+4*y(i)+y(i+1) );
end
simp = simp * (2*h)/6;
```

sin_simpson.m

```
f = @(x) sin(x);
a = 0; b=pi; n = 10;

%--------------------------
simp = simpson_rule(f,a,b,n);

fprintf('f= '); disp(f)
fprintf('   [a, b]=[%g, %g]; n=%d; simpson=%.8g\n',a,b,n,simp)
```

Output

```
f= @(x) sin (x)
   [a, b]=[0, 3.14159]; n=10; simpson=2.0001095
   [a, b]=[0, 3.14159]; n=20; simpson=2.0000068
```

What You Will Do

1. Implement the two methods for numerical integration, saved to:
   - `trapezoid_rule.m`
   - `simpson_rule.m` (It is already given.)
2. Select functions and intervals:
   (1) $f_1(x) = \dfrac{4}{1+x^2}$, $[a,b]=[0,1]$ $\Rightarrow$ Exact integral: $I_1 = \pi$
   (2) $f_2(x) = 1/x$, $[a,b]=[1,e]$ $\Rightarrow$ Exact integral: $I_2 = 1$
   (3) Select another function f_3 and interval on your own for which the exact integral (I_3) is known.
3. For each function f_i and interval, perform numerical integration with
   (a) Trapezoid Rule, with $n = 4$ and 8 (The result are $T_{i,4}$ and $T_{i,8}$)
   (b) Simpson's Rule, with $n = 4$ and 8 (The result are $S_{i,4}$ and $S_{i,8}$)
4. Check the error and compare. That is,
   - $|(T_{i,4} - I_i)/(T_{i,8} - I_i)|$, for $i = 1,2,3$
   - $|(S_{i,4} - I_i)/(S_{i,8} - I_i)|$, for $i = 1,2,3$

   Are the error ratios around 4 or 16?
5. Now, compute $RT_i = \dfrac{1}{3}(4 * T_{i,8} - T_{i,4})$, for $i = 1,2,3$. Compare accuracy between R_i and $S_{i,8}$. Which one is more accurate?
6. Finally, compute $RS_i = \dfrac{1}{15}(16 * S_{i,8} - S_{i,4})$, for $i = 1,2,3$. Are they more accurate? What are the errors?

   Report your experiences, including your code and results.

Note: The technique used in 5 and 6 is called **Richardson extrapolation** which has been employed for higher-order accurate estimations in various applications.

P.3. The Euler's Number e

Recall: The **Euler's number** e is introduced in Section 1.5, p. 42, and mathematically formulated in Remark 3.67, p. 154, in Section 3.8.3.

Problem P.13. In this project, we will verify and apply the following equations

$$e = \lim_{n\to\infty}(1+\frac{1}{n})^n = \sum_{n=0}^{\infty}\frac{1}{n!} \tag{P.3.1a}$$

$$e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \tag{P.3.1b}$$

You will later learn (P.3.1b) as the **Taylor series** of e^x. Here you will verify it using integration techniques.

Recall: You most likely learned how to formulate **compound interest** from your high school. Compound interest is *interest on interest*.

- Let P be the principal balance and r the annual interest.
- When interest is allowed to *compound annually*, after x years the investment will be worth

$$A = P(1+r)^x. \tag{P.3.2}$$

- In the real world, interest on an investment is more often compounded than one per year.
- When interest is *compound n times per year*, after x years the investment will be worth

$$A = P\Big(1+\frac{r}{n}\Big)^{nx}. \tag{P.3.3}$$

Example P.14. For the computation of compound interest, we consider a special case:

$$P = 1 \text{ and } r = 1\,(= 100\%). \tag{P.3.4}$$

- When interest is *compound n times per year*, after a year ($x = 1$) the investment will be worth

$$S_n = \left(1 + \frac{1}{n}\right)^n. \tag{P.3.5}$$

- When interest is *compound continuously*, after a year ($x = 1$) the investment will be worth

$$S = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n = e. \tag{P.3.6}$$

Example P.15. As aforementioned, compound interest is interest on interest. So, when interest is *compound continuously*, the net value of the investment in a year ($x = 1$) can be computed as

$$1 + \underbrace{\int_0^1 1\,dt}_{\text{interest on } 1} + \underbrace{\int_0^1 \Big(\int_0^u 1\,dt\Big)\,du}_{\text{interest on } \int_0^u 1\,dt} + \underbrace{\int_0^1 \Big(\int_0^v \Big(\int_0^u 1\,dt\Big)\,du\Big)\,dv}_{\text{interest on } \int_0^v \left(\int_0^u 1\,dt\right)du} + \cdots, \tag{P.3.7}$$

where $0 \le v \le u \le 1$.

What You Will Do

1. Verify the second equality in (P.3.1a) by evaluating integrals in (P.3.7).
2. The term e^x can be considered as the net value of the investment in x years. Verify (P.3.1b) by appropriately replacing 1 with x in (P.3.7) and evaluating the resulting integrals.
3. Use a computer program to find n such that $(1+1/n)^n$ approximates e to **eight** decimal places.
4. Let $p_n(x)$ be the nth-order **Taylor polynomial** of e^x:

$$p_n(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}. \tag{P.3.8}$$

   Plot $p_n(x)$, for $n = 2, 4, 6$, together with e^x over the interval $[-3, 3]$.
5. Let $f(x) = e^x$.
   (a) Find $f(0)$, $f'(0)$, and $\displaystyle\int_{-\infty}^{0} f(x)\,dx \stackrel{\text{def}}{=\!=} \lim_{a\to-\infty}\int_a^0 f(x)\,dx$
   (b) Find $f(1)$, $f'(1)$, and $\displaystyle\int_{-\infty}^{1} f(x)\,dx \stackrel{\text{def}}{=\!=} \lim_{a\to-\infty}\int_a^1 f(x)\,dx$
   (c) What can you say about $f(b)$, $f'(b)$, and $\displaystyle\int_{-\infty}^{b} f(x)\,dx$, for arbitrary $b \in \mathbb{R}$? Interpret it geometrically.
6. Recall the Taylor series of e^x in (P.3.1b). Let $x = i\theta$, where $i = \sqrt{-1}$. Then

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{2!} + \frac{i^3\theta^3}{3!} + \frac{i^4\theta^4}{4!} + \frac{i^5\theta^5}{5!} + \frac{i^6\theta^6}{6!} + \cdots \tag{P.3.9}$$

   It is known that

$$e^{i\theta} = \cos\theta + i\sin\theta, \tag{P.3.10}$$

   which is called the **Euler's identity**. Particularly when $\theta = \pi$, the above equation reads

$$e^{i\pi} + 1 = 0, \tag{P.3.11}$$

   which is also called the Euler's identity.

   Find the Taylor series of $\sin\theta$ and $\cos\theta$.

Report your work and results, including code and plots.

P.4. Area Estimation of A Region: An Application of Green's Theorem

Region, saved via a Simple Closed Curve

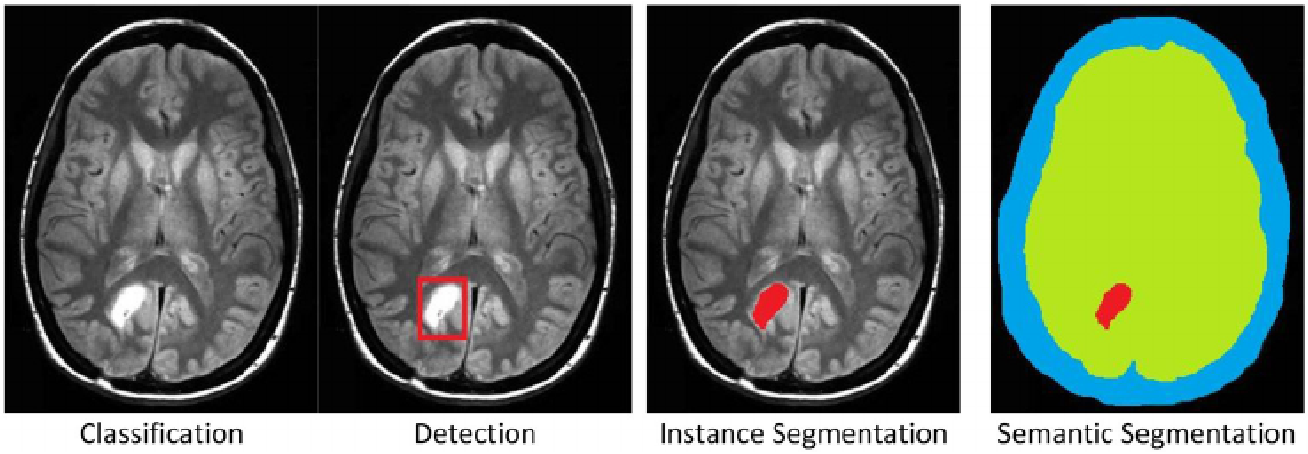


Figure P.2: A **brain tumor** example in an **MRI scan** (Bangert *et al.*, 2021)[1].

Green's Theorem

Let C be a positively oriented, piecewise-smooth, simple closed curve in the plane and D be the region bounded by C. If $\boldsymbol{F} = \langle P, Q \rangle$ has **continuous partial derivatives** on an open region including D, then

$$\oint_C P dx + Q dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA. \tag{P.4.1}$$

A choice:

$$\begin{cases} P(x, y) = 0 \\ Q(x, y) = x \end{cases} \implies \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 1 \implies A(D) = \iint_D 1\, dA = \oint_C x\, dy$$

Project Objectives

- Derive a numerical method for the line integral $\oint_C x\, dy$.
- Implement a code for the computation of areas.

Problem P.16. It is common in reality that a **region of interest** is saved by *a sequence of points*: For some $n > 0$,

$$(x_0, y_0),\ (x_1, y_1),\ \cdots,\ (x_n, y_n), \qquad (x_n, y_n) = (x_0, y_0). \tag{P.4.2}$$

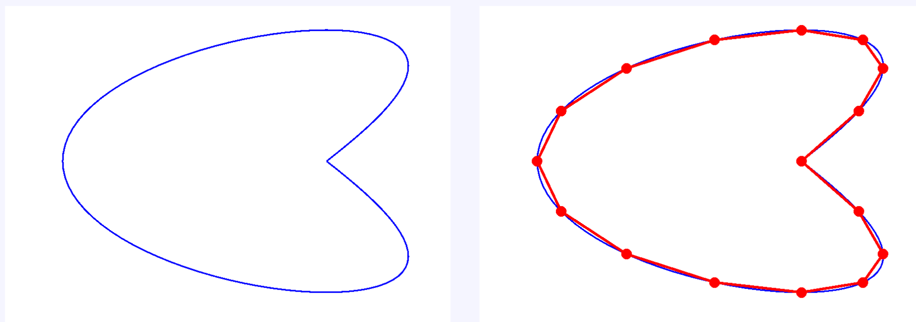


Figure P.3: A region and its approximation.

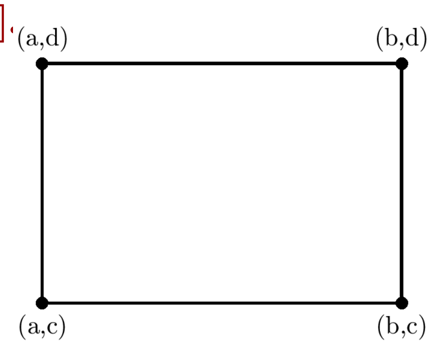
Here the question is:

> *If a sequence of points (P.4.2) represents a region, how can we compute its area accurately?*

Derivation of Computational Formulas

Example P.17. Let's begin with a very simple example.

(a) **Find the area of the rectangle $[a, b] \times [c, d]$.**



Solution. We know the area = $(b - a) \cdot (d - c)$. It can be rewritten as

$$b \cdot (d - c) - a \cdot (d - c) = b \cdot (d - c) + a \cdot (c - d)$$

from which we may guess that

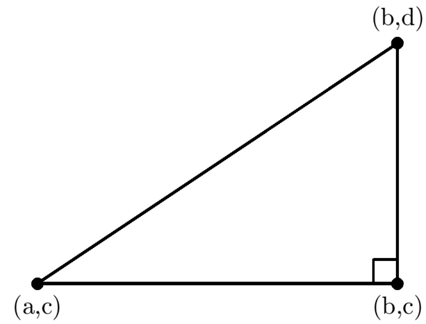
$$\text{Area} = \sum_i x_i^* \cdot \Delta y_i, \tag{P.4.3}$$

where the sum is carried out over line segments L_i and x_i^* denotes the mid value of x on L_i.

(b) **Find the area of the triangle**.

Solution. We know the area = $\frac{1}{2}(b-a)\cdot(d-c)$.
Now, let's try to find the area using the formula (P.4.3):

$$\text{Area} = \sum_i x_i^* \cdot \Delta y_i.$$



Let L_1, L_2, L_3 be the bottom side, vertical side, and the hypotenuse, respectively.
Then

$$\begin{aligned}\text{Area} &= \frac{a+b}{2}\cdot(c-c)+\frac{b+b}{2}\cdot(d-c)+\frac{b+a}{2}\cdot(c-d)\\ &= 0 \quad + b\cdot(d-c) \quad + \frac{b+a}{2}\cdot(c-d)\\ &= \left(b-\frac{b+a}{2}\right)\cdot(d-c) = \frac{1}{2}(b-a)\cdot(d-c).\end{aligned}$$

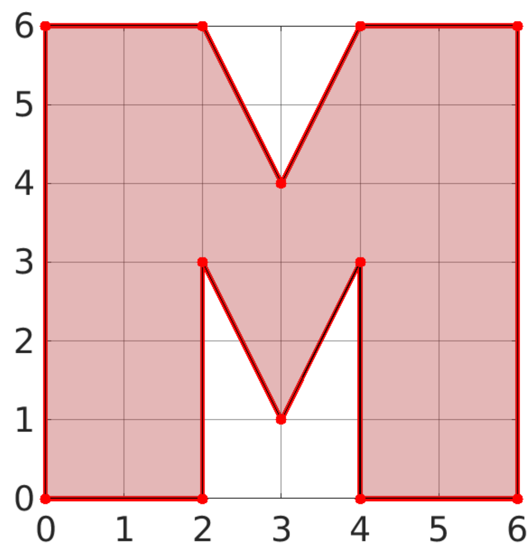
Okay. The formula is correct!

Note: Horizontal line segments makes **no** contribution to the area.

(c) **Let's verify the formula once more.**

The area of the M-shaped is 30. Let's collect only nonzero values:

$$\begin{aligned}&2\cdot 3 - 2.5\cdot 2 + 3.5\cdot 2 - 4\cdot 3\\ &\quad +6\cdot 6\\ &\quad -3.5\cdot 2 + 2.5\cdot 2\\ &= 6-5+7-12\\ &\quad +36\\ &\quad -7+5\\ &= 30\end{aligned}$$



Again, the formula is correct!!

Summary P.18. The above work can be summarized as follows.

- Let a region D be represented as a sequence of points

$$(x_0, y_0),\ (x_1, y_1),\ \cdots,\ (x_n, y_n), \qquad (x_n, y_n) = (x_0, y_0). \tag{P.4.4}$$

- Let L_i be the i-th line segment connecting (x_{i-1}, y_{i-1}) and (x_i, y_i), $n = 1, 2, \cdots, n$. Then the area of D can be computed using the formula

$$\text{Area}(D) = \sum_{i=1}^{n} x_i^* \cdot \Delta y_i, \tag{P.4.5}$$

where

$$x_i^* = \frac{x_{i-1} + x_i}{2}, \quad \Delta y_i = y_i - y_{i-1}.$$

Accuracy Analysis for the Numerical Method

Claim P.19. The numerical formula (P.4.5) is **exact**, when the closed curve C consists of line segments.

Proof. Lect C_1 be the first line segment, connecting from $P_0(x_0, y_0)$ to $P_1(x_1, y_1)$.

- **Numerical Formula**. Along C_1, the formula (P.4.5) results in

$$\frac{x_0 + x_1}{2} \cdot (y_1 - y_0). \tag{P.4.6}$$

- **Exact Evaluation**. The line segment C_1 can be parametrized as

$$\begin{aligned} \mathbf{r}_1(t) &= (1-t)P_0 + tP_1 \ = \ P_0 + t(P_1 - P0) \\ &= \langle x_0 + t(x_1 - x_0),\ y_0 + t(y_1 - y_0) \rangle, \quad 0 \le t \le 1. \end{aligned} \tag{P.4.7}$$

  Thus

$$\begin{aligned} \int_{C_1} x\,dy &= \int_0^1 x(t)y'(t)\,dt \ = \ \int_0^1 [x_0 + t(x_1 - x_0)](y_1 - y_0)\,dt \\ &= (y_1 - y_0)\Big[x_0 t + \frac{t^2}{2}(x_1 - x_0)\Big]_0^1 \ = \ (y_1 - y_0) \cdot \frac{x_0 + x_1}{2}, \end{aligned} \tag{P.4.8}$$

  which is the same as (P.4.6).

The above argument can be applied for each of line segments. □

Example P.20. We will generate a dataset, plot it, and measure its area.

(a) Generate a dataset that represents an ellipse, e.g., $\dfrac{x^2}{4} + y^2 = 1$.
For $i = 0, 1, 2, \cdots, n$,

$$(x_i, y_i) = (2\cos\theta_i, \sin\theta_i), \quad \theta_i = i \cdot \frac{2\pi}{n}. \tag{P.4.9}$$

Note that $(x_n, y_n) = (x_0, y_0)$.

(b) Analyze accuracy improvement of the area as n grows. The larger n you choose, the more accurately the data would represent the region.

Solution. You should implement the function area_closed_curve.

Ellipse_Approximate_Area.m

```
close all

a=2; b=1; true_area = a*b*pi;
Partition = [10,20,40,80];
for n = Partition
    %%---- Data generation -----------------
    theta = linspace(0,2*pi,n+1)';   % a column vector
    data  = [a*cos(theta),b*sin(theta)];

    %%---- Area computation ----------------
    area = area_closed_curve(data);
    fprintf('n = %3d; estimation = %.10f, rel-error = %.10f\n', ...
        size(data,1)-1,area, abs(true_area-area)/true_area);

    %%---- Plot & Save ---------------------
    figure, plot(data(:,1),data(:,2),'r-','linewidth',2);
    daspect([1 1 1]); axis tight;
    xlim([-a a]), ylim([-b b]);
    title(['Approximate Ellipse: n=' int2str(n)])
    image_name = strcat('ellipse-n=',int2str(n),'.png');
    exportgraphics(gca,image_name,'Resolution',100);
end
```

Output

```
n =  10; estimation = 5.8778525229, rel-error = 0.0645107162
n =  20; estimation = 6.1803398875, rel-error = 0.0163683569
n =  40; estimation = 6.2573786016, rel-error = 0.0041072648
n =  80; estimation = 6.2767276582, rel-error = 0.0010277668
```

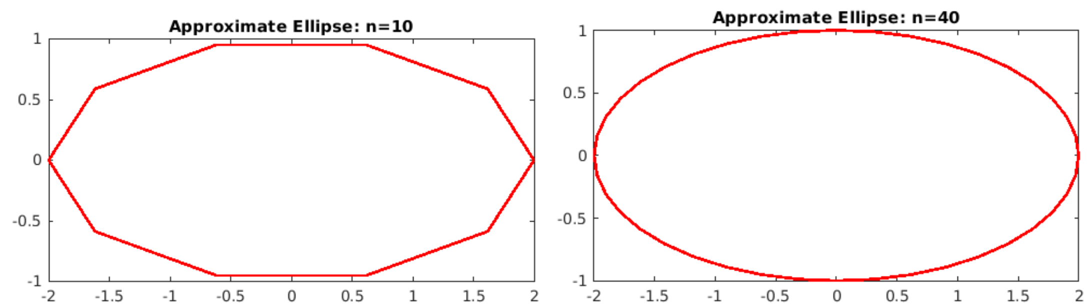


Figure P.4: Approximate Ellipse: $n = 10, 40$.

What You Will Do

1. **A given dataset**: First, download a dataset saved in `heart-data.txt`: https://skim.math.msstate.edu/LectureNotes/heart-data.txt
   - Draw a figure for it.
   - Use the formula (P.4.5) to find the area.

2. **Repeat Part 1 for another dataset**: For a dataset, you may choose one of the following.
   - Search the internet
   - Generate a dataset for, e.g. a **cardioid**:

$$\begin{cases} x = 2a\cos\theta \cdot (1-\cos\theta) \\ y = 2a\sin\theta \cdot (1-\cos\theta) \end{cases} \quad 0 \le \theta \le 2\pi, \tag{P.4.10}$$

   of which the area is $6\pi a^2$.
     - The last point must be the same as the first one: $(x_n, y_n) = (x_0, y_0)$
     - The larger n you choose, the more accurately the data would represent the region.
     - Analyze accuracy improvement of the area as n grows.

Your report should include your code, outputs, and a concluding remark.

P.5. Linear and Quadratic Approximations

This project is designed for students to experience computer algebra, while solving some calculus problems with computer coding. Although it includes examples written in Maple only, students can finish the project using Maple, Mathematica, or MathCad.

Getting familiar with Computer Algebra CAS

For a smooth function of one variable, f, its **Taylor series** about a is given as

$$f(x) \sim \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots. \quad \text{(P.5.1)}$$

As with any convergent series, $f(x)$ is the limit of the sequence of partial sums. That is,

$$f(x) = \lim_{n\to\infty} T_n(x), \quad \text{(P.5.2)}$$

where $T_n(x)$ is called the **Taylor polynomial of degree n**:

$$T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Example P.21. Let

$$f(x) = \arctan(x) - \frac{1}{3}. \quad \text{(P.5.3)}$$

Then, when it is expanded about $a = 1/2$, $T_n(x)$ can be obtained using Maple:

```
a  := 1/2:
Tn := x-> convert(taylor(f(x),x=a,n+1),polynom):
```

See Figure P.5[1] (p. 717). For the function in (P.5.3),

$$\begin{aligned} T_1(x) &= \arctan\left(\frac{1}{2}\right) - \frac{1}{3} + \frac{4}{5}\left(x - \frac{1}{2}\right), \\ T_2(x) &= \arctan\left(\frac{1}{2}\right) - \frac{1}{3} + \frac{4}{5}\left(x - \frac{1}{2}\right) - \frac{8}{25}\left(x - \frac{1}{2}\right)^2. \end{aligned} \quad \text{(P.5.4)}$$

[1] In Maple, `taylor(f(x),x=a,n+1)` returns a polynomial of $(n+1)$ terms plus the remainder, $T_n(x) + \mathcal{O}((x-a)^{n+1})$; while the command `convert(g,polynom)` converts `g` into a polynomial form, which is $T_n(x)$. In Mathematica, `Series[f[x],x,a,n]` produces the same result as for `taylor(f(x),x=a,n+1)` in Maple. Now, the Mathematica-command `Normal` can be used to convert the result into normal expressions of polynomials.

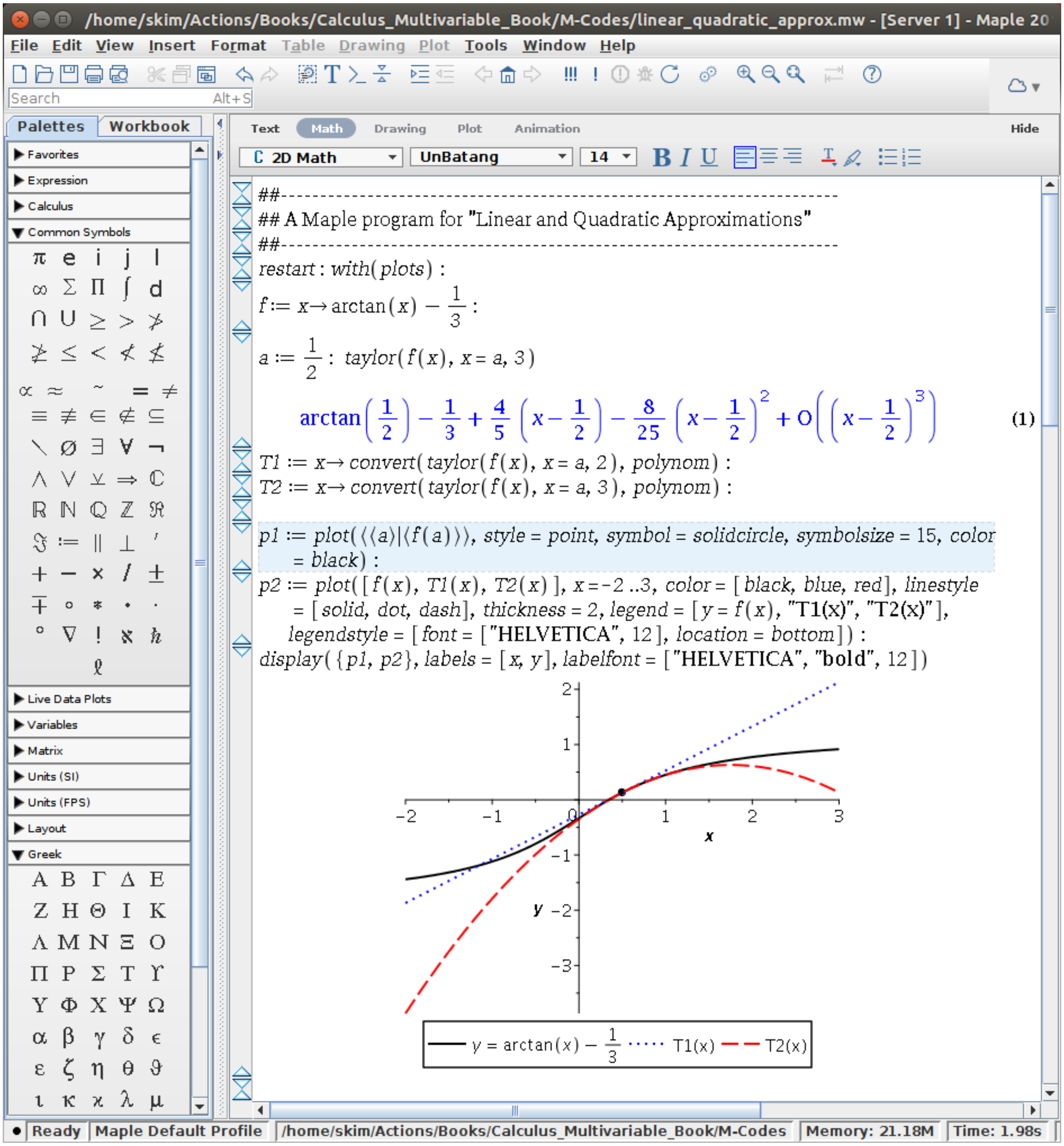


Figure P.5: Screen-shot of Maple window, which plots linear and quadratic approximations of $f(x) = \arctan(x) - \frac{1}{3}$ about $a = 1/2$.

Maple: 3D Plots

First load the `plots` package, along with other frequently used packages, using the entry:

```
with(plots): with(plottools):
with(VectorCalculus): with(Student[MultivariateCalculus]):
```

1. Plot $z = f(x, y)$ in **Cartesian coordinates**, using

$$plot3d(\ f(x,y), x = a..b, y = c..d, options)$$

Consider the options
   (a) *style* = *patchcontour* Puts contour curves on the surface.
   (b) *axes* = *boxed* Puts the axes on the edges of a box enclosing the surface.
   (c) *scaling* = *constrained* Makes the scale on the three axes the same.
   (d) *orientation* =$[40, 70]$ Orients the viewpoint so it is closer to what you see in your text.

2. Plot $F(x, y, z) = 0$ in **Cartesian coordinates**, using

$$implicitplot3d(\ F(x,y,z) = 0, x = a..b, y = c..d, z = s..t, options)$$

Consider the options listed above along with the following.
   (a) *grid* = $[m, n, k]$ Where m, n, k are positive integers, try $[30, 30, 30]$ for example. This plots 30 points in each direction for a smoother surface.
   (b) *axes* = *framed* Puts axes along the edges of a frame around the plot.
   (c) *orientation* = $[-50, 60]$ Another nice viewing angle.

3. Plot $r = f(\theta, z)$ in **cylindrical coordinates**, using

$$plot3d(\ f(\theta,z), \theta = a..b, z = s..t, coords = cylindrical, options)$$

To plot $z = g(r, \theta)$, use

$$plot3d(\ [r,\theta,g(r,\theta)], r = a..b, \theta = \alpha..\beta, coords = cylindrical, options)$$

Options are more or less the same as the above.

4. Plot $\rho = f(\theta, \phi)$ in **spherical coordinates**, using

$$plot3d(\ f(\theta,\phi), \theta = \alpha..\beta, \phi = \gamma..\delta, coords = spherical, options)$$

5. Implicit plots can also be made in **cylindrical or spherical coordinates**. For example, to plot the equation $r^2 + 2z^2 = r\cos\theta$ in cylindrical coordinates, use

$$implicitplot3d(\ r^2 + 2\,z^2 = r\cos(\theta), r = a..b, \theta = \alpha..\beta, z = s..t, coords = cylindrical, options)$$

6. **(Contour plots in** $2D$**)**. For $z = f(x, y)$, use

$$contourplot(f(x,y), x = a..b, y = c..d, options)$$

P.5.1. Newton's method

As one can see from Figure P.5, the first-degree Taylor series $T_1(x)$ is the tangent line to the curve $y = f(x)$ at the point $(a, f(a))$. One of popular applications exploiting the tangent line is **Newton's method** for the problem of root-finding.

$$\boxed{\text{Given a differentiable function } f(x)\text{, find } r \text{ such that } f(r) = 0,} \tag{P.5.5}$$

where r is an x-intercept of the curve $y = f(x)$.

The idea behind Newton's method:

- The tangent line is close to the curve and so its x-intercept must be close to the x-intercept of the curve.
- Let x_0 be the initial approximation close to r. Then, the tangent line at $(x_0, f(x_0))$ reads
$$L(x) = f'(x_0)(x - x_0) + f(x_0). \tag{P.5.6}$$
Let x_1 be the x-intercept of $y = L(x)$. Then,
$$0 = f'(x_0)(x_1 - x_0) + f(x_0).$$
Solving the above equation for x_1 becomes
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}, \tag{P.5.7}$$
which hopefully is a better approximation for the root r.
- Repeat the above till the convergence.

Algorithm P.22. ***(Newton's method for solving*** $\mathrm{f(x) = 0}$***).*** *For x_0 chosen close to a root r, compute $\{x_n\}$ repeatedly satisfying*

$$\boxed{x_n = x_{n-1} - \frac{f(x_{n-1})}{f'(x_{n-1})}, \quad n \geq 1.} \tag{P.5.8}$$

Problem P.23. Consider the function $f(x) = \arctan(x) - \frac{1}{3}$ in (P.5.3).

1. Implement a code for Newton's method to approximate a root of $f(x) = 0$.
(You can use Maple, Mathematica, or MathCad.)
2. Run a few iterations, starting from $x_0 = 0.5$, and measure how the error decreases as the iteration count grows.
(Note that the exact root $r = \tan(1/3) \approx 0.346253549510575491 03.$)

P.5.2. Estimation of critical points

The second part of the project involves a min-max analysis of a function in (x, y) that is based on each student's ID number, so that each student has his/her own function to work with. If a student's ID number is 123-45-6789, then he/she will study the behavior of the function

$$f(x, y) = 1 * \sin(x - 2) + 3 * \cos(y - 4) + 5 * x^2 - 6 * xy + 7 * y^2 - 8 * x + 9 * y, \quad \text{(P.5.9)}$$

where the alternating signs are used to create a little more "action". We will call such a function the **ID function**.

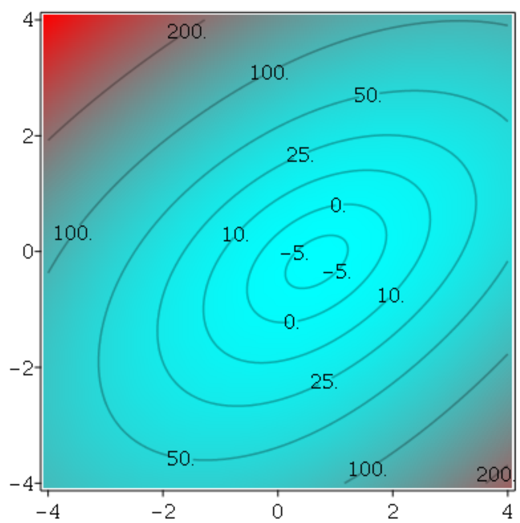


Figure P.6: Contour plot of $f(x, y)$ in (P.5.9).

Problem P.24. Create your ID function. Then,

1. Include a variety of surface plots with different views and contour plots with different windows to provide a good picture of the behavior of your ID function.[2]
2. Label the figures and refer to them in your write-up, as you discuss the kinds of critical points you observe. (If you have no or one critical point, change the signs and/or shuffle the digits in your ID function to get more action.)
3. Zoom in sufficiently so that you can estimate the coordinates of each of the critical points.

[2]In Maple, you can use the commands `plot3d` and `countourplot`. In Mathematica, `Plot3D` and `CountourPlot` are available.

P.5.3. Quadratic approximations

We have discussed the **linear approximation** (or, **linearization**) of a function f of two variables at a point (a,b):

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b), \tag{P.5.10}$$

which is also called the **first-degree Taylor polynomial** of f at (a,b). If f has continuous second-order partial derivatives at (a,b), then the **second-degree Taylor polynomial** of f at (a,b) is

$$\begin{aligned} Q(x,y) \;=\;& f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+\frac{1}{2}f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \frac{1}{2}f_{yy}(a,b)(y-b)^2, \end{aligned} \tag{P.5.11}$$

and the approximation $f(x,y) \approx Q(x,y)$ is called the **quadratic approximation** of f at (a,b).

Problem P.25. Answer the following.

1. Verify that the quadratic approximation Q has the same first- and second-order partial derivatives as f at (a,b). (This is the only portion of the project that you can finish without using computer implementation.) ***Hint***: The partial derivatives evaluated at (a,b), appeared in Q, are all constant.
2. Use computer algebra to find the first- and second-degree Taylor polynomials L and Q for your ID function f at a critical point $C(x_0, y_0)$ that you estimated from Problem P.24.
3. Compare the values of f, L, and Q at $(x_0 + 0.1, y_0 - 0.1)$.
4. Graph f, L, and Q; comment on how well L and Q approximate f.

Report. Submit hard copies of your experiences.

- Solve Problems P.23, P.24, and P.25, using computer programming.
- Make hard copies of your work, and **collect them in order**.
- Attach a "summary" or "conclusion" page at the beginning of report.

You may work in a small group; however, you must report individually.

P.6. The Volume of the Unit Ball in n-Dimensions

In this project, we will find formulas for the **volume of the unit ball** in n-dimensions (nD). From your high school, you learned volumes of unit balls for $n = 1, 2, 3$.

n	B_n	V_n
1	$\{x \mid x^2 \le 1\} = [-1, 1]$	2
2	$\{(x, y) \mid x^2 + y^2 \le 1\}$	π
3	$\{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$	$4\pi/3$

(P.6.1)

Define the 4D **unit ball (hypersphere)** as

$$B_4 = \{(x, y, z, w) \mid x^2 + y^2 + z^2 + w^2 \le 1\}. \tag{P.6.2}$$

Before finding its volume, V_4, let's try to verify $V_3 = \dfrac{4\pi}{3}$ by using a **specific integration technique**.

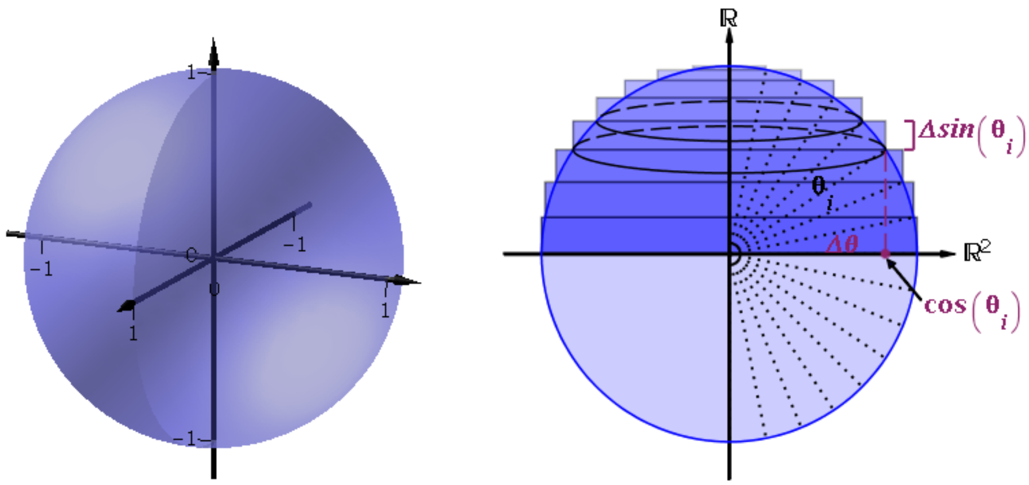


Figure P.7: B_3 and its projection to $\mathbb{R}^2 \times \mathbb{R}$: the volume V_3 approximates the sum of the volume of circular slices having radius $\cos\theta_i$ and thickness $\Delta \sin\theta_i := \sin\theta_{i+1} - \sin\theta_i = \dfrac{\sin\theta_{i+1} - \sin\theta_i}{\Delta\theta}\,\Delta\theta \approx \cos\theta_i\,\Delta\theta$.

The computation of V_3: We first partition B_3 into horizontal circular slices. Let, for $k > 0$,

$$\Delta\theta = \frac{\pi}{2} \cdot \frac{1}{k} \text{ and } \theta_i = i\Delta\theta, \quad i = 0, 1, \cdots, k. \tag{P.6.3}$$

One can see from Figure P.7 that the volume V_3 approximates the sum of the volume of **circular slices**. The i-th circular slice S_i has radius $\cos\theta_i$; its area is

$$A(S_i) = V_2 \cdot \cos^2\theta_i = \pi\, \cos^2\theta_i. \tag{P.6.4}$$

Since S_i has thickness $\Delta\sin\theta_i = \sin\theta_{i+1} - \sin\theta_i$, we have

$$V_3 \approx 2\sum_{i=0}^{k-1}(\pi\cos^2\theta_i)\,\Delta\sin\theta_i. \tag{P.6.5}$$

Therefore,

$$\begin{aligned} V_3 &= \lim_{k\to\infty} 2\sum_{i=0}^{k-1}(\pi\cos^2\theta_i)\,\Delta\sin\theta_i \\ &= 2\pi\int_0^{\pi/2}\cos^2\theta\, d(sin\theta) = 2\pi\int_0^{\pi/2}\cos^3\theta\, d\theta = 2\pi\cdot\frac{2}{3}. \end{aligned} \tag{P.6.6}$$

Note: Equation (P.6.6) can be rewritten as

$$V_3 = 2\,V_2\int_0^{\pi/2}\cos^3\theta\, d\theta. \tag{P.6.7}$$

The computation of V_4: We are now ready for it! First image B_4 and its projection to $\mathbb{R}^3 \times \mathbb{R}$. With the same partitioning of the last dimension, the i-th horizontal piece S_i now becomes a **spherical slice**, rather than a circular slice, but having the same radius $\cos\theta_i$ and thickness $\Delta\sin\theta_i$. Thus, the **volume** of the i-th spherical slice reads

$$V(S_i) = V_3\, \cos^3\theta_i \cdot \Delta\sin\theta_i \approx V_3\, \cos^4\theta_i\,\Delta\theta. \tag{P.6.8}$$

Recall that $\Delta\sin\theta_i = \sin\theta_{i+1} - \sin\theta_i \approx \cos\theta_i\,\Delta\theta$. By summing up for $i = 0, 1, \cdots, k-1$, and multiplying the result by 2 (due to symmetry), we have

$$V_4 \approx 2\,V_3\sum_{i=0}^{k-1}\cos^4\theta_i\,\Delta\theta. \tag{P.6.9}$$

Problem P.26.

1. Complete a formula for V_4, by applying $k \to \infty$ to (P.6.9).
   Hint: Your result must be similar to (P.6.7).
2. Apply the above arguments recursively to find formulas for V_n, $n \geq 2$.
3. Use a computer algebra system (e.g., Maple) to evaluate exact values of V_n, for $n = 1, 2, \cdots, 20$.
4. Plot $\{(n, V_n) \mid n = 1, 2, \cdots, 20\}$.
   Hint: You may use Maple-code P.27 and your plot must look like Figure P.8.

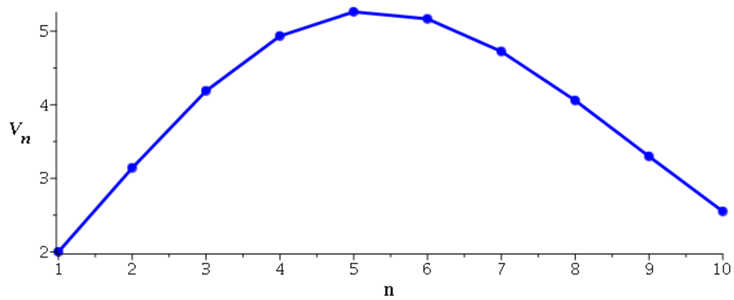


Figure P.8: A plot for V_n, where $\max(V) = V_5 = \dfrac{8\pi^2}{15} \approx 5.263789$.

Maple-code P.27. Assume you have a formula for V_n of the form

$$V_n = V_{n-1}\, g(n). \tag{P.6.10}$$

Then you may implement a Maple code:

Maple Script for the Computation of Vn and Plotting

```
with(plots): with(plottools):
with(VectorCalculus): with(Student[MultivariateCalculus]):

m := 20:
V := Vector(m):
V[1] := 2:
for n from 2 to m do V[n] := V[n-1]*g(n); end do:
max[index](V); max(V); evalf(%);

X := [seq(n, n = 1..m)]:
pp := pointplot(Vector(X), Vector(V), color = blue, symbol = solidcircle, symbolsize = 12):
pl := plot(Vector(X), Vector(V), color = blue, thickness = 3):
display(pp, pl, scaling = constrained, labels = ["n", V__n], labelfont = ["times", "bold", 13])
```

Figure P.8 is constructed using the above code, with `m := 10:` and `g(n)` defined appropriately. □

Report. Submit hard copies of your experiences.

- Derive a formula for V_n of the form in (P.6.10).
- Implement a code to evaluate V_n, $n = 1, 2, \cdots, 20$, exactly.
- Plot the results.
- Collect all your work, in order.
- Attach a "summary" or "conclusion" page at the beginning of report.

You may work in a small group; however, you must report individually.

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